Type-Contingent Perfect Public Ex-Post Equilibria[*]

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Abstract

We consider repeated games with incomplete information where players observe imperfect public signals of the actions and the map from actions to signal distributions is itself unknown. To do so, we introduce the concept of type-contingent perfect public ex-post equilibrium or T-PPXE, which reduces to the PPXE of Fudenberg and Yamamoto (2009) in complete-information games, and reduces to the belief-free equilibrium of Hörner and Lovo (2009) when actions are perfectly observed. We provide a sufficient condition for the folk theorem, and a characterization of the T-PPXE payoffs in games with a known monitoring structure. Under a "sufficient rank" condition, we show that the theorems of Hörner and Lovo (2009) on games with perfectly observed actions extend to imperfect monitoring, and that the folk theorem holds if each pair of states can be distinguished by the private information of at least three players.

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1 Introduction

The fact that repeated interactions can allow new and more efficient equilibrium outcomes is one of game theory's most important insights. It has been shown to apply in a range of settings, including games with imperfect public information about opponents' actions and games with private information about the payoff functions.[1] This paper studies games with the combination of these two features, where in addition the "monitoring structure"- the map from actions to signal distributions- is itself unknown, and the private information can be about the monitoring structure as well as about the payoff functions. This describes, for example, a repeated partnership game where players observe group output but do not observe each other's effort, and each player has private information about the effect of her effort on the probability distribution of output. In some cases these informational imperfections make it impossible for players to approximate efficient outcomes with equilibrium play; in other cases there are approximately efficient equilibria when players are sufficiently patient. Our main goals in this paper are first to understand when efficient equilibria exist, and second to gain some insight about the set of equilibrium payoffs when asymptotic efficiency is impossible.

One special case of the repeated games we consider here is the class of repeated games with unknown payoff functions and perfectly observed actions, notably Forges (1984), Sorin (1984), Hart (1985), Sorin (1985), Aumann and Maschler (1995), Cripps and Thomas (2003), Gossner and Vieille (2003), Wiseman (2005), Hörner and Lovo (2009), Wiseman (2008), and Hörner, Lovo, and Tomala (2009).[2] Our work extends this literature both to the case of imperfectly

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[1] Other extensions include games with long-run and short-run players (Fudenberg, Kreps, and Maskin (1990)), games with overlapping generations of players (Kandori (1992b)), community enforcement (Kandori (1992a) and Ellison (1993)), and games with imperfect private monitoring (Compte (1998) and Kandori and Matsushima (1998)).

[2] Cripps and Thomas (2003), Gossner and Vieille (2003), and Wiseman (2005) study symmetric-information settings. In Forges (1984), Sorin (1984), Hart (1985), Sorin (1985), Aumann and Maschler (1995), Hörner and Lovo (2009), Wiseman (2008), and Hörner, Lovo, and Tomala (2009), players receive private signals about the payoff functions and so can have different beliefs. (In Wiseman (2008) the players privately observe their own realized payoff each period, in the other papers the players do not observe their own realized payoffs, and the private signals are the players' initial information or "type.")

observed actions and and a known monitoring technology, and to the case where the monitoring structure is itself unknown. Like Hörner and Lovo (2009) and Hörner, Lovo, and Tomala (2009), we restrict attention to a subset of the sequential equilibria that is both tractable and has desirable robustness properties. Specifically, we consider the set of *type-contingent perfect public ex-post equilibrium* or *T-PPXE*; these are ex-post equilibria where each player's strategy depends only on the realized public outcomes and his initial private information (hence "type-contingent") but not on the player's private information about his own past actions. T-PPXE reduces to the belief-free equilibria of Hörner and Lovo (2009) and Hörner, Lovo, and Tomala (2009) when actions are observed, [3] and reduces to the perfect public equilibrium (PPE) of Fudenberg, Levine, and Maskin (1994, hereafter FLM) in complete information games with a known monitoring structure; in games with with complete information and an unknown monitoring structure, T-PPXE reduces to the perfect public ex-post equilibria of Fudenberg and Yamamoto (2009). As with ex-post equilibria more generally, these equilibria are robust to the specification of the players' prior beliefs: a T-PPXE for a given prior distribution is a T-PPXE for an arbitrary prior.[4] T-PPXE also has the practical advantages of having a recursive structure, and not requiring the specification of the players' beliefs after each history. This allows us to characterize the payoffs of T-PPXE with an extension of the linear programming techniques introduced in Fudenberg and Levine (1994, hereafter FL) and generalized in Fudenberg and Yamamoto (2009) to repeated games with unknown outcomes.

To put this paper into perspective, note that any PPXE of the symmetric information game (where no player has initial private information about the state) in-

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[3] T-PPXE does not require that players be indifferent, and so unlike belief-free equilibria of repeated games with private monitoring (e.g. Piccione (2002), Ely and Välimäki (2002), Ely, Hörner, and Olszewski (2005), Yamamoto (2007), Kandori (2008), and Yamamoto (2009)) it is not subject to the robustness critiques that Bhaskar, Mailath, and Morris (2008) argue. Miller (2007) analyzes a different sort of ex-post equilibrium: he considers repeated games of adverse selection, where players report their types each period, as in Section 8 of FLM, and requires that announcing truthfully should be optimal regardless of the announcements of the other players.

[4] See Bergmann and Morris (2007) for a discussion of various definitions of ex-post equilibrium. Miller (2007) analyzes a different sort of ex-post equilibrium: he considers repeated games of adverse selection, where players report their types each period, as in Section 8 of FLM, and adds the restriction that announcing truthfully should be optimal regardless of the announcements of the other players.

duces a PPXE of the game where some players do have private information: these PPXE correspond to pooling equilibria of the incomplete-information game. Thus the analysis of our earlier paper applies to incomplete information games, and in particular that paper's sufficient conditions for the folk theorem are still sufficient here. However, those conditions require that the distribution of signals vary with the state in a sufficiently rich way, essentially so that the state can be learned from the signals generated by some fixed action profile. This condition is unduly restrictive when some players have private information. For example, if one player knows the state, he may be able to communicate it to the others using a strategy that conditions on the player's private information. The sufficient conditions of this paper takes the possibility of such implicit communication into account, and so they apply to cases where the conditions of our previous paper do not.

Using our linear programming techniques, we are able to provide a simple sufficient condition for a folk theorem in general games. While the exact conditions are complicated to state, the following are the key assumptions for games with three or more players: For each pair of players i and j (where possibly $i = j$) and each pair of states $\omega \neq \omega'$, either (i) there is a player $l \neq i, j$ whose private information distinguishes ω and ω', and player l can reveal this information to the opponents by choosing α_l for state ω and α'_l for state ω', or (ii) there is an action profile α (independent of the private information) that distinguishes (more formally, "statewise identifies") ω from ω'.

We then consider a few cases with additional structure that simplifies the characterization of the set of limit payoffs. We begin with the case where the state space has one component that only influences payoffs and a second component that only influences the monitoring structure; here we show that when the full rank conditions are satisfied the limit set can be determined for each payoff function separately. Next we consider games with a product structure, where there is a separate and independent signal associated with each player's action, and moreover each player knows the effect of his action on the signal distribution while the others do not. For example, in a game of bilateral production and exchange, the public signal might be the quality of a player's output, with each player having private information about the probability that she will make a high-quality good when she exerts high effort. Here we find that there are approximately efficient equilibria under fairly mild conditions. As a detailed application, we re-examine the repeated partnership example of Fudenberg and Yamamoto (2009), where only group output is observed, and the state determines the productivity of player 2. We show that if player 1's private information reveals player 2's productivity while 2 has no private information (i.e. "1 knows 2's productivity"), then the folk theorem holds in general, while if only player 2 knows player 2's productivity, the folk theorem can fail, and moreover the limit equilibrium payoffs can be bounded away from efficiency. Intuitively, player 2 cannot be induced to reveal the state when doing so would lower his equilibrium payoff, and this leads to a bound on the extent to which equilibria can trade off player 2's payoffs between the two states; in some cases this bound is so strong that it rues out the efficient outcome.

Finally, we specialize to the case of a known monitoring structure, where we show that the set of limit equilibrium payoffs with imperfectly observed actions is the same as in the observed-action case studied by Hörner and Lovo (2009) and Hörner, Lovo, and Tomala (2009) provided that the monitoring structure satisfies a full-rank condition. Hörner, Lovo, and Tomala (2009) provide a characterization of the limit equilibrium payoffs that is equivalent to ours but has a much different form; each characterization may be better suited for some applications. Our more general results show that their conclusions about limit payoffs extend to imperfectly observed actions; their work is complementary and more informative because it also explictly constructs equilibrium strategies. The assumption of a known monitoring structure also lets us provide a sufficient condition for the folk theorem that is easier to verify: the key is that for every pair of states ω and ω', there be at least three players whose private information distinguishes between ω and ω'; Hörner, Lovo, and Tomala (2009) use this same condition to obtain a folk theorem for games with observed actions. In the case of one-sided incomplete information, we are able to further extend and refine their results; for example, we find a simpler sufficient condition for the existence of T-PPXE.[5]

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[5] Hörner, Lovo, and Tomala (2009) give tight conditions ensuring that the set Q we define below is non-empty; this set equals the set of limit payoffs of T-PPXE when it has non-empty interior. The definition of the set involves the players' information and incentive constraints, while the sufficient condition we provide does not.

2 Framework

2.1 Model

Let $\boldsymbol{I} = \{1, \cdots, I\}$ represent the set of players. At the beginning of the game, Nature chooses the state of the world ω from a finite set $\Omega = \{\omega_1, \cdots, \omega_O\}$. Then each player observes a private signal, which gives (possible imperfect) information about the true state ω. The set of player i's private signals, Θ_i, is a partition of Ω, and given the true state $\omega \in \Omega$, he observes a private signal $\theta_i \in \Theta_i$ that contains ω. For notational convenience, let $\theta_i(\omega)$ denote this θ_i, i.e., $\omega \in \theta_i(\omega)$, and let $\theta(\omega) = (\theta_i(\omega))_{i \in \boldsymbol{I}}$. Given $\theta_i \in \Theta_i$, player i forms a prior about the true state ω, which is denoted by $\mu_i(\theta_i) \in \triangle\theta_i$.

