Representing Preferences with a Unique Subjective State Space: Corrigendum[1]

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1 Introduction

Dekel, Lipman, and Rustichini [2001] (henceforth DLR) contains two incomplete proofs, each of which requires stronger assumptions than are given in the paper. In one case, the version of the independence axiom used has to be strengthened to hold not only for strict comparisons, as assumed, but also for indifference. Section 2 shows the need for this strengthening by example and presents the strengthening. The other incomplete proof requires us to strengthen continuity to include Lipschitz continuity. In Section 3, we explain the gap, the new axiom, and prove that it closes the gap. The appendix provides an example showing that the original axioms were not sufficient. We also show that the strengthening of continuity is not needed in two useful subclasses of preferences, namely preferences with a finite representation and preferences satisfying monotonicity (preference for flexibility). We demonstrate these results, in the process providing an axiomatic characterization of the former class. The remainder of this section summarizes the key points from DLR.

DLR consider a preference relation $\succ$ defined over the set of nonempty subsets of $\Delta(B)$, endowed with the Hausdorff topology, where B is a finite set and $\Delta(B)$ is the set of probability distributions over B. They characterize three types of representations of $\succ$. Each representation consists of three objects: a (nonempty) state space S, a state–dependent utility function $U : \Delta(B) \times S \to \mathbf{R}$, and an aggregator $u : \mathbf{R}^S \to \mathbf{R}$. For each representation, DLR require two properties. First, the function V defined by

$$V(x) = u\left(\left(\sup_{\beta\in x} U(\beta, s)\right)_{s\in S}\right) \tag{1}$$

is continuous and represents $\succ$. Second, each $U(\cdot, s)$ is an expected–utility function in the sense that for every $\beta \in \Delta(B)$,

$$U(\beta, s) = \sum_{b\in B} \beta(b)U(b, s).$$

(As usual, $U(b, s)$ is short–hand for $U(\delta_b, s)$ where δ_b is a degenerate lottery with probability 1 on b.) DLR interpret $\sup_{\beta\in x} U(\beta, s)$ as a possible ex post utility level from menu x and refer to $\left(\sup_{\beta\in x} U(\beta, s)\right)_{s\in S}$ as the vector of ex post utilities.

The most general representation is what DLR call a weak EU representation. In this case, the only requirements added to the above are some nonredundancy conditions which are not relevant for the issues discussed here.

While S is called a "state space," it is just an index set. DLR use the phrase "subjective state space" to refer to the collection of preferences over $\Delta(B)$ induced by U and S. They show that this set is essentially unique for a weak EU representation and hence for any of the representations they consider. In the case where the subjective state space is finite, "essentially unique" is the same as unique.

2 Existence of an Ordinal EU Representation

The second type of representation considered in DLR is an ordinal EU representation. Such a representation adds to the properties discussed above the requirement that u is strictly increasing on $\{(\sup_{\beta \in x} U(\beta, s))_{s \in S} \mid x \subseteq \Delta(B)\}$.

We thank Jacob Sagi for the following example. Let $B = \{a, b, c\}$. Define expected utility preferences U_1 and U_2 by

	U_1	U_2
a	0	0
b	0	1
c	2	2

Define a preference over menus by

$$V(x) = \left[\max_{\beta \in x} U_1(\beta)\right] \left[\max_{\beta \in x} U_2(\beta)\right].$$

While this preference satisfies the axioms DLR state in Theorem 3.A, the conclusion of that theorem does not hold. Note that $V(\{a\}) = V(\{a, b\}) = 0$ even though the menu $\{a, b\}$ yields strictly higher ex post utility in subjective state U_2. Hence there is no representation of this preference which aggregates the ex post utilities with a strictly increasing u.[1] Hence this preference does not have an ordinal EU representation.

The problem is the definition of weak independence, defined by DLR as

Axiom 1 (Weak Independence — Original.) *If $x' \subset x$ and $x \succ x'$, then for all $\lambda \in (0, 1]$ and all $\bar{x}$,*

$$\lambda x + (1 - \lambda)\bar{x} \succ \lambda x' + (1 - \lambda)\bar{x}.$$

DLR's footnote 31 states that this implies the usual "indifference version" of independence (see statement below). This claim is incorrect as Sagi's example shows. For example, we obtain a contradiction for $x' = \{a\}$, $x = \{a, b\}$, $\bar{x} = \{c\}$, and $\lambda = 1/2$. It is not hard to show that the indifference version of independence is also a necessary condition for an ordinal EU representation. Hence DLR's Theorem 3.A holds as stated if we strengthen weak independence to

[1] The uniqueness of the subjective state space shown by DLR implies that we cannot escape this conclusion by using some other set of possible utility functions to represent this preference.

Axiom 2 (Weak Independence — New.) *If $x' \subset x$, then for all $\lambda \in (0,1]$ and all $\bar{x}$,*

$$x \succ x' \implies \lambda x + (1-\lambda)\bar{x} \succ \lambda x' + (1-\lambda)\bar{x}$$
$$x \sim x' \implies \lambda x + (1-\lambda)\bar{x} \sim \lambda x' + (1-\lambda)\bar{x}.$$

3 Existence of an Additive EU Representation

Additive EU representations differ from ordinal EU representations in two ways. First, the function u is integration with respect to a measure over S. Second, u is not required to be increasing. In other words, the measure is not required to be positive. More precisely, an additive EU representation is a weak EU representation with the requirement that there is a finitely additive measure μ on S such that, for all $x \subseteq \Delta(B)$,

$$u((\sup_{\beta\in x} U(\beta,s))_{s\in S}) = \int_S \sup_{\beta\in x} U(\beta,s)\mu(ds).$$