Each period, players move simultaneously, and player $i \in \boldsymbol{I}$ chooses an action a_i from a finite set A_i. Given an action profile $a = (a_i)_{i \in \boldsymbol{I}} \in A \equiv \times_{i \in \boldsymbol{I}} A_i$, players observe a public signal y from a finite set Y according to the probability function $\pi^\omega(a) \in \triangle Y$; we call the function π^ω the "monitoring technology." Player i's realized payoff is $u_i^\omega(a_i, y)$, so that her expected payoff conditional on $\omega \in \Omega$ and $a \in A$ is $g_i^\omega(a) = \sum_{y \in Y} \pi_y^\omega(a) u_i^\omega(a_i, y)$; $g^\omega(a)$ denotes the vector of expected payoffs associated with action profile a. If there are $\omega' \neq \omega$ such that $\theta_i(\omega) = \theta_i(\omega')$ and $u_i^\omega(a_i, y) \neq u_i^{\omega'}(a_i, y)$ for some $a_i \in A_i$ and $y \in Y$, then we assume that player i does not observe the realized value of u_i as the game is played; if not then it is immaterial whether or not u_i is observed, as player i can compute it from a_i, y, and θ_i.[6]

In the infinitely repeated game, players have a common discount factor $\delta \in (0, 1)$. Let (a_i^τ, y^τ) be the realized pure action and observed signal in period τ, and denote player i's private history from period one to period period $t \geq 1$ by $h_i^t = (a_i^\tau, y^\tau)_{\tau=1}^t$. Let $h_i^0 = \emptyset$, and for each $t \geq 0$, let H_i^t be the set of all h_i^t. Likewise, a public history up to period $t \geq 1$ is denoted by $h^t = (y^\tau)_{\tau=1}^t$, and H^t denotes the set of all h^t. A strategy for player i is defined to be a mapping $s_i : \Theta_i \times \bigcup_{t=0}^\infty H_i^t \to \triangle A_i$. Let S_i be the set of all strategies for player i, and let $S = \times_{i \in \boldsymbol{I}} S_i$.

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[6] The equilibria we construct would still be equilibria if players observed u_i^ω and it revealed information about ω, but the private history would be larger than described in the paper. Note also that our theorems also apply if A_i depends on θ_i, all this entails is additional notation.

We define the set of feasible payoffs in a given state ω to be

$$V^{\omega} \equiv \text{co}\{(g^{\omega}(a))|a \in A\} = \{g^{\omega}(\eta)|\eta \in \Delta(A)\};$$

where $\Delta(A)$ is the set of all probability distributions over A: As in the standard case of a game with a known monitoring structure, the feasible set is both the set of feasible average discounted payoffs in the infinite-horizon game when players are sufficiently patient and the set of expected payoffs of the stage game that can be obtained when players use of a public randomizing device to implement distribution η over the action profiles.

Next we define the set of feasible payoffs of the overall game to be

$$V \equiv \times_{\omega \in \Omega} V^{\omega},$$

so that a point $v \in V = (v^{\omega_1}, \cdots, v^{\omega_O})$..

Note that a given $v \in V$ may be generated using different action distributions η^{ω} in each state ω. If players observe ω at the start of the game and are very patient, then any payoff in V can be obtained by a state-contingent strategy of the infinitely repeated game. Looking ahead, there will be equilibria that approximate payoffs in V if the state is *identified* by the signals, so that players learn it over time.

2.2 Type-Contingent Perfect Public Ex-Post Equilibria

This paper studies a special class of Nash equilibria called *type-contingent perfect public ex-post equilibria* or T-PPXE; this is an extension of the concept of perfect public ex-post equilibrium that was introduced by Fudenberg and Yamamoto (2009).

Definition 1. A strategy profile $s \in S$ is a *type-contingent perfect public ex-post equilibrium* if s_i depends only on $\theta_i \in \Theta_i$ and $h^t \in H^t$ for each $i \in \boldsymbol{I}$, and if $s|_{(\theta(\omega),h^t)}$ is a Nash equilibrium for any $\omega \in \Omega$ and $h^t \in H^t$. Here, $s_i|_{(\theta_i,h^t)}$ denotes player i's continuation strategy after he observed a signal θ_i and the past public history was h^t, and $s|_{(\theta,h^t)} = (s_i|_{(\theta_i,h^t)})_{i \in \boldsymbol{I}}$.

Note that T-PPXE coincides with PPXE if there is no asymmetric information, i.e., $\Theta_i = \{(\Omega)\}$ for all $i \in \boldsymbol{I}$. Also, for games with perfectly observed actions (which implies a known monitoring structure), T-PPXE corresponds to the belief-free equilibrium of Hörner and Lovo (2009) and Hörner, Lovo, and Tomala (2009). We say more about their work when we specialize to games with a known monitoring structure.

Given a discount factor $\delta \in (0,1)$, let $E(\delta)$ denote the set of T-PPXE payoffs, i.e., $E(\delta)$ is the set of all vectors $v = (v_i^\omega)_{(i,\omega)\in \boldsymbol{I}\times\Omega} \in \boldsymbol{R}^{I\times|\Omega|}$ such that there is a T-PPXE $s \in S$ satisfying

$$(1-\delta)E\left[\sum_{t=1}\delta^{t-1}g_i^\omega(a^t)\middle| s,\omega\right] = v_i^\omega$$

for all $i \in \boldsymbol{I}$ and $\omega \in \Omega$. Note that $v \in E(\delta)$ specifies the equilibrium payoff for all players and for all possible states.

Let $\vec{\alpha}_i = (\alpha_i^{\theta_i})_{\theta_i\in\Theta_i}$ where $\alpha_i^{\theta_i} \in \triangle A_i$ for each $\theta_i \in \Theta_i$, and let $\vec{\alpha} = (\vec{\alpha}_i)_{i\in\boldsymbol{I}}$. In words, $\vec{\alpha}$ is an action profile contingent on private information; it specifies a mixed action for each private signal θ_i of each player i. For example, if the true state is ω, then players receive the private information $\theta(\omega)$, so that $\vec{\alpha}$ says to play $\alpha^{\theta(\omega)} = (\alpha_i^{\theta_i(\omega)})_{i\in\boldsymbol{I}}$. Let $g(\vec{\alpha})$ denote the payoff vector of a state-contingent profile $\vec{\alpha}$, that is, $g(\vec{\alpha}) = (g_i^\omega(\alpha^{\theta(\omega)}))_{(i,\omega)}$. On the other hand, when players play an action profile α independently of private information, we simply denote its payoff vector by $g(\alpha)$, that is, $g(\alpha) = (g_i^\omega(\alpha))_{(i,\omega)}$.

By definition, any continuation strategy $s|_{h^t} = (s|_{\theta(\omega),h^t})_{\omega\in\Omega}$ of a T-PPXE is also a T-PPXE. Thus any T-PPXE specifies T-PPXE continuation play after each signal y, where the continuation payoffs $w(y) = (w_i^\omega(y))_{(i,\omega)\in\boldsymbol{I}\times\Omega}$ corresponding to this signal specify the payoffs for every player and every state. We will write $\pi^\omega(\alpha)\cdot w_i^\omega$ for the the expected continuation payoff at state ω under action profile α, where w_i^ω is the vector $(w_i^\omega(y))_{y\in Y}$. This recursive structure of the equilibrium payoff set motivates the following definition.

Definition 2. For $\delta \in (0,1)$ and $W \subseteq \boldsymbol{R}^{I\times|\Omega|}$, a pair $(\vec{\alpha}, v) \in (\times_{i\in\boldsymbol{I}} \times_{\theta_i\in\Theta_i} \triangle A_i) \times \boldsymbol{R}^{I\times|\Omega|}$ is *ex-post enforceable with respect to δ and W* if there is a function $w : Y \to W$ such that

$$v_i^\omega = (1-\delta)g_i^\omega(\alpha^{\theta(\omega)}) + \delta\pi^\omega(\alpha^{\theta(\omega)})\cdot w_i^\omega$$

for all $i \in \boldsymbol{I}$ and $\omega \in \Omega$, and

$$v_i^\omega \geq (1-\delta)g_i^\omega(a_i, \alpha_{-i}^{\theta_{-i}(\omega)}) + \delta\pi^\omega(a_i, \alpha_{-i}^{\theta_{-i}(\omega)})\cdot w_i^\omega$$

for all $i \in \boldsymbol{I}$, $\omega \in \Omega$, and $a_i \in A_i$.

The equality condition in this definition is a version of the familiar "adding-up" condition: It says that if the continuation payoffs are given by the function w, and current actions are given by $\vec{\alpha}$, then expected payoffs in each state ω are given by v. Note however that this definition takes into account the possibility that α varies with the private information θ. The second condition is ex-post incentive compatibility: it requires that player i cannot obtain a higher payoff than v_i^ω in any state ω by any choice of action. Since this condition is imposed at every state ω, and tests each of player i's pure actions, it does not depend on player i's private information θ_i.

For each $\delta \in (0,1)$, $W \subseteq \boldsymbol{R}^{I \times |\Omega|}$, and $\vec{\alpha} \in \times_{i \in \boldsymbol{I}} \times_{\theta_i \in \Theta_i} \triangle A_i$, let $B(\delta, W, \vec{\alpha})$ denote the set of all payoff vectors $v \in \boldsymbol{R}^{I \times |\Omega|}$ such that $(\vec{\alpha}, v)$ is ex-post enforceable with respect to δ and W. Let $B(\delta, W)$ be a union of $B(\delta, W, \vec{\alpha})$ over all $\vec{\alpha} \in \times_{i \in \boldsymbol{I}} \times_{\theta_i \in \Theta_i} \triangle A_i$.