DLR's existence proof begins by observing that their continuity and independence axioms imply that one can restrict attention to the set of closed, convex, nonempty subsets of $\Delta(B)$, denoted X. Next, DLR construct an affine function V on X which represents the preference. They give an artificial state space S^K and show that X is one–to–one with a certain set of functions, C, mapping S^K to the reals. The set x is mapped to its *support function*, the function $\sigma_x(s)$ which gives the maximal expected utility from the set x in state s. I.e., $\sigma_x(s) = \max_{\beta\in x} \beta \cdot s$. Since C is one–to–one with X, we can define a function $W : C \to \mathbf{R}$ so that $W(\sigma_x) = V(x)$. Note that an additive EU representation has $V(x) = \int_S \sigma_x(s)\mu(ds)$ for some measure μ, so the connection to support functions is natural.

The remainder of the proof extends W to a larger space H^*, which is dense in $C(S^K)$, the set of continuous real–valued functions on S^K. The only important facts about W and H^* which are used below are the following. First, for all $f \in H^*$, there exists $\sigma^1, \sigma^2 \in C$ and a number $r > 0$ such that $f = r(\sigma^1 - \sigma^2)$. Second, W is linear in the sense that $W(r_1 f - r_2 g) = r_1 W(f) - r_2 W(g)$. Finally, W is normalized so that $W(\mathbf{0}) = 0$ where $\mathbf{0}$ denotes the zero function.[2]

In Lemma 12, DLR show that W is bounded on H^* in order to use the Hahn–Banach theorem to extend it to $C(S^K)$. However, the proof assumes that W is continuous on H^*, a hypothesis which is not justified. We thank Christopher Chambers for first pointing out this error. In the Appendix, we show that this gap in the proof cannot be filled without

[2]For proofs and more details, see the supplementary appendix.

additional axioms by giving an example of a preference satisfying the DLR axioms which does not have an additive EU representation.

The problem is that DLR need W to be Lipschitz continuous on C to ensure that it is continuous on H^*. Since W on C is essentially equivalent to V, DLR require V to be Lipschitz continuous. Given sets x and y, let $d_h(x, y)$ denote the Hausdorff distance between x and y.

Definition 1 *$V : X \to \mathbf{R}$ is* Lipschitz continuous *if there is a $\bar{K}$ such that*

$$V(y) - V(x) \leq \bar{K} d_h(x, y), \quad \forall x, y.$$

The following axiom yields this property. To state it, let x^b denote a best set and x^w a worst set according to $\succ$. Continuity of $\succ$ and compactness of the domain ensures such sets exist. By DLR's nontriviality axiom, $x^b \succ x^w$.[3]

Axiom 3 (L–Continuity) *There exists a $K > 0$ such that for every $\varepsilon \in (0, 1/K)$, for every x and y with $d_h(x, y) \leq \varepsilon$,*

$$(1 - K\varepsilon)x + K\varepsilon x^b \succeq (1 - K\varepsilon)y + K\varepsilon x^w.$$

To see the intuition, note that continuity and $x^b \succ x^w$ implies that for any sets x and y, there is a $\lambda \in (0, 1)$ such that $\lambda x + (1 - \lambda)x^b \succeq \lambda y + (1 - \lambda)x^w$. If $x \succeq y$, then independence implies this preference for all $\lambda \in (0, 1)$. So suppose $y \succ x$. Then there will be some largest λ for which $\lambda x + (1 - \lambda)x^b \succeq \lambda y + (1 - \lambda)x^w$. Intuitively, as we consider x and y closer together, this maximum λ "should" be larger and "should" converge to 1 as the distance between x and y converges to 0. Moreover, it seems plausible that this convergence should be "gradual" in the sense that this λ does not jump to 1 when the distance between x and y reaches zero. L–continuity corresponds to this intuition.

Before showing that L–continuity delivers the appropriate Lipschitz continuity of the representation, we briefly comment on an insignificant separate error in DLR. DLR define S^K, the domain for their support functions, to be $\{(s_1, \ldots, s_K) \in \mathbf{R}^K \mid \sum_i s_i = 0, \ \sum_i |s_i| = 1\}$. DLR's Lemma 8 says that

$$d_h(x, y) = \| \sigma_x - \sigma_y \| \tag{2}$$

where $\| \cdot \|$ denotes sup norm and σ_x is the support function for x. This statement is not correct. The error is easily fixed either by redefining S^K by replacing $\sum_i |s_i| = 1$ with

[3] As the argument below will make clear, one could replace these by any pair of sets with $x^b \succ x^w$. We use the best and worst sets to make the axiom as weak as possible.

$\sum_i s_i^2 = 1$ or by retaining the definition of S^K and showing that there are constants M_1 and M_2 such that

$$M_1 \parallel \sigma_x - \sigma_y \parallel \leq d_h(x, y) \leq M_2 \parallel \sigma_x - \sigma_y \parallel .$$

Either approach, with some minor and obvious changes in proof details, eliminates this problem. For simplicity, we adopt the former approach, redefining S^K so that equation (2) does hold. (For proof of (2), see Theorem 1.8.11 in Schneider [1993], page 53.) We use (2) repeatedly below.