Because T-PPXE has a recursive structure, the concepts of ex-post self-generation and local ex-post generation extend in a natural way. The proofs are omitted, since they are straightforward generalizations of Fudenberg and Yamamoto (2009).

Definition 3. A subset W of $\boldsymbol{R}^{I \times |\Omega|}$ is *ex-post self-generating with respect to* δ if $W \subseteq B(\delta, W)$.

Proposition 1. *If a subset W of $\boldsymbol{R}^{I \times |\Omega|}$ is bounded and ex-post self-generating with respect to δ, then $W \subseteq E(\delta)$.*

Definition 4. A subset W of $\boldsymbol{R}^{I \times |\Omega|}$ is *locally ex-post generated* if for each $v \in W$, there exist $\delta_v \in (0,1)$ and an open neighborhood U_v of v such that $W \cap U_v \subseteq B(\delta_v, W)$.

Proposition 2. *If a subset W of $\boldsymbol{R}^{I \times |\Omega|}$ is compact, convex, and locally ex-post generated, then there is $\overline{\delta} \in (0,1)$ such that $W \subseteq E(\delta)$ for all $\delta \in (\overline{\delta}, 1)$.*

3 Linear Programming Characterization

Let $\vec{\alpha} \in \times_i \times_{\theta_i \in \Theta_i} \triangle A_i$, $\lambda \in R^{I \times |\Omega|}$, and $\delta \in (0,1)$. Note that $\vec{\alpha}$ corresponds to a strategy of the static Bayesian game, as opposed to an element of $\times_i \triangle A_i$; this is the key difference between the linear programming problem below and the one in our earlier paper. Now consider the linear program

$$k^*(\vec{\alpha},\lambda,\delta) = \max_{\substack{v\in \boldsymbol{R}^{I\times|\Omega|}\\ w:Y\to \boldsymbol{R}^{I\times|\Omega|}}} \lambda\cdot v \qquad \text{subject to}$$

$$\begin{aligned}
&\text{(i)} \quad v_i^\omega = (1-\delta)g_i^\omega(\alpha^{\theta(\omega)}) + \delta\pi^\omega(\alpha^{\theta(\omega)})\cdot w_i^\omega \\
&\qquad\qquad \text{for all } i,\ \omega, \\
&\text{(ii)} \quad v_i^\omega \geq (1-\delta)g_i^\omega(a_i,\alpha_{-i}^{\theta_{-i}(\omega)}) + \delta\pi^\omega(a_i,\alpha_{-i}^{\theta_{-i}(\omega)})\cdot w_i^\omega \\
&\qquad\qquad \text{for all } i,\ \omega, \text{ and } a_i\in A_i, \\
&\text{(iii)} \quad \lambda\cdot v \geq \lambda\cdot w(y) \qquad\quad \text{for all } y\in Y.
\end{aligned}$$

If there is no (v,w) satisfying the constraints, let $k^*(\vec{\alpha},\lambda,\delta) = -\infty$. If for every $K > 0$ there is (v,w) satisfying all the constraints and $\lambda\cdot v > K$, then let $k^*(\vec{\alpha},\lambda,\delta) = \infty$.

Here condition (i) is the "adding-up" condition, condition (ii) is ex-post incentive compatibility, and condition (iii) requires that the continuation payoffs lie in half-space corresponding to direction vector λ and payoff vector v. Note that when $\lambda_i^\omega \neq 0$ and $\lambda_j^{\omega'} \neq 0$ for some $\omega \neq \omega'$, condition (iii) allows "utility transfer" across states.

It is easy to see that $k^*(\vec{\alpha},\lambda,\delta)$ is independent of δ, so henceforth we denote it by $k^*(\vec{\alpha},\lambda)$. Let

$$k^*(\lambda) = \sup_{\vec{\alpha}} k(\vec{\alpha},\lambda)$$

be the highest score attainable in direction λ for any choice of $\vec{\alpha}$, and for each $\lambda \in \boldsymbol{R}^{I\times|\Omega|}\setminus\{0\}$ and $k\in\boldsymbol{R}$, let $H(\lambda,k) = \{v\in\boldsymbol{R}^{I\times|\Omega|}|\lambda\cdot v\leq k\}$. Let $H(\lambda,k) = \boldsymbol{R}^{I\times|\Omega|}$ for $k=\infty$ or $\lambda = 0$, and $H(\lambda,k) = \emptyset$ for $k=-\infty$ and $\lambda\neq 0$. Now let

$$H^*(\lambda) = H(\lambda,k^*(\lambda))$$

be the maximal half-space in direction λ, and let

$$Q = \bigcap_{\lambda\in\boldsymbol{R}^{I\times|\Omega|}} H^*(\lambda).$$

As the following proposition shows, this set Q equals the limit set of T-PPXE payoffs if it is full dimensional. The proof is omitted, as it is very similar to the analogous result in Fudenberg and Yamamoto (2009).

Proposition 3. *If* $\dim Q = I\times|\Omega|$, *then* $\lim_{\delta\to 1} E(\delta) = Q$.

4 Explicit Examples

Example 1. Suppose that there are two players and two states, so that $\boldsymbol{I} = \{1,2\}$ and $\Omega = \{\omega_1, \omega_2\}$, and that neither player knows the state, i.e., $\Theta_1 = \{(\omega_1, \omega_2)\}$ and $\Theta_2 = \{(\omega_1, \omega_2)\}$. The payoffs for state ω_1 are shown in the left panel, and those for state ω_2 in the right.

	L	R
U	1, 1	−1, 2
D	2, −1	0, 0

	C	D
C	0, 0	2, −1
D	−1, 2	1, 1

Note that the stage game is prisoner's dilemma for each state, but the role of actions are reversed; specifically, (U,L) is efficient for state ω_1 while (D,R) is efficient for state ω_2.

Assume that $Y = A \times \Omega$ and $\pi_y^\omega(a) = 1$ if $y = (a, \omega)$. Thus players can learn the state from the observed signals, and can attain the efficient outcome, using strategies with the following four phases:

- Phase "Regular ω_1." Players play (U,L), which gives the efficient payoffs for state ω_1. If $y = ((U,L), \omega_1)$, stay. If $y = ((D,L), \omega_1)$, $y = ((U,R), \omega_1)$, or $y = ((D,R), \omega_1)$, then go to "Punish ω_1." If $y = ((U,L), \omega_2)$, then go to "Regular ω_2." Otherwise, go to "Punish ω_2."

- Phase "Punish ω_1." Players play (D,R), which gives the minimax payoffs for state ω_1. If $y = ((D,R), \omega_2)$, then go to "Regular ω_2." If $y = ((U,R), \omega_2)$, $y = ((D,L), \omega_2)$, or $y = ((U,L), \omega_2)$, then go to "Punish ω_2." Otherwise, stay.

- Phase "Regular ω_2." Players play (D,R), which gives the efficient payoffs for state ω_2. If $y = ((D,R), \omega_2)$, stay. If $y = ((U,R), \omega_2)$, $y = ((D,L), \omega_2)$, or $y = ((U,L), \omega_2)$, then go to "Punish ω_2." If $y = ((D,R), \omega_1)$, then go to "Regular ω_1." Otherwise, go to "Punish ω_1."

- Phase "Punish ω_2." Players play (U,L), which gives the minimax payoffs for state ω_2. If $y = ((U,L), \omega_1)$, then go to "Regular ω_1." If $y = ((D,L), \omega_1)$, $y = ((U,R), \omega_1)$, or $y = ((D,R), \omega_1)$, then go to "Punish ω_1." Otherwise, stay.

It is straightforward to check that this strategy profile with initial state "Regular ω_1" is a T-PPXE and approximates $((2,2),(2,2))$.

Example 2. As above, $\boldsymbol{I} = \{1,2\}$ and $\Omega = \{\omega_1, \omega_2,\}$, but now player 1 knows the state while player 2 does not, i.e., $\Theta_1 = \{(\omega_1),(\omega_2)\}$ and $\Theta_2 = \{(\omega_1,\omega_2)\}$. Player 1 chooses either U or D, and player 2 chooses either L or R. The payoffs for state ω_1 are in the left panel, and those for state ω_2 are in the right.

	L	R
U	2, 2	0, 1
D	1, 0	1, 1

	L	R
U	1, 1	0, 1
D	1, 0	2, 2

In this example, both (U,L) and (D,R) are static ex-post equilibria.

Assume that $Y = A$ and $\pi_y^\omega(a) = \varepsilon$ if $y \neq a$. Note that the signal distribution does not depend on the state here, so that players cannot learn the state from state-independent actions. Instead, the efficient outcome $((2,2),(2,2))$ can be achieved if player 1 reveals his private information to player 2 through his actions. Specifically, consider the following three-phase automaton.

- Phase 1. Player 1 chooses U if $\theta_1 = (\omega_1)$, and D if $\theta_1 = (\omega_2)$. Player 2 chooses L. If the observed signal is $y = (U,L)$ or $y = (D,R)$, then go to Phase 2. If the observed signal is $y = (D,L)$, then go to Phase 3. Otherwise, stay.
- Phase 2. Players choose (U,L) in the rest of the game.
- Phase 3. Players choose (D,R) in the rest of the game.

We claim that the strategy profile with initial state Phase 1 is a T-PPXE if δ is close to one and ε is close to zero. First, players do not want to deviate in Phase 2 or Phase 3, as (U,L) and (D,R) are static ex-post equilibria. Also, player 1 with $\theta_1 = (\omega_1)$ does not want to deviate in Phase 1. Indeed, if he deviates to D, then players are likely to go to Phase 3 and play (D,R) forever, while if he does not deviate, then players are likely to go to Phase 2 so that (U,L) is played thereafter. Likewise, we can check that player 1 with $\theta_1 = (\omega_2)$ does not want to deviate in Phase 1. Player 2's prescribed play is always a static best response, and since 2's play has no effect on the transitions between stages 2 does not want to deviate either.

5 Ex-Post Folk Theorems

In this section we provide two sorts of folk theorem in T–PPXE: The first shows that all feasible individually rational payoffs can be approximated by payoffs of T-PPXE, and the second uses weaker conditions to obtain a ”static-threats” version. In both cases, the key is finding the appropriate conditions on the combination of initial private information and the information revealed by the public outcomes.

5.1 A T-PPXE Folk Theorem

For each $i \in \boldsymbol{I}$, $\omega \in \Omega$, and $\vec{\alpha} \in \times_{i \in \boldsymbol{I}} \times_{\theta_i \in \Theta_i} \triangle A_i$, let $\Pi_{(i,\omega)}(\vec{\alpha})$ represent a matrix with rows $(\pi_y^{\omega}(a_i, \alpha_{-i}^{\theta_{-i}(\omega)}))_{y \in Y}$ for all $a_i \in A_i$. Let $\Pi_{(i,\omega)(j,\omega')}(\vec{\alpha})$ be a matrix constructed by stacking two matrices, $\Pi_{(i,\omega)}(\vec{\alpha})$ and $\Pi_{(j,\omega')}(\vec{\alpha})$. With an abuse of notation, we write $\Pi_{(i,\omega)}(\alpha)$ and $\Pi_{(i,\omega)(j,\omega')}(\alpha)$ when $\vec{\alpha}$ corresponds to a state-independent action profile α.

Definition 5. Profile $\vec{\alpha}$ has *individual full rank for* (i,ω) if $\Pi_{(i,\omega)}(\vec{\alpha})$ has rank $|A_i|$. Profile $\vec{\alpha}$ has *individual full rank* if it has individual full rank for all players and all states.

This condition implies that at each state, every possible deviation of any one player leads to a statistically different distribution on outcomes.

Definition 6. For each $i \in \boldsymbol{I}$, $j \neq i$, and $\omega \in \Omega$, profile $\vec{\alpha}$ has *pairwise full rank for* (i,ω) *and* (j,ω) if $\Pi_{(i,\omega)(j,\omega)}(\vec{\alpha})$ has rank $|A_i| + |A_j| - 1$.

Note that pairwise full rank implies individual full rank; it implies that deviations by one player can be distinguished from deviations by another.

Definition 7. For each $i \in \boldsymbol{I}$, $j \in \boldsymbol{I}$, $\omega \in \Omega$, and $\tilde{\omega} \neq \omega$, profile $\vec{\alpha}$ has *statewise full rank for* (i,ω) *and* (j,ω') if $\Pi_{(i,\omega)(j,\omega')}(\vec{\alpha})$ has rank $|A_i| + |A_j|$. Likewise, α has *statewise full rank for* (i,ω) *and* (j,ω') if $\Pi_{(i,\omega)(j,\omega')}(\alpha)$ has rank $|A_i| + |A_j|$.

Statewise full rank implies that deviations by player i at state ω are distinguishable from deviations by player j in state ω'.

Condition IFR. Every pure action profile $\vec{\alpha}$ has individual full rank.

Condition PFR. For each (i,ω) and (j,ω) satisfying $i \neq j$, there is a profile $\vec{\alpha}$ that has pairwise full rank for (i,ω) and (j,ω).

Condition SFR1. For each pair of states (ω,ω') satisfying $\omega \neq \omega'$, at least one of the following two conditions holds.

(i) for each i, j, there is a profile $\vec{\alpha}$ that has statewise full rank for (i,ω) and (j,ω').

(ii) $\theta_l(\omega) \neq \theta_l(\omega')$ for all $l \in \boldsymbol{I}$.

(SFR1) requires that for any pair of states $\omega \neq \omega'$, either (i) for every (i,j) there is a profile that lets players distinguish state ω from state ω', regardless of whether player i deviates in state ω or player j deviates in state ω', or (ii) players can distinguish these ω and ω' using their private information θ. Note that (SFR1) includes pairs (i,ω) and (j,ω') where $i=j$. For such pairs, it is easier to find an $\vec{\alpha}$ that satisfied condition (i) as the partitions Θ_l for $l \neq i$ become finer. The intuition is that if player l has more information, then it is easier for the players to learn the true state through player l's state-contingent play. This following lemma gives a formal statement of this idea.

Condition SFR2. For each (i,ω), (j,ω'), and l satisfying $\omega \neq \omega'$ and $l \neq i,j$, there are α and α_l' such that a matrix constructed by stacking $\Pi_{(i,\omega)}(\alpha)$ and $\Pi_{(j,\omega')}(\alpha_l',\alpha_{-l})$ has rank $|A_i|+|A_j|$. Also, for each (i,ω) and (j,ω') satisfying $\omega \neq \omega'$, either (i) there is a profile α that has statewise full rank, (ii) there is $l \neq i,j$ such that $\theta_l(\omega) \neq \theta_l(\omega')$, or (iii) $\theta_l(\omega) \neq \theta_l(\omega')$ for all $l \in \boldsymbol{I}$.

The condition stated in the first sentence assures that if player l can distinguish ω from ω' using his private information θ_l, then he can reveal this information to the opponents by choosing α_l for state ω and α' for state ω'.[7] Note that clause (iii) implies clause (ii) for games with three or more players. But when there are only two players and $i \neq j$, (iii) does not imply (ii).

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[7] If (IFR), (PFR), and the condition stated in the first sentence of (SFR2) holds, then we can show that introducing cheap-talk communication does not change the limit set of equilibrium payoffs, because the players can use their actions to "communicate" their private information. When (IFR) or (PFR) fail, then cheap-talk communication might enlarge the set of limit equilibrium payoffs, as in Kandori (2003).

Lemma 1. *If (SFR2) holds, then (SFR1) holds.*

Proof. It suffices to show that clause (ii) of (SFR2) implies clause (i) of (SFR1). Consider a pair (i,ω) and (j,ω') satisfying $\omega \neq \omega'$, and let $l \neq i,j$ be such that $\theta_l(\omega) \neq \theta_l(\omega')$. Note that by assumption, there are α and α'_l such that a matrix constructed by stacking $\Pi_{(i,\omega)}(\alpha)$ and $\Pi_{(j,\omega')}(\alpha'_l,\alpha_{-l})$ has rank $|A_i|+|A_j|$. Letting $\vec{\alpha}$ be such that $\alpha^{\theta(\omega)} = \alpha$ and $\alpha^{\theta(\omega')} = (\alpha'_l,\alpha_{-l})$, this $\vec{\alpha}$ has statewise full rank for (i,ω) and (j,ω'), so that clause (i) of (SFR1) follows. *Q.E.D.*

On the other hand, (SFR1) fails for (i,ω) and (i,ω') if π^{ω} is independent of ω (so that the monitoring structure is known) and $\theta_j(\omega) = \theta_j(\omega')$ for all j (so no player's private information distinguishes between ω and ω'). We say more about the case of a known monitoring structure in the next section.

The next proposition establishes a general folk theorem in T-PPXE.

Proposition 4. *Suppose that (IFR), (PFR), and (SFR1) hold. Let* $V^* \equiv \{v \in V | \forall i \in \boldsymbol{I} \forall \omega \in \Omega \ v_i^{\omega} \geq \underline{v}_i^{\omega}\}$ *where* $\underline{v}_i^{\omega} = \min_{\alpha_{-i}} \max_{a_i} g_i^{\omega}(a_i,\alpha_{-i})$. *Then, for any smooth strict subset* W *of* V^*, *there exists* $\overline{\delta} \in (0,1)$ *such that* $W \subseteq E(\delta)$ *for all* $\delta \in (\overline{\delta},1)$.

The proof is in the Appendix. The key point is that (SFR1) implies the the maximal score for cross-state directions can be made arbitrarily large, so that the set Q is determined by λ that has nonzero components only for a single state. Assumptions (IFR) and (PFR) imply that the maximal score for such directions is $\max_{v \in V^*} \lambda \cdot v$, so $Q = V^*$.

5.2 Static-Threats T-PPXE Folk Theorem

Next we provide weaker conditions for a static-threats folk theorem that can be satisfied with fewer signals.

Definition 8. For each $i \in \boldsymbol{I}$, $j \neq i$, and $\omega \in \Omega$, $\vec{\alpha}$ is *pairwise identifiable for* (i,ω) *and* (j,ω) if $\operatorname{rank}\Pi_{(i,\omega)(j,\omega)}(\vec{\alpha}) = \operatorname{rank}\Pi_{(i,\omega)}(\vec{\alpha}) + \operatorname{rank}\Pi_{(j,\omega)}(\vec{\alpha}) - 1$.

We say that $\vec{\alpha}$ is ex-post enforceable if it is ex-post enforceable with respect to $\boldsymbol{R}^{I\times|\Omega|}$ and δ for some $\delta \in (0,1)$. This is equivalent to $\alpha^{\theta(\omega)}$ being enforceable with respect to $\boldsymbol{R}^{I}$ and δ for each information structure π^{ω} in isolation. Also, $\vec{\alpha}$ is a pure action profile if $\alpha^{\theta(\omega)}$ is a pure action profile for all $\omega \in \Omega$

Condition X-Eff. If a pure action profile $a \in A$ gives a Pareto-efficient payoff vector for some $\omega \in \Omega$, then there is a pure action profile $\vec{\alpha}$ being ex-post enforceable and pairwise identifiable for $(i, \omega)$and (j, ω) for each $i \in \boldsymbol{I}$ and $j \neq i$, and such that $\alpha^{\theta(\omega)} = a$.

Condition U-Eff. If a pure action profile $a \in A$ gives a Pareto-efficient payoff vector for some $\omega \in \Omega$, then there is a pure action profile $\vec{\alpha}$ such that $\alpha^{\theta(\omega)} = a$, $\alpha^{\theta(\omega')}$ gives a Pareto-efficient payoff vector for each $\omega' \in \Omega$, and $\alpha^{\theta(\omega')}$ and is pairwise identifiable for (i, ω') and (j, ω') for all $i \in \boldsymbol{I}$, $j \neq i$, and $\omega' \in \Omega$.