Lemma 1 *Assume $\succ$ has an affine representation V. Then V is Lipschitz continuous if and only if $\succ$ satisfies L–continuity.*

Proof. Suppose $\succ$ satisfies L–continuity. Fix the K of the axiom, any $D < 1/K$, and any x and y with $d_h(x, y) \leq D$. Let $\delta = d_h(x, y)$. Then

$$(1 - K\delta)x + K\delta x^b \succeq (1 - K\delta)y + K\delta x^w.$$

Using the representation and the fact that it is affine, this implies

$$V(y) - V(x) \leq \frac{K}{1 - K\delta}[V(x^b) - V(x^w)]d_h(x, y).$$

By hypothesis, $K\delta = Kd_h(x, y) \leq KD < 1$. Hence $K/(1 - K\delta) \leq K/(1 - KD) < \infty$. Let $\bar{K} = [K/(1 - KD)][V(x^b) - V(x^w)]$. Then for any x and y with $d_h(x, y) \leq D$, we have

$$V(y) - V(x) \leq \bar{K}d_h(x, y).$$

To complete the proof that V is Lipschitz continuous, we show that the same holds for arbitrary x and y. So fix any x and y. Fix any sequence

$$0 = \lambda_0 < \lambda_1 < \ldots < \lambda_M < \lambda_{M+1} = 1$$

such that $(\lambda_{m+1} - \lambda_m)d_h(x, y) \leq D$. Obviously, such a sequence exists. Let $x_m = \lambda_m x + (1 - \lambda_m)y$. Then

$$\begin{aligned} d_h(x_{m+1}, x_m) &= \parallel \sigma_{x_{m+1}} - \sigma_{x_m} \parallel \\ &= (\lambda_{m+1} - \lambda_m) \parallel \sigma_x - \sigma_y \parallel \\ &= (\lambda_{m+1} - \lambda_m)d_h(x, y). \end{aligned}$$

Hence from the previous part, we see that

$$V(x_{m+1}) - V(x_m) \leq \bar{K}(\lambda_{m+1} - \lambda_m)d_h(x, y).$$

Summing both sides over m from $m = 0$ to $m = M$ gives

$$V(y) - V(x) \leq \bar{K} d_h(x, y),$$

so V is Lipschitz continuous.

For the converse, suppose there is a $\bar{K}$ such that $V(y) - V(x) \leq \bar{K} d_h(x, y)$ for all x and y. Let $K = \bar{K}/[V(x^b) - V(x^w)]$. So for all x and y, we have

$$V(y) - V(x) \leq K[V(x^b) - V(x^w)] d_h(x, y).$$

So for all x and y with $d_h(x, y) < 1/K$,

$$V(y) - V(x) \leq \frac{K d_h(x, y)}{1 - K d_h(x, y)} [V(x^b) - V(x^w)].$$

So for every $\varepsilon \in [d_h(x, y), 1/K)$,

$$V(y) - V(x) \leq \frac{K\varepsilon}{1 - K\varepsilon} [V(x^b) - V(x^w)].$$

Rearranging by reversing the steps above, we see that $\succ$ is L–continuous. ▌

We obtain the following corrected version of DLR's Theorem 4.A.

Theorem 1 *The ex ante preference $\succ$ has an additive EU representation if and only if it satisfies weak order, continuity, nontriviality, independence, and L–continuity.*

Proof. Necessity of the first four axioms is obvious. To show necessity of L–continuity, we show that Lipschitz continuity of V is necessary, implying from Lemma 1 that L–continuity of $\succ$ is necessary. So suppose V is an additive EU representation. That is, there is a measure μ such that

$$V(x) = \int_{S^K} \sigma_x \, \mu(ds).$$

We can write μ as $\mu^+ - \mu^-$ where both of these measures are positive. Let

$$K = \int_{S^K} \mu^+(ds) + \int_{S^K} \mu^-(ds).$$

Then

$$\begin{aligned} V(y) - V(x) &\leq |\textstyle\int_{S^K} (\sigma_y - \sigma_x) \, \mu(ds)| \\ &= |\textstyle\int_{S^K} (\sigma_y - \sigma_x) \, \mu^+(ds) - \int_{S^K} (\sigma_y - \sigma_x) \, \mu^-(ds)| \\ &\leq |\textstyle\int_{S^K} (\sigma_y - \sigma_x) \, \mu^+(ds)| + |\int_{S^K} (\sigma_y - \sigma_x) \, \mu^-(ds)| \\ &\leq K \parallel \sigma_y - \sigma_x \parallel \\ &= K d_h(x, y). \end{aligned}$$

Thus V is Lipschitz continuous. Since it is affine, Lemma 1 implies $\succ$ satisfies L–continuity.

For sufficiency, recall that the missing step in DLR's proof is to show that there is a κ such that for all $f \in H^*$, $W(f) \leq \kappa \parallel f \parallel$. For any set x, $W(\sigma_x) = V(x)$. Hence there exists K such that for all $\sigma^1, \sigma^2 \in C$,

$$|W(\sigma^1) - W(\sigma^2)| \leq K \parallel \sigma^1 - \sigma^2 \parallel .$$

For any $f \in H^*$, there exists $\sigma^1, \sigma^2 \in C$ and a number $r > 0$ such that $f = r(\sigma^1 - \sigma^2)$. Since W is linear,

$$W(f) = r[W(\sigma^1) - W(\sigma^2)].$$

Hence

$$W(f) \leq |W(f)| = r|W(\sigma^1) - W(\sigma^2)|,$$

so

$$W(f) \leq Kr \parallel \sigma^1 - \sigma^2 \parallel \ = K \parallel f \parallel .$$

Setting $\kappa = K$, we have the required bound. ▌

While we require an axiom to ensure Lipschitz continuity in general, there are two special cases of interest where a separate axiom is not needed. First, when the state space is finite, the standard continuity condition is sufficient. Demonstrating this is made more complex by the fact that the state space is a subjective construct in the DLR representation so whether it is finite or not depends on the preference.