Obviously, (U-Eff) holds if any pure action profile $a \in A$ that attains a Pareto-efficient payoff vector for some $\omega \in \Omega$ gives a Pareto-efficient payoff vector for each state $\omega' \in \Omega$ and is pairwise identifiable for (i, ω') and (j, ω') for all $i \in \boldsymbol{I}$, $j \neq i$, and $\omega' \in \Omega$. This property is likely to be satisfied ω has less impacts on players' payoffs.

Lemma 2. *If $u_i(a_i, y, \omega)$ is independent of ω and (U-Eff) holds, then (X-Eff) holds.*

Proof. Let $\vec{\alpha}$ be a pure action profile such that $\alpha^{\theta(\omega)}$ gives a Pareto-efficient payoff vector for every $\omega \in \Omega$ and is pairwise identifiable for (i, ω) and (j, ω) for all $i \in \boldsymbol{I}$, $j \neq i$, and $\omega \in \Omega$. Since a player's payoff depends only on his own action and a public signal, Lemma 6.1 of FLM applies to each state ω in isolation, and hence $\vec{\alpha}$ is ex-post enforceable. *Q.E.D.*

Definition 9. Profile $\vec{\alpha}$ *statewise distinguishes* (i, ω) from (j, ω') if there is a vector $\xi = (\xi(y))_{y \in Y} \in \boldsymbol{R}^{|Y|}$ such that

(i) $\xi \cdot \pi^{\omega}(\alpha^{\theta(\omega)}) > \xi \cdot \pi^{\omega'}(\alpha^{\theta(\omega')})$,

(ii) $\xi \cdot \pi^{\omega}(\alpha^{\theta(\omega)}) = \xi \cdot \pi^{\omega}(a_i, \alpha_{-i}^{\theta_{-i}(\omega)}) \geq \xi \cdot \pi^{\omega}(a_i', \alpha_{-i}^{\theta_{-i}(\omega)})$ for all $a_i \in \text{supp}\alpha_i^{\theta_i(\omega)}$ and $a_i' \in A_i$,

(iii) $\xi \cdot \pi^{\omega'}(\alpha^{\theta(\omega')}) = \xi \cdot \pi^{\omega'}(a_j, \alpha_{-j}^{\theta_{-j}(\omega')})$ for all $a_j \in A_j$.

Condition SD. For each $i \in \boldsymbol{I}$, $j \in \boldsymbol{I}$, $\omega \in \Omega$, and $\omega' \neq \omega$, either (i) there is $\vec{\alpha}$ that is ex-post enforceable and statewise distinguishes (i, ω) from (j, ω'), or (ii) $\theta_l(\omega) \neq \theta(\omega')$ for all $l \in \boldsymbol{I}$.

Clause (i) of this condition implies that the signals generated by $\vec{\alpha}$ statistically distinguish ω from ω'. Clause (ii) says that changing player i's continuation payoff function in state ω from $w_i^{\omega}(y)$ to $w_i^{\omega}(y) + \xi(y)$ preserves incentive compatibility for player i, and clause (iii) says that the change in player i's continuation payoff (of $\Delta w_i^{\omega}(y) \equiv \xi(y)$) can be offset to preserve the feasibility constraint ($\lambda_i^{\omega} \Delta w_i^{\omega}(y) + \lambda_j^{\omega'} \Delta w_j^{\tilde{\omega}}(y) = 0$) without changing player j's expected continuation payoff to any action. (Note that this definition is not symmetric between i and j because condition (ii) is an inequality and condition (iii) is an equality. When this condition is satisfied, scaling up the vector ξ can generate arbitrarily large scores for cross-state directions λ, so (SD) implies "enough cross state information" can be revealed for a static-threat folk theorem.

The folk theorem holds with even weaker conditions, because different profiles can be used in different directions.

Definition 10. Profile $\vec{\alpha}$ *n-statewise distinguishes* (i, ω) from (j, ω') if there is a vector $\xi = (\xi(y))_{y \in Y} \in \boldsymbol{R}^{|Y|}$ such that

(i) $\xi \cdot \pi^{\omega}(\alpha^{\theta(\omega)}) > \xi \cdot \pi^{\omega'}(\alpha^{\theta(\omega')})$,

(ii) $\xi \cdot \pi^{\omega}(\alpha^{\theta(\omega)}) = \xi \cdot \pi^{\omega}(a_i, \alpha_{-i}^{\theta_{-i}(\omega)}) \geq \xi \cdot \pi^{\omega}(a_i', \alpha_{-i}^{\theta_{-i}(\omega)})$ for all $a_i \in \mathrm{supp}\alpha_i^{\theta_i(\omega)}$ and $a_i' \in A_i$,

(iii) $\xi \cdot \pi^{\omega'}(\alpha^{\theta(\omega')}) = \xi \cdot \pi^{\omega'}(a_j, \alpha_{-j}^{\theta_{-j}(\omega')}) \geq \xi \cdot \pi^{\omega'}(a_j', \alpha_{-j}^{\theta_{-j}(\omega')})$ for all $a_j \in \mathrm{supp}\alpha_j^{\theta_j(\omega')}$ and $a_j' \in A_j$.

Note that this condition relaxes statewise distinguishability by replacing the last equality in (iii) with an inequality. Lemma A9 in the appendix shows that a profile that n-statewise distinguishes (i, ω) from (j, ω') can be used to generate an infinite score for all λ such that $\lambda_i^{\omega} > 0$ and $\lambda_j^{\omega'} < 0$; the "n" in the conditions name refers to the fact that in these directions there is a negative trade-off between the utilities of the two players.

Definition 11. A profile $\vec{\alpha}$ *p-statewise distinguishes* (i, ω) from (j, ω') if there is a vector $\xi = (\xi(y))_{y \in Y} \in \boldsymbol{R}^{|Y|}$ such that

(i) $\xi \cdot \pi^{\omega}(\alpha^{\theta(\omega)}) > \xi \cdot \pi^{\omega'}(\alpha^{\theta(\omega')})$,

(ii) $\xi\cdot\pi^{\omega}(\alpha^{\theta(\omega)})=\xi\cdot\pi^{\omega}(a_i,\alpha_{-i}^{\theta_{-i}(\omega)})\geq\xi\cdot\pi^{\omega}(a'_i,\alpha_{-i}^{\theta_{-i}(\omega)})$ for all $a_i\in\operatorname{supp}\alpha_i^{\theta_i(\omega)}$ and $a'_i\in A_i$,

(iii) $\xi\cdot\pi^{\omega'}(\alpha^{\theta(\omega')})=\xi\cdot\pi^{\omega'}(a_j,\alpha_{-j}^{\theta_{-j}(\omega')})\leq\xi\cdot\pi^{\omega'}(a'_j,\alpha_{-j}^{\theta_{-j}(\omega')})$ for all $a_j\in\operatorname{supp}\alpha_j^{\theta_j(\omega')}$ and $a'_j\in A_j$.

Lemma A9 in the appendix shows that a profile that p-statewise distinguishes (i,ω) from (j,ω') can be used to generate an infinite score for all "positive" directions λ such that $\lambda_i^{\omega}>0$ and $\lambda_j^{\omega'}>0$. As this suggests, the condition is symmetric:

Lemma 3. *Suppose that $\vec{\alpha}$ p-statewise distinguishes (i,ω) from (j,ω'). Then $\vec{\alpha}$ p-statewise distinguishes (j,ω') from (i,ω).*

Proof. Let ξ be a vector utilized to p-statewise distinguish (i,ω) from (j,ω'). Then the vector $-\xi$ satisfies all the conditions of p-statewise distinguishability of (j,ω') from (i,ω). *Q.E.D.*

Note that if $\vec{\alpha}$ statewise distinguishes (i,ω) from (j,ω'), then it n-statewise distinguishes this pair and p-statewise distinguishes this pair. So the following condition is weaker than (SD).

Condition Weak-SD. For each (i,ω) and (j,ω') satisfying $\omega\neq\omega'$, either (i) there is an ex-post enforceable action profile $\vec{\alpha}$ that n-statewise distinguishes (i,ω) from (j,ω') and there is an ex-post enforceable action profile $\vec{\alpha}'$ that p-statewise distinguishes (i,ω) from (j,ω'), or (ii) $\theta_l(\omega)\neq\theta(\omega')$ for all $l\in\boldsymbol{I}$.

This is sufficient for the static-threat folk theorem as shown by the following proposition.

Proposition 5. *Suppose (PFR) or (X-Eff) holds. Suppose also that (Weak-SD) holds. Assume that there is an ex-post equilibrium $\vec{\alpha}^0$, i.e., $\vec{\alpha}$ such that $\alpha_i^{\theta_i(\omega)}\in\arg\max_{\alpha_i}g_i^{\omega}(\alpha_i,\alpha_{-i}^{\theta_{-i}(\omega)})$ for all $i\in\boldsymbol{I}$ and $\omega\in\Omega$, and let $V^0\equiv\{v\in V|\forall i\in\boldsymbol{I}$, $\forall\omega\in\Omega\ \ v_i^{\omega}\geq g_i^{\omega}(\alpha^0)\}$. Then, for any smooth strict subset W of V^0, there is $\overline{\delta}\in(0,1)$ such that $W\subseteq E(\delta)$ for all $\delta\in(\overline{\delta},1)$.*

The proof is in the Appendix, it is similar to the proof of the analogous result in Fudenberg and Yamamoto (2009).

6 Applications and Examples

This section explores the effect of some plausible assumptions on the monitoring structure. The first two cases are fairly general; the third and fourth are increasingly specific and illustrate how to apply the general results.