Below we give an axiom which is necessary and sufficient for the state space to be finite. We show that when this axiom holds, we do not need to add L–continuity. Given $x \in X$, say that $\hat{x} \subseteq x$ is *critical* for x if for all $y \in X$ such that $\hat{x} \subseteq y \subseteq x$, we have $y \sim x$. Intuitively, a critical subset of x contains all the "relevant" points in x. It is easy to use the DLR axioms to show that the boundary of x is critical for x, so every set has at least one critical subset.

Axiom 4 (Finiteness) *Every set $x \in X$ has a finite critical subset.*

The following is proved in the Appendix.

Theorem 2 *$\succ$ has an additive EU representation with a finite state space if and only if it satisfies weak order, continuity, nontriviality, independence, and finiteness.*

The basic idea behind the proof of Theorem 2 is very simple. Fix a sphere, say x^*, in the interior of the simplex and a finite critical subset. The proof shows that this set can be taken to be a subset of the boundary of the sphere. For each point in the critical set, there is a unique expected utility preference over lotteries such that this point is the best point in x^*. In this sense, the critical set identifies a subjective state space. Note that this set of expected utility preferences "works" when restricted to subsets of x^* which contain the critical set: any such set will generate the same maximal expected utility for each state. Hence a representation with this space will have the agent indifferent among all such sets, just as the definition of critical implies the agent must be. We then use independence to show that the agent must be indifferent between any two sets generating the same maximal expected utilities in each of these states. Hence we can restrict attention to a finite set of states.

Theorem 2 clarifies the relationship between DLR and the main representation in Gul–Pesendorfer [2001]. Their axioms are weak order, continuity, independence, and set betweenness. They do not rule out trivial representations and so do not use a nontriviality axiom. Obviously, DLR could do likewise. One can show that their set betweenness axiom implies our finiteness axiom, explaining why they do not require L–continuity.[4] For more on the relationship, see Dekel, Lipman, and Rustichini [2005].

A second situation in which L–continuity is not needed is when the preference is monotonic. It is easy to show

Theorem 3 *$\succ$ has an additive EU representation with a measure μ which is always positive iff it satisfies weak order, continuity, nontriviality, independence, and monotonicity.*

Proof. Necessity is straightforward. For sufficiency, note that monotonicity implies that W is increasing in the pointwise order on $\sigma \in C$. So consider $f, g \in H^*$ with $f \geq g$ pointwise or $f - g \geq \mathbf{0}$. Since H^* is a vector subspace $f - g \in H^*$. Hence there exists $\sigma^1, \sigma^2 \in C$ and $r > 0$ such that $r[\sigma^1 - \sigma^2] = f - g \geq \mathbf{0}$. So $\sigma^1 \geq \sigma^2$ pointwise. Hence $W(\sigma^1) \geq W(\sigma^2)$, so $W(r[\sigma^1 - \sigma^2]) \geq W(\mathbf{0}) = 0$ implying $W(f - g) \geq 0$ or $W(f) \geq W(g)$. Hence W is increasing on H^*.

For any $f \in H^*$, $f \leq \| f \| \mathbf{1}$, where $\mathbf{1}$ denotes the function identically equal to 1, so $W(f) \leq \| f \| W(\mathbf{1})$. Hence letting $\kappa = W(\mathbf{1})$, we again have the upper bound needed to complete the proof in DLR. ▌

[4] In addition, Gul–Pesendorfer assume B is compact, rather than finite.

Appendix

A Example

We give an example of a preference satisfying the axioms DLR state for Theorem 4.A but which violates L–continuity and hence does not have an additive EU representation. We define the preference by constructing a function W on C, the space of support functions. This induces a preference on the closed, convex sets by $x \succ y$ iff $W(\sigma_x) > W(\sigma_y)$. Since any set is indifferent to its closure and its convex hull, this is sufficient to define a preference over all menus. This preference will satisfy independence iff W is affine, continuity iff W is continuous with respect to the sup norm, and L–continuity iff W is Lipschitz continuous.

Let $\| \cdot \|_E$ denote the Euclidean norm. Assume B has at least three elements. To construct W, first choose an arbitrary $s^* \in S^K$ and a sequence $\{s^n\} \subset S^K$ such that $\| s^n - s^* \|_E = \frac{1}{n^2}$ for $n = 1, 2, \ldots$. Since B has at least three elements, it is easily verified that such a sequence exists. Define $W : C \to \mathbf{R}$ by

$$W(\sigma) = \sum_{n=1}^{\infty} [\sigma(s^n) - \sigma(s^*)].$$

It is not hard to show that for any $\sigma \in C$ and $s, s' \in S^K$,[5]

$$|\sigma(s) - \sigma(s')| \leq \| s - s' \|_E .$$

Using this, we see that W is well–defined since for any $\sigma \in C$,

$$W(\sigma) \leq \sum_{n=1}^{\infty} |\sigma(s^n) - \sigma(s^*)| \leq \sum_{n=1}^{\infty} \| s^n - s^* \|_E = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

It is also easy to see that W is affine in σ. Hence the preference satisfies independence.