6.1 Separable State Space

In general. the set of limit payoffs depends on the state's impact on both the monitoring structure and the payoff functions. When these dependencies are separable, the characterization of the limit set can be simplified. To see this, suppose that the state consists of two components, one that influences payoffs but not the monitoring structure and one that influences the monitoring structure but has no effect on payoffs. That is, $\Omega = \Phi \times \Psi$, where $u_i^{\omega}(a_i, y) = u_i^{\omega'}(a_i, y)$ if $\phi = \phi'$, and $\pi_y^{\omega}(a) = \pi_y^{\omega'}(a)$ if $\psi = \psi'$.

Condition SFR3. For each (i,ω) and (j,ω') satisfying $\psi \neq \psi'$, there is an ex-post enforceable profile $\vec{\alpha}$ that has statewise full rank for (i,ω) and (j,ω').

For each $\psi \in \Psi$, let $Q(\psi)$ denote the set Q for the known monitoring structure game corresponding to ψ, i.e., the game where the state space is restricted to $\Omega = \Phi \times \{\psi\}$ and the payoff functions u_i^{ω} and the monitoring structure π^{ω} for a given $\omega \in \Omega$ are the same as those of the original game.

Proposition 6. *Suppose that the state space Ω is separable and (SFR3) holds. Then $Q = \times_{\psi \in \Psi} Q(\psi)$. In particular, when π^{ω} has strong full rank for each ω, each $Q(\psi)$ is calculated using the formula in Proposition 7.*

Proof. As Lemma A5 shows, if a profile $\vec{\alpha}$ is ex-post enforceable and has statewise full rank for (i,ω) and (j,ω') satisfying $\omega \neq \omega'$, then $k^*(\vec{\alpha},\lambda) = \infty$ for direction λ such that $\lambda_i(\omega) \neq 0$ and $\lambda_j(\omega') \neq 0$. Thus from (SFR3), $k^*(\lambda) = \infty$ for all λ such that $\lambda_i(\omega) \neq 0$ and $\lambda_j(\omega') \neq 0$ for (ω,ω') satisfying $\psi \neq \psi'$. This proves $Q = \times_{\psi \in \Psi} Q(\psi)$. *Q.E.D.*

6.2 Games with a Product Structure

In this section, we consider games with a product structure. By this we mean first that, as in FLM, there is a separate signal y_i associated with the action of each player i, and moreover that each player i knows the distribution of “her” signal, and that no player $j \neq i$ has any private information about the distribution of y_i. There are a number of economic situations that have this extra structure; it applies for example to bilateral production and exchange, where the public signal is the quality of a player’s output, and each player has private information about the probability that she will make a high-quality good when she exerts high effort.

Formally, we assume that $Y = \times_{i\in \boldsymbol{I}} Y_i$, $\Omega = \times_{i\in \boldsymbol{I}} \Omega_i$,

$$\sum_{y_{-i}\in Y_{-i}} \pi_y^{\omega}(a) = \sum_{y_{-i}\in Y_{-i}} \pi_y^{\omega'}(a)$$

for each $i \in \boldsymbol{I}$, $a \in A$, $y_i \in Y_i$, $\omega \in \Omega$, and $\omega' \in \Omega$ such that $\omega_i = \omega_i'$, and

$$\pi_y^{\omega}(a) = \prod_{i\in \boldsymbol{I}} \sum_{y_{-i}\in Y_{-i}} \pi_y^{\omega}(a)$$

for each $a \in A$, $y \in Y$, and $\omega \in \Omega$. Note that the distribution of y_i depends only on a_i and ω_i here. We also assume that $\Theta_i = \{\theta_i^{\omega_i} | \omega_i \in \Omega_i\}$ where $\theta_i^{\omega_i} = \{\omega' | \omega_i' = \omega_i\}$; that is, player i knows the distribution of y_i but not the distribution of y_{-i}. We also assume that every state has some impact on the distribution of signals in the following sense: for each $\omega \in \Omega$ and $\omega' \neq \omega$, there is $a \in A$ such that $\pi^{\omega}(a) \neq \pi^{\omega'}(a)$. Note that this rules out the case where the signal distribution is known and the states refer only to the player’s payoffs.

Intuitively, in this setup each player j is able to signal his private information ω_j, as no other player’s action can be confused with his own. Thus we might expect that the main obstacle to information revelation comes when player j’s information will be used to lower his payoff. This corresponds to the fact that in general the product structure assumption only implies p-statewise distinguishability, and n-statewise distinguishability can fail for (i,ω) and (j,ω') such that $\omega_j \neq \omega_j'$ and $\omega_{-j} = \omega_{-j}'$.

Conversely, the next lemma shows that there is full statewise distinguishability for all of the other cross-state directions. Part (a) shows this is true for (i,ω) and (i,ω') with $\omega_{-i} \neq \omega_{-i}'$: Here the state is revealed by the actions of players other than i and because of the product structure there is nothing that player i can do to prevent it. Part (b) applies to (i,ω) and (j,ω') such that $i \neq j$ and $\omega_i \neq \omega_i'$. Then there is a type-contingent profile $\vec{\alpha}$ that statewise distinguishes (i,ω) from (j,ω'), as player i can signal his private information and other players cannot jam the signal. Finally, part (c) says that neither player i nor player j can stop other players from revealing their information.

Lemma 4.

(a) *Let* (i,ω) *and* (j,ω') *be such that* $i = j$ *and* $\omega_{-i} \neq \omega'_{-i}$. *Then there is a type-independent profile* a^* *that statewise distinguishes* (i,ω) *from* (j,ω').

(b) *Let* (i,ω) *and* (j,ω') *be such that* $i \neq j$ *and* $\omega_i \neq \omega'_i$. *Then there is a type-contingent profile* $\vec{\alpha}$ *that statewise distinguishes* (i,ω) *from* (j,ω').

(c) *Let* (i,ω) *and* (j,ω') *be such that* $i \neq j$ *and* $\omega_{-ij} \neq \omega'_{-ij}$. *Then there is a type-independent profile* a^* *that statewise distinguishes* (i,ω) *from* (j,ω').

Proof. Part (a). Because every state has an impact on signals, there is a type-independent profile a^* such that the probability of some y_{-i} is different in state ω than in ω': $\sum_{y_i \in Y_i} \pi_y^{\omega}(a^*) \neq \sum_{y_i \in Y_i} \pi^{\omega'}(a^*)$ for some $y_{-i} \in Y_{-i}$. Then there is ξ such that $\xi(y)$ depends only on y_{-i} (so that $\xi(y) = \xi(y')$ if $y_{-i} = y'_{-i}$) and $\xi \cdot \pi^{\omega}(a^*) > \xi \cdot \pi^{\omega'}(a^*)$. This ξ satisfies all the conditions of statewise distinguishability: (i) comes from $\xi \cdot \pi^{\omega}(a^*) > \xi \cdot \pi^{\omega'}(a^*)$, and (ii) and (iii) follow from the facts that player i's action only influences the distribution of y_i and that ξ does not depend on y_i .

Part (b). Let a^* be a profile such that $\sum_{y_{-i} \in Y_{-i}} \pi_y^{\omega}(a^*) \neq \sum_{y_{-i} \in Y_{-i}} \pi^{\omega'}(a^*)$ for some $y_i \in Y_i$. Let ξ depend only on y_i : $\xi(y) = \xi(y')$ if $y_i = y'_i$ and $\xi \cdot \pi^{\omega}(a^*) > \xi \cdot \pi^{\omega'}(a^*)$. Let $a_i^{\omega} \in \arg\max_{a_i \in A_i} \xi \cdot \pi^{\omega}(a_i, a^*_{-i})$ and $a_i^{\omega'} \in \arg\min_{a_i \in A_i} \xi \cdot \pi^{\omega'}(a_i, a^*_{-i})$. Let $\vec{\alpha}$ be such that players play (a_i^{ω}, a^*_{-i}) for state ω and $(a_i^{\omega'}, a^*_{-i})$ for state ω'. As in the proof of part (a), condition (i) of statewise distinguishability is satisfied by the definition of ξ; condition (ii) now comes from the fact that player i's action in state ω maximizes $\xi \cdot \pi^{\omega}(a_i, a^*_{-i})$, and condition (iii) comes from the fact that ξ does not depend on y_j.

Part (c). Let a^* be a profile such that $\sum_{y_i \in Y_i} \sum_{y_j \in Y_j} \pi_y^{\omega}(a^*) \neq \sum_{y_i \in Y_i} \sum_{y_j \in Y_j} \pi^{\omega'}(a^*)$ for some $y_{-ij} \in Y_{-ij}$. Let ξ be such that the value $\xi(y)$ depends only on y_{-ij} (so that $\xi(y) = \xi(y')$ if $y_{-ij} = y'_{-ij}$) and $\xi \cdot \pi^{\omega}(a^*) > \xi \cdot \pi^{\omega'}(a^*)$. Then a^* and ξ satisfy all the conditions of statewise distinguishability. *Q.E.D.*

The above lemmas cover every cross-state comparison except the case (i,ω) and (j,ω') with $\omega_j \neq \omega'_j$ and $\omega_{-j} = \omega'_{-j}$. As the next lemma shows, p-statewise distinguishability is satisfied in this case. Intuitively, player j can signal his information by varying his action with his type, and this is consistent with incentive compatibility in positive directions because the player is rewarded for revealing the information. Note that the lemma does not assert n-statewise distinguishability for (i,ω) and (j,ω') with $\omega_j \neq \omega'_j$.