We now show that W is continuous on C. It is not hard to show that for any $\sigma_x, \sigma_y \in C$,

$$\begin{array}{rl} |(\sigma_x(s^n) - \sigma_y(s^n)) - (\sigma_x(s^*) - \sigma_y(s^*))| & \leq \min\{2 \| \sigma_x - \sigma_y \|, \, 2 \| s^n - s^* \|_E\} \\ & = 2 \min\{\| \sigma_x - \sigma_y \|, \, 1/n^2\} \end{array} .$$

Hence

$$\begin{array}{rcl} |W(\sigma_x) - W(\sigma_y)| & \leq & \sum_{n=1}^{\infty} |(\sigma_x(s^n) - \sigma_y(s^n)) - (\sigma_x(s^*) - \sigma_y(s^*))| \\ & \leq & 2 \sum_{n=1}^{\infty} \min\{\| \sigma_x - \sigma_y \|, \, 1/n^2\} \\ & = & 2 \sum_{n \leq \| \sigma_x - \sigma_y \|^{-1/2}} \| \sigma_x - \sigma_y \| + \sum_{n > \| \sigma_x - \sigma_y \|^{-1/2}} \frac{2}{n^2} \\ & \leq & 2 \| \sigma_x - \sigma_y \|^{1/2} + \sum_{n > \| \sigma_x - \sigma_y \|^{-1/2}} \frac{2}{n^2} \end{array}$$

[5] Proof: Fix $\beta \in x$ with $\sigma_x(s) = \beta \cdot s$. Then $\sigma_x(s) - \sigma_x(s') \leq \beta \cdot s - \beta \cdot s' \leq \| \beta \|_E \cdot \| s - s' \|_E$. Since $\| \beta \|_E \leq 1$, the last term is less than $\| s - s' \|_E$. Reversing the role of s and s' completes the proof.

Both terms converge to 0 as $\| \sigma_x - \sigma_y \|$ converges to 0. Thus W is continuous.

Finally, we show that W is not Lipschitz continuous. First, we note two useful facts. For any $s, s' \in S^K$, $s \cdot s' \leq 1$. Also, one can rearrange $\| s^n - s^* \|_E = 1/n^2$ to show that $s^n \cdot s^* = 1 - 1/2n^4$.

Let $x = \{\beta \mid \beta \cdot s^* \leq 0\}$. Let $\beta^* = (1/K, \ldots, 1/K)$ where K is the cardinality of B. $\sum_k s_k = 0$ for all $s \in S^K$ implies $\beta^* \in x$. For any $\varepsilon \in (0, 1/K)$, define $x(\varepsilon) = \mathrm{conv}(x \cup \{\beta^* + \varepsilon s^*\})$. It is not hard to show $\beta^* + \varepsilon s^* \in \Delta(B)$ for all $\varepsilon \in (0, 1/K)$.

It is easy to see that

$$\sigma_{x(\varepsilon)}(s) = \max\{(\beta^* + \varepsilon s^*) \cdot s, \sigma_x(s)\} = \max\{\beta^* \cdot s + \varepsilon s^* \cdot s, \sigma_x(s)\}.$$

For $s = s^*$,

$$\sigma_{x(\varepsilon)}(s^*) = \max\{\beta^* \cdot s^* + \varepsilon s^* \cdot s^*, \sigma_x(s^*)\} = \max\{0 + \varepsilon, 0\} = \varepsilon,$$

where the second equality follows from $\sum_k s_k^* = 0$, $\sum_k (s_k^*)^2 = 1$, and the definition of x. Hence $\sigma_{x(\varepsilon)}(s^*) = \varepsilon = \varepsilon + \sigma_x(s^*)$.

For any $s \in S^K$, note that $\beta^* \cdot s \leq \sigma_x(s)$ as $\beta^* \in x$. Also, from the second useful fact above, $s^* \cdot s \leq 1$. Hence

$$\beta^* \cdot s + \varepsilon s^* \cdot s \leq \sigma_x(s) + \varepsilon,$$

implying $\sigma_x(s) \leq \sigma_{x(\varepsilon)}(s) \leq \sigma_x(s) + \varepsilon$.

We now show that the first inequality holds with equality at $s = s^n$ for all $n \leq (\frac{1}{2} + \frac{1}{2K\varepsilon})^{1/4}$. To see this, for each n, let $\beta_n = \beta^* + \frac{1}{K}(s^n - s^*)$. It is not hard to show that $\beta_n \in \Delta(B)$ for all n. In addition,

$$\beta_n \cdot s^* = \frac{1}{K}(s^n - s^*) \cdot s^* = 0 + \frac{1}{K}(1 - 1/2n^4 - 1) < 0.$$

Hence $\beta_n \in x$ for all n. Finally, note that

$$\begin{aligned} \beta_n \cdot s^n \geq (\beta^* + \varepsilon s^*) \cdot s^n \quad & \text{iff} \quad \tfrac{1}{K}(s^n \cdot s^n - s^* \cdot s^n) \geq \varepsilon s^* \cdot s^n \\ & \text{iff} \quad \tfrac{1}{K}\left(\tfrac{1}{2n^4}\right) \geq \varepsilon\left(1 - \tfrac{1}{2n^4}\right). \end{aligned}$$

Rearranging, this holds iff $n \leq (\frac{1}{2} + \frac{1}{2K\varepsilon})^{1/4}$. Hence when this inequality holds, $\sigma_x(s^n) = \sigma_{x(\varepsilon)}(s^n)$.

Then, we have

$$\begin{array}{rcl} W(\sigma_x) - W(\sigma_{x(\varepsilon)}) & = & \sum_{n=1}^{\infty} \Big[(\sigma_x(s^n) - \sigma_{x(\varepsilon)}(s^n)) - (\sigma_x(s^*) - \sigma_{x(\varepsilon)}(s^*)) \Big] \\ & = & \sum_{n=1}^{\infty} \Big[(\sigma_x(s^n) - \sigma_{x(\varepsilon)}(s^n)) - (-\varepsilon) \Big] \\ & = & \sum_{n \le (\frac{1}{2}+\frac{1}{2K\varepsilon})^{1/4}} [0+\varepsilon] + \sum_{n > (\frac{1}{2}+\frac{1}{2K\varepsilon})^{1/4}} \Big[(\sigma_x(s^n) - \sigma_{x(\varepsilon)}(s^n)) + \varepsilon \Big] \\ & \ge & \left[\left(\frac{1}{2} + \frac{1}{2K\varepsilon} \right)^{1/4} - 1 \right] \varepsilon + \sum_{n > (\frac{1}{2}+\frac{1}{2K\varepsilon})^{1/4}} [(-\varepsilon) + \varepsilon] \\ & = & \left[\left(\frac{1}{2} + \frac{1}{2K\varepsilon} \right)^{1/4} - 1 \right] \parallel \sigma_x - \sigma_{x(\varepsilon)} \parallel \end{array}$$

Since $(\frac{1}{2} + \frac{1}{2K\varepsilon})^{1/4} \to \infty$ as $\varepsilon \to 0$, W cannot be Lipschitz continuous on C.

B Proof of Theorem 2

Necessity is straightforward. For sufficiency, fix any sphere, say x^*, in the interior of $\Delta(B)$. By Axiom 4, x^* has a finite critical subset. Let x_c denote such a subset. We claim that we may as well assume x_c is contained in the boundary of x^*. To see this, suppose it is not. For every point in x_c, associate any line through this point. Let $\hat{x}_c$ denote the collection of intersections of these lines with the boundary of x^*. Obviously, $\hat{x}_c$ is finite. Also, it is easy to see that $\text{conv}(x_c) \subseteq \text{conv}(\hat{x}_c)$. In light of this, consider any convex $y \subseteq x^*$ and suppose $\hat{x}_c \subseteq y$. Then

$$x_c \subseteq \text{conv}(x_c) \subseteq \text{conv}(\hat{x}_c) \subseteq y \subseteq x^*.$$

So $y \sim x^*$. Hence $\hat{x}_c$ is a finite critical subset of x^* which is contained in the boundary of x^*. So without loss of generality, we assume x_c is contained in the boundary of x^*.

Since x^* is a sphere, there is a one–to–one mapping, say g, from the boundary of x^* to S^K where $g(\beta)$ is the s such that β is the unique maximizer of $\alpha \cdot s$ over $\alpha \in x$. That is, $g(\beta)$ is the s for which we have an indifference curve tangent to x^* at β. Let

$$S^* = g(x_c) = \{s \in S^K \mid g(\beta) = s \text{ for some } \beta \in x_c\}.$$

Let

$$x = \bigcap_{\beta \in x_c} \{\alpha \in \Delta(B) \mid \alpha \cdot g(\beta) \le \beta \cdot g(\beta)\}.$$

That is, x is the polytope bounded by the hyperplanes tangent to x^* at the points in x_c.

Lemma 2 *x_c is critical for x.*

Proof. Obviously, $x_c \subset x$. Fix any convex y such that $x_c \subseteq y \subseteq x$. We show that $y \sim x$.

To show this, fix any $\varepsilon > 0$ and let

$$y^\varepsilon = \text{conv}\left(x_c \ \cup \ \left[\bigcap_{\beta \in x_c} \{\alpha \in y \mid \alpha \cdot g(\beta) \leq \beta \cdot g(\beta) - \varepsilon\} \right] \right).$$

Note that $x_c \subseteq y^\varepsilon \subseteq y$. Also, $y^\varepsilon \to y$ as $\varepsilon \downarrow 0$ since $x_c \subseteq y \subseteq x$.

We claim that

Claim 1 *For every $\varepsilon > 0$, there exists $\lambda < 1$ such that*

$$\lambda \text{conv}(x_c) + (1 - \lambda) y^\varepsilon \subseteq x^*.$$

We establish this geometric property shortly. First, note that with this claim, the proof of the lemma can be completed as follows. Fix any $\varepsilon > 0$ and $\lambda \in (0, 1)$ such that $\lambda \text{conv}(x_c) + (1 - \lambda) y^\varepsilon \subseteq x^*$. Because $x_c \subseteq y^\varepsilon$, we have

$$x_c \subseteq \lambda \text{conv}(x_c) + (1 - \lambda) y^\varepsilon \subseteq x^*.$$

Since x_c is critical for x^* and $\lambda \text{conv}(x_c) + (1-\lambda) y^\varepsilon \in X$, this implies $\lambda \text{conv}(x_c) + (1-\lambda) y^\varepsilon \sim x^*$. The fact that x_c is critical for x^* also implies $\text{conv}(x_c) \sim x^*$. Hence independence requires $y^\varepsilon \sim x^*$. Since this is true for all $\varepsilon > 0$, continuity implies $y \sim x^*$. But this argument also works for the case of $y = x$, so we see that $x \sim x^*$. Hence $y \sim x$, so x_c is critical for x.

Proof of Claim 1. First, note that it is sufficient to prove this for the case of $y = x$ since this makes the set on the left–hand side the largest possible. Next, note that it is then sufficient to show that for every $\varepsilon > 0$, there exists $\lambda < 1$ such that every extreme point of $\lambda \text{conv}(x_c) + (1 - \lambda) x^\varepsilon$ is contained in x^*. Since each such extreme point must be a convex combination of extreme points in x_c and x^ε, this implies that a sufficient condition is that there is a $\lambda < 1$ such that for every $\alpha_1 \in x_c$ and $\alpha_2 \in \text{ext}(x^\varepsilon)$, $\lambda \alpha_1 + (1-\lambda) \alpha_2 \in x^*$ where $\text{ext}(\cdot)$ denotes the set of extreme points. Since x^ε is a convex polyhedron, it has finitely many extreme points. Also, x_c is finite. Since there are finitely many α_1 and α_2 to handle, it is sufficient to show that for every $\alpha_1 \in x_c$ and $\alpha_2 \in \text{ext}(x^\varepsilon)$, there is a $\lambda \in (0, 1)$ such that $\lambda \alpha_1 + (1 - \lambda) \alpha_2 \in x^*$.