Lemma 5. *Let (i,ω) and (j,ω') be such that $\omega_j \neq \omega'_j$ (but possibly $i = j$). Then there is a type-contingent profile $\vec{\alpha}$ that p-statewise distinguishes (i,ω) from (j,ω').*

Proof. Let a^*be a type-independent profile such that the probability of some y_j is different in state ω than in ω': $\sum_{y_{-j}\in Y_{-j}} \pi_y^{\omega}(a^*) \neq \sum_{y_{-j}\in Y_{-j}} \pi^{\omega'}(a^*)$ for some $y_j \in Y_j$. Let ξ be such that $\xi \cdot \pi^{\omega}(a^*) > \xi \cdot \pi^{\omega'}(a^*)$ and $\xi(y) = \xi(y')$ for y and y' such that $y_j = y'_j$. Let $a_j^{\omega} \in \arg\max_{a_j\in A_j} \xi \cdot \pi^{\omega}(a_j, \vec{\alpha}_{-j})$ and $a_j^{\omega'} \in \arg\min_{a_j\in A_j} \xi \cdot \pi^{\omega'}(a_j, \vec{\alpha}_{-j})$. Let $\vec{\alpha}$ prescribe (a_j^{ω}, a_{-j}^*) for state ω and $(a_j^{\omega'}, a_{-j}^*)$ for all other states, including state ω'. (Note that only player j's play varies with the state. Then by construction, $\vec{\alpha}$ and ξ satisfy all the conditions of p-statewise distinguishability. *Q.E.D.*

Because p-statewise distinguishability is satisfied for all cross-state pairs, we know that the maximal score $k^*(\lambda)$ is infinitely large for direction λ such that there are (i,ω) and (j,ω') such that $\omega \neq \omega'$, $\lambda_i^{\omega} > 0$, and $\lambda_j^{\omega'} > 0$. Thus we expect that an efficient outcome can be approximated by T-PPXE payoffs if the constraints related to the other directions are not too tight.

In general, though, product structure does not guarantee n-statewise distinguishability for (j,ω) and (j,ω') with $\omega_j \neq \omega'_j$. Perhaps the simplest example where this condition fails is a bilateral gift-exchange game:There are two players, and $\Omega = \Omega_1 \times \Omega_2$ where $\Omega_i = \{\omega_{i1}, \omega_{i2}\}$. Player i chooses high effort ($a_i = C_i$) or low effort ($a_i = D_i$), and the quality of player i's output is good ($y_i = G$) or bad ($y_i = B$). We assume that the probability $p_i^{\omega_i}(a_i)$ that $y_i = G$ depends only on ω_i and a_i, that the two signals are independent, and that $p_i^{\omega_{i1}}(C_i) > p_i^{\omega_{i2}}(C_i) > p_i^{\omega_{i1}}(D_i) = p_i^{\omega_{i2}}(D_i)$. Player i's realized payoff is given by $u_i(a_i, y_i) = r_i(y_{-i}) - c_i(a_i)$, where $r_i(y_{-i})$ is the utility from consumption of y_{-i}, and $c_i(a_i)$ is the effort cost. Suppose that $\Theta_i = \{\theta(\omega_{i1}), \theta(\omega_{i2})\}$ where $\theta(\omega_{ik}) = \{\omega = (\omega_1, \omega_2) \in \Omega | \omega_i = \omega_{ik}\}$ for each $k = 1, 2$.

Claim 1. *No profile $\vec{\alpha}$ n-statewise distinguishes $((1, (\omega_{12}, \omega_{21}))$ from $(1, (\omega_{11}, \omega_{21}))$.*

Proof. Note that for any type-contingent profile $\vec{\alpha}$, the distribution of y_2 is the same for state $\omega = (\omega_{12}, \omega_{21})$ and for state $\omega' = (\omega_{11}, \omega_{21})$, since $\omega_2 = \omega_2'$. Thus it suffices to show that there is no profile $\vec{\alpha}$ such that all the conditions of n-statewise distinguishability are satisfied by a ξ that depends only on y_1. Let ξ^G denote the value of $\xi(y)$ when $y_1 = G$, and ξ^B the value of $\xi(y)$ when $y_1 = B$.

Let $\vec{\alpha}$ be such that player 1 chooses α_1 for state ω and α_1' for state ω'; since $\omega_2 = \omega_2'$, player 2 chooses the same action α_2 in both states. Suppose that $\vec{\alpha}$ n-statewise distinguishes (i, ω) from (j, ω'). Then, from clause (i) of the definition, we have $((p_1^{\omega_{12}}(C_1) - p_1^{\omega_{12}}(D_1))\alpha_1'(C_1) - (p_1^{\omega_{11}}(C_1) - p_1^{\omega_{11}}(D_1))\alpha_1(C_1))(\xi^G - \xi^B) > 0$ If $\xi^G - \xi^B > 0$, then clause (iii) implies that $\alpha_1(C_1) = 1$, and hence $((p_1^{\omega_{12}}(C_1) - p_1^{\omega_{12}}(D_1))\alpha_1'(C_1) - (p_1^{\omega_{11}}(C_1) - p_1^{\omega_{11}}(D_1))\alpha_1(C_1))(\xi^G - \xi^B) < 0$. Therefore $\xi^G - \xi^B < 0$. But then clause (iii) implies that $\alpha_1(C_1) = 0$, and thus $((p_1^{\omega_{12}}(C_1) - p_1^{\omega_{12}}(D_1))\alpha_1'(C_1) - (p_1^{\omega_{11}}(C_1) - p_1^{\omega_{11}}(D_1))\alpha_1(C_1))(\xi^G - \xi^B) \leq 0$, a contradiction.
Q.E.D.

A similar issue arises in the more specialized models of the next subsections, as there too n-statewise distinguishability fails. In section 6.4 we explicitly determine the impact of this failure on the set of limit payoffs.

6.3 Known Own Productivity and Linear Uncertainty

The next two subsections explore a related two-player model with known productivity. We maintain the assumption that the state space has a product structure, $\Omega = \Omega_1 \times \Omega_2$; the main difference is that instead of the signals having a product structure, the signal corresponds to the output of a partnership, and is influenced by both players' actions. We also assume that there is a "null" action profile $a^0 \in A$ such that

$$\pi^{\omega}(a^0) = \pi^{\omega'}(a^0)$$

for all ω and $\omega' \neq \omega$, and that for each $i \in \boldsymbol{I}$, $a_i \neq a^0$, $a_{-i} \in A_{-i}$, and $\omega_i \in \Omega_i$, there are $\eta_i^{\omega_i}(a) =\in \boldsymbol{R}^{|Y|}$ and $\beta_i^{\omega_i}(a) > 0$ such that for each $\omega_{-i} \in \Omega_{-i}$,

$$\pi^{\omega}(a) - \pi^{\omega}(a_i^0, a_{-i}) = \beta_i^{\omega_i}(a)\eta_i^{\omega_i}(a).$$

In words, the marginal productivity of action a_i (as opposed to the default) when the opponent plays a_{-i} is $\beta_i^{\omega_i}(a)\eta_i^{\omega_i}(a)$, which depends only on ω_i. Finally, and importantly, this marginal productivity is linear with respect to i's information ω_i in the sense that for each ω and $\omega' \neq \omega$,

$$\pi^{\omega}(a) - \pi^{\omega}(a_i^0, a_{-i}) = \gamma(\pi^{\omega'}(a) - \pi^{\omega'}(a_i^0, a_{-i})).$$

for some $\gamma > 0$; this greatly simplifies the analysis.

We assume that there is no redundant state, so that for each ω and $\omega' \neq \omega$, there is a such that $\pi^{\omega}(a) \neq \pi^{\omega'}(a)$. Assume also that each pure action profile has pairwise full rank, so that for each $a \in A$ and $\omega = (\omega_1, \omega_2) \in \Omega$, the set of vectors

$$\{\pi^{\omega}(a)\} \bigcup \{\eta_1^{\omega_1}(a_1', a_2) | a_1' \neq a_1^0\} \bigcup \{\eta_2^{\omega_2}(a_1, a_2') | a_2' \neq a_2^0\}$$

is linearly independent. Suppose that $\Theta_i = \{\theta_i^{\omega_i} | \omega_i \in \Omega_i\}$ where $\theta_i^{\omega_i} = \{\omega' | \omega_i' = \omega_i\}$; that is, each player knows his own productivity but not the opponent's.

The assumption of linear uncertainty implies that a type-independent action profile α can not have statewise full rank. However, using type-contingent action profiles, we have p-statewise distinguishability for all cross-state pairs (i, ω) and (j, ω') such that only one player can distinguish ω from ω'. For ω and ω' such that $\omega_i \neq \omega_i'$ for both $i = 1, 2$, we do not need statewise distinguishability as all players can distinguish these two states using private information.

Lemma 6. *Let (i, ω) and (j, ω') be such that $i = j$ and $\omega_{-i} = \omega'_{-i}$.*

- **(a)** *Suppose that there are $a_i^* \neq a_i^0$ and $a_{-i}^* \in A_{-i}$ such that $\beta_i^{\omega_i}(a^*) > \beta_i^{\omega_i'}(a^*)$. Let $\vec{\alpha}$ be such that players play a^* for state ω and (a_i^0, a_{-i}^*) for state ω'. Then $\vec{\alpha}$ p-statewise distinguishes (i, ω) from (j, ω').*

- **(b)** *Suppose that there are $a_{-i}^* \neq a_{-i}^0$ and $a_i^* \in A_i$ such that $\beta_{-i}^{\omega_{-i}}(a^*) > \beta_{-i}^{\omega'_{-i}}(a^*)$. Then the type-independent action profile a^* p-statewise distinguishes (i) from (j, ω').*

Proof. Part (a). Let $\kappa > 0$, and let ξ be such that $\xi \cdot \pi^{\omega}(a^*) = \beta_i^{\omega}(a^*)\kappa$, $\xi \cdot \eta_i(a^*) = \kappa$, and $\xi \cdot \eta_i(a_i, a_{-i}^*) = 0$ for all $a_i \neq a_i^0, a_i^*$. Such a ξ exists, as each pure action has pairwise full rank. Then we have $\xi \cdot \pi^{\omega}(a^*) = \beta_i^{\omega}(a^*)\kappa$, $\xi \cdot \pi^{\omega'}(a^*) = \beta_i^{\omega'}(a^*)\kappa$, and $\xi \cdot \pi^{\omega}(a_i, a_{-i}^*) = \xi \cdot \pi^{\omega'}(a_i, a_{-i}^*) = 0$ for all $a_i \neq a_i^*$. Thus all the conditions of p-statewise distinguishability is satisfied.