Equivalently, we show that for every $\alpha_1 \in x_c$ and $\alpha_2 \in x^\varepsilon$, there exists $\lambda \in (0, 1)$ such that $(\lambda \alpha_1 + (1 - \lambda) \alpha_2) \cdot s \leq \sigma_{x^*}(s)$ for all $s \in S^K$. That is,

$$(1 - \lambda)(\alpha_2 \cdot s - \alpha_1 \cdot s) \leq \sigma_{x^*}(s) - \alpha_1 \cdot s, \quad \forall s \in S^K. \tag{3}$$

Since $\alpha_1 \in x^*$, we have $\sigma_{x^*}(s) \geq \alpha_1 \cdot s$ for all $s \in S^K$. By construction, there is a unique s, say $\hat{s} = g(\alpha_1)$, such that this inequality holds with equality. For all $s \neq \hat{s}$, $\sigma_{x^*}(s) > \alpha_1 \cdot s$. Also, by definition of x^ε, $\alpha_2 \in x^\varepsilon$ implies that $\alpha_2 \cdot \hat{s} \leq \alpha_1 \cdot \hat{s} - \varepsilon$. Hence for any $\lambda \in [0,1]$, equation (3) holds at $s = \hat{s}$. For any $s \neq \hat{s}$, if $\alpha_2 \cdot s \leq \alpha_1 \cdot s$, again, equation (3) holds for all $\lambda \in [0,1]$. Hence we can restrict attention to s such that $\alpha_2 \cdot s > \alpha_1 \cdot s$ and $\sigma_{x^*}(s) > \alpha_1 \cdot s$. Given this restriction, it is clear that if $\alpha_2 \cdot s \leq \sigma_{x^*}(s)$, again, equation (3) holds for all $\lambda \in [0,1]$.

Let $\hat{S} = \{s \in S^K \mid \alpha_2 \cdot s > \sigma_{x^*}(s) > \alpha_1 \cdot s\}$. From the above, it is sufficient to show the existence of a $\lambda \in (0,1)$ satisfying equation (3) for all $s \in \hat{S}$. A sufficient condition for this is that there exists $\lambda \in (0,1)$ such that

$$(1-\lambda)(\sigma_{\Delta(B)}(s) - \alpha_1 \cdot s) \leq \sigma_{x^*}(s) - \alpha_1 \cdot s, \quad \forall s \in \hat{S}.$$

Obviously, $\sigma_{\Delta(B)}(s) - \alpha_1 \cdot s$ is bounded from above. Hence it is sufficient to show that the right–hand side of the inequality is bounded away from zero for $s \in \hat{S}$.

To see that this must hold, suppose there is a sequence $\{s^n\}$ with $s^n \in \hat{S}$ for all n with $\sigma_{x^*}(s^n) - \alpha_1 \cdot s^n \to 0$. Clearly, this implies $s^n \to \hat{s}$. But then

$$\lim_{n\to\infty} \alpha_2 \cdot s^n = \alpha_2 \cdot \hat{s} \leq \sigma_{x^*}(\hat{s}) - \varepsilon = \lim_{n\to\infty} \sigma_{x^*}(s^n) - \varepsilon,$$

implying that we cannot have $s^n \in \hat{S}$ for all n, a contradiction. Hence such a λ must exist. ▌

Lemma 3 *If y is any set with $\sigma_y(s) = \sigma_x(s)$ for all $s \in S^*$, then $y \sim x$.*

Proof: Fix any such y. Without loss of generality, assume y is convex. (Otherwise, we can replace y with its convex hull.) Clearly,

$$y \subseteq \{\beta \mid \beta \cdot s \leq \sigma_x(s) \;\; \forall s \in S^*\}$$

since otherwise y would contain points giving it a higher value of the support function for some $s \in S^*$. But the set on the right–hand side is x, so $y \subseteq x$. Obviously, then if $x_c \subseteq y$, the fact that x_c is critical for x implies $y \sim x$.

So suppose $x_c \not\subseteq y$. As noted, we must have $y \subseteq x$. So let $y_\lambda = \lambda x + (1-\lambda)y$. Obviously, y_λ converges to x as $\lambda \to 1$. For each $\beta \in x_c$, there is a face of the polyhedron x such that β is in the (relative) interior of the face. Also, y must intersect the face of the polyhedron and so y_λ must intersect the face. As λ increases, the intersection of y_λ with the face enlarges as it is pulled out toward the boundaries of the face. Clearly, for λ sufficiently large, β will be contained in the intersection of y_λ with the face of x which contains β. Take any λ larger than the biggest such λ over the finitely many $\beta \in x_c$. Then $x_c \subseteq y_\lambda \subseteq x$. Since x_c is critical for x, this implies $\lambda x + (1-\lambda)y \sim x$. By independence, then, $y \sim x$. ▌

Lemma 4 *For any y and $\hat{y}$ such that $\sigma_y(s) = \sigma_{\hat{y}}(s)$ for all $s \in S^*$, we have $y \sim \hat{y}$.*

Proof. Fix any such y and $\hat{y}$. For any $\lambda \in [0, 1)$, define $u_\lambda : S^* \to \mathbf{R}$ by

$$u_\lambda(s) = \frac{\sigma_x(s) - \lambda\sigma_y(s)}{1 - \lambda}.$$

Because $\sigma_y(s) = \sigma_{\hat{y}}(s)$ for all $s \in S^*$, it would be equivalent to use $\sigma_{\hat{y}}$ instead of σ_y. Let

$$z_\lambda = \{\beta \in \Delta(B) \mid \beta \cdot s \leq u_\lambda(s), \quad \forall s \in S^*\}.$$

Obviously, $\lambda\sigma_y(s)+(1-\lambda)u_\lambda(s) = \sigma_x(s)$ for all $s \in S^*$. This implies that for all $\lambda \in (0, 1)$, $\lambda y + (1 - \lambda)z_\lambda \subseteq x$. To see this, note that for any $\alpha \in y$ and $\beta \in z_\lambda$,

$$\lambda\alpha \cdot s + (1 - \lambda)\beta \cdot s \leq \lambda\sigma_y(s) + (1 - \lambda)u_\lambda(s) = \sigma_x(s), \quad \forall s \in S^*.$$

But $x = \cap_{s\in S^*}\{\gamma \mid \gamma \cdot s \leq \sigma_x(s)\}$, so $\lambda\alpha + (1 - \lambda)\beta \in x$.

Note also that $u_\lambda(s) \to \sigma_x(s)$ as $\lambda \downarrow 0$. We claim that this implies that there is a $\lambda \in (0, 1)$ such that for every $s \in S^*$, there exists $\beta \in z_\lambda$ with $\beta \cdot s = u_\lambda(s)$. To see this, suppose it is not true. Then for all $\lambda \in (0, 1)$, there exists $\hat{s}_\lambda \in S^*$ such that for all $\beta \in z_\lambda$, $\beta \cdot \hat{s}_\lambda < u_\lambda(\hat{s}_\lambda)$ so

$$\bigcap_{s\in S^*\setminus\{\hat{s}_\lambda\}} \{\beta \mid \beta \cdot s \leq u_\lambda(s)\} = \bigcap_{s\in S^*} \{\beta \mid \beta \cdot s \leq u_\lambda(s)\}.$$

Because S^* is finite, this implies that there exists $\hat{s} \in S^*$, a sequence $\{\lambda_n\}$ with $\lambda_n \in (0, 1)$ for all n, $\lambda_n \to 0$ such that for all n,

$$\bigcap_{s\in S^*\setminus\{\hat{s}\}} \{\beta \mid \beta \cdot s \leq u_{\lambda_n}(s)\} = \bigcap_{s\in S^*} \{\beta \mid \beta \cdot s \leq u_{\lambda_n}(s)\}.$$

But $u_{\lambda_n} \to \sigma_x$ as $n \to \infty$. Hence the limit as $n \to \infty$ of the right–hand side, namely x, cannot equal the limit of the left–hand side, a contradiction.

Hence there is a $\lambda \in (0, 1)$ such that for every $s \in S^*$, there is a $\beta \in z_\lambda$ with $\beta \cdot s = u_\lambda(s)$. Choose such a λ and let $u = u_\lambda$ and $z = z_\lambda$. Obviously, for every $s \in S^*$, there is $\alpha \in y$ with $\beta \cdot s = \sigma_y(s)$. Hence given our choice of λ, for every $s \in S^*$, there is $\gamma \in \lambda y+(1-\lambda)z$ such that $\gamma \cdot s = \lambda\sigma_y(s)+(1-\lambda)u(s) = \sigma_x(s)$. Hence $\sigma_{\lambda y+(1-\lambda)z}(s) = \sigma_x(s)$ for all $s \in S^*$. Hence Lemma 3 implies $\lambda y + (1 - \lambda)z \sim x$. The symmetric argument with $\hat{y}$ replacing y implies $\lambda\hat{y} + (1 - \lambda)z \sim x$. So $\lambda y + (1 - \lambda)z \sim \lambda\hat{y} + (1 - \lambda)z$. By independence, then, $y \sim \hat{y}$. ▌

Define W on C as in DLR. Let $\mathcal{U} = \{(\sigma_x(s))_{s\in S^*} \mid x \in X\} \subset \mathbf{R}^M$ where M is the cardinality of S^*. Let $\sigma|S^*$ denote the restriction of σ to S^*. Define a function

$\hat{W} : \mathcal{U} \to \mathbf{R}$ by $\hat{W}(U) = W(\sigma)$ for any σ such that $\sigma | S^* = U$. From Lemma 4, we see that if $\sigma | S^* = \sigma' | S^*$, then $W(\sigma) = W(\sigma')$ so $\hat{W}$ is well–defined. It is easy to see that $\hat{W}$ is affine and continuous and that $\mathcal{U}$ is closed, convex, and contains the 0 vector. It is easy to show that $\hat{W}$ has a well–defined extension to a continuous, linear function on the linear span of $\mathcal{U}$. Since $\mathcal{U}$ is finite dimensional, $\hat{W}$ has an extension to a continuous linear functional on $\mathbf{R}^M$. (See Lemma 6.13 in Aliprantis and Border [1999], for example.) Since a linear function on a finite dimensional space has a representation by means of a matrix, we can write

$$\hat{W}(U) = \sum_{s \in S^*} c_s U_s$$

where the c_s's are constants and $U = (U_s)_{s \in S^*}$. Hence

$$V(x) = \hat{W}((\sigma_x(s))_{s \in S^*}) = \sum_{s \in S^*} c_s \max_{\beta \in x} \beta \cdot s.$$

Hence we have an additive EU representation with finitely many states.

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