Part (b). Let $\kappa > 0$, and let ξ be such that $\xi \cdot \eta_{-i}(a^*) = \kappa$ and $\xi \cdot \eta_i(a_i, a_{-i}^*) = 0$ for all $a_i \neq a_i^0$. Thus all the conditions of p-statewise distinguishability is satisfied.

Q.E.D.

Lemma 7.

(a) *Let (i,ω) and (j,ω') be such that $i \neq j$, $\omega_j = \omega'_j$, and there are $a_i^* \neq a_i^0$ and $a_{-i}^* \in A_{-i}$ such that $\beta_i^{\omega_i}(a^*) > \beta_i^{\omega'_i}(a^*)$. Then the type-independent action profile a^* p-statewise distinguishes (i,ω) from (j,ω').*

(b) *Let (i,ω) and (j,ω') be such that $i \neq j$, $\omega_i = \omega'_i$, and there are $a_j^* \neq a_j^0$ and $a_{-j}^* \in A_{-j}$ such that $\beta_j^{\omega_j}(a^*) > \beta_j^{\omega'_j}(a^*)$. Let $\vec{\alpha}$ be such that players play a^* for state ω and (a_j^0, a_{-j}^*) for state ω'. Then this $\vec{\alpha}$ p-statewise distinguishes (i,ω) from (j,ω').*

Proof. Part (a). Let $\kappa > 0$, and let ξ be such that $\xi \cdot \pi^\omega(a^*) = \beta_i^\omega(a^*)\kappa$, $\xi \cdot \eta_i(a_i^*) = \kappa$, $\xi \cdot \eta_i(a_i, a_{-i}^*) = 0$ for all $a_i \neq a_i^0, a_i^*$, and $\xi \cdot \eta_j(a_j, a_{-j}^*) = 0$ for all $a_j \neq a_j^0,$. Then this ξ satisfies all the conditions of p-statewise distinguishability.

Part (b). Let $\kappa > 0$, and let ξ be such that $\xi \cdot \pi^\omega(a^*) = \beta_j^{\omega_j}(a^*)\kappa$, $\xi \cdot \eta_i(a_i, a_{-i}^*) = 0$ for all $a_i \neq a_i^0$, $\xi \cdot \eta_j(a^*) = \kappa$, $\xi \cdot \eta_j(a_j, a_{-j}^*) = 0$ for all $a_j \neq a_j^0, a_j^*$. Then this ξ satisfies all the conditions of p-statewise distinguishability. *Q.E.D.*

Because these games satisfy the p-statewise distinguishability condition, they have efficient limit equilibrium payoffs provided that the scores in the negative cross-state directions are not too low.

6.4 A Two-Player, Two-Actions Partnership

Here we revisit Example 4 of Fudenberg and Yamamoto (2009) to illustrate the effect of players knowing their own productivity. In this example, there are two players, two actions $A_i = \{C_i, D_i\}$, two states, and three outcomes $Y = \{H, M, L\}$. The state only influences the productivity of player 2's effort: If player 1 chooses C_1 instead of D_1, then the probabilities of H and M increase by p_H and p_M, independent of the state. In contrast, if player 2 chooses C_2 instead of D_2, then the probabilities of H and M increase by q_H and q_M in state ω_1, but they increase only by βq_H and βq_M in state ω_2. We assume that the vectors (p_H, p_M) and (q_H, q_M) are linearly independent; this implies that individual full rank and the pairwise full rank are satisfied at every profile and every state. However, no type-independent profile p-statewise distinguishes $(1, \omega_1)$ and $(2, \omega_2)$, and as a result, the set of

PPXE payoffs is bounded away from efficiency uniformly in the discount factor. The following table shows whether statewise distinguishability holds for each pair (i,ω) and (j,ω'), when we restrict attention to type-independent profiles.

(i,ω), (j,ω')	p-statewise	n-statewise
$(1,\omega_1)$, $(1,\omega_2)$	(C_1,C_2)	(C_1,C_2)
$(1,\omega_2)$, $(1,\omega_1)$	(C_1,C_2)	(C_1,C_2)
$(2,\omega_1)$, $(2,\omega_2)$	No	(C_1,C_2)
$(2,\omega_2)$, $(2,\omega_1)$	No	No
$(1,\omega_1)$, $(2,\omega_2)$	No	(C_1,C_2)
$(2,\omega_2)$, $(1,\omega_1)$	No	No
$(2,\omega_1)$, $(1,\omega_2)$	(C_1,C_2)	(C_1,C_2)
$(1,\omega_2)$, $(2,\omega_1)$	(C_1,C_2)	No

However, if a player can distinguish ω_1 from ω_2 using his private information, then it is often possible to approximate an efficient outcome using T-PPXE.

If player 1 knows the state and player 2 does not, then the distinguishability conditions in the various cross-state directions as satisfied by the following profiles:

(i,ω), (j,ω')	p-statewise	n-statewise
$(1,\omega_1)$, $(1,\omega_2)$	(C_1,C_2)	(C_1,C_2)
$(1,\omega_2)$, $(1,\omega_1)$	(C_1,C_2)	(C_1,C_2)
$(2,\omega_1)$, $(2,\omega_2)$	$((C_1,D_1),C_2)$	(C_1,C_2)
$(2,\omega_2)$, $(2,\omega_1)$	$((C_1,D_1),C_2)$	$((C_1,D_1),C_2)$
$(1,\omega_1)$, $(2,\omega_2)$	$((C_1,D_1),C_2)$	(C_1,C_2)
$(2,\omega_2)$, $(1,\omega_1)$	$((C_1,D_1),C_2)$	Not satisfied
$(2,\omega_1)$, $(1,\omega_2)$	(C_1,C_2)	(C_1,C_2)
$(1,\omega_2)$, $(2,\omega_1)$	(C_1,C_2)	$((C_1,D_1),D_2)$

Since n-statewise distinguishability does not hold for $((i,\omega),(j,\omega')) = (2,\omega_2),(1,\omega_1)$, the maximal scores for the corresponding directions are not infinitely large. Nevertheless, as shown by Proposition 12 in the appendix, these scores are high enough to achieve the perfect folk theorem for any $\beta \in (0,1)$. This is the first example in which the folk theorem holds despite the fact that statewise conditions fail for some cross-state directions.

If player 2 knows the state and player 1 does not, then distinguishability is satisfied (or not) as in the following table:

(i,ω), (j,ω')	p-statewise	n-statewise
$(1,\omega_1)$, $(1,\omega_2)$	(C_1,C_2)	(C_1,C_2)
$(1,\omega_2)$, $(1,\omega_1)$	(C_1,C_2)	(C_1,C_2)
$(2,\omega_1)$, $(2,\omega_2)$	$(C_1,(C_1,D_2))$	(C_1,C_2)
$(2,\omega_2)$, $(2,\omega_1)$	$(C_1,(C_1,D_2))$	Not satisfied
$(1,\omega_1)$, $(2,\omega_2)$	$(C_1,(C_1,D_2))$	(C_1,C_2)
$(2,\omega_2)$, $(1,\omega_1)$	$(C_1,(C_1,D_2))$	$(C_1,(C_1,D_2))$
$(2,\omega_1)$, $(1,\omega_2)$	(C_1,C_2)	(C_1,C_2)
$(1,\omega_2)$, $(2,\omega_1)$	(C_1,C_2)	Not satisfied

In this case Proposition 13 shows that the folk theorem fails because the maximum score in direction $\lambda = ((0,-1),(0,1))$ is too low. Moreover, if the cost of effort is high, then for $\lambda = ((0,-\varepsilon),(1,0))$ the maximal score can be so low that it rules out equilibrium with the payoffs of the efficient action profile (C_1,C_2). Specifically, this is the case if player 1's effort cost is high enough so that $g_1^{\omega_2}(D_1,C_2) - g_1^{\omega_2}(C_1,C_2)$ is close to zero.[8] Intuitively, player 2 cannot be induced to reveal the state when doing so would lower his equilibrium payoff, and as a result the maximal score for direction λ with $\lambda_2^{\omega_1} < 0$ is lower than $\lambda \cdot g(C_1,C_2)$.

7 Known Monitoring Structure

So far we have studied a general model, where both payoffs and monitoring structure can depend on the state of the world, and provided sufficient conditions for the folk theorems. However, these sufficient conditions may not be satisfied in some games. One notable example is the case of a known monitoring structure; here a state-independent profile α cannot induce different signal distributions for different states, so for players to distinguish the states they must have "enough" private information θ. In this section we characterize the limit equilibrium payoffs in games with a known monitoring structure and then for the subcase of a known

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[8] The derivation of this bound on the maximal score is very similar to the proof of Claims 13 and 14 which are used to prove Proposition 13; all of these proofs are in the supplementary on-line materials.

monitoring structure and one-sided incomplete information. This lets us compare our results with those of Hörner and Lovo (2009) and Hörner, Lovo, and Tomala (2009) for games with observed actions.

Formally, the *monitoring structure is known* if $\pi_y^{\omega}(a) = \pi_y^{\omega'}(a)$ for all $y \in Y$, $a \in A$, $\omega \in \Omega$, and $\omega' \neq \omega$. We maintain this assumption throughout this section. Since π^{ω} does not depend on ω, we denote it by π.

In this section, we often impose the following strong full rank condition. As we will see, under this condition the case of a known but imperfect monitoring structure is very similar to that where actions are perfectly observed. Let $\Pi_i(a)$ denote the matrix with rows $(\pi_y(a_i', a_{-i}))_{y \in Y}$ for all $a_i' \in A_i$. Also, for each $i \in \boldsymbol{I}$, $j \in \boldsymbol{I}$, $a \in A$, and $a' \in A$, let $\Pi_{(i,a)(j,a')}$ denote the matrix constructed by stacking two matrices $\Pi_i(a)$ and $\Pi_j(a')$.

Definition 12. Monitoring structure π has *strong full rank* if

(i) $\Pi_{(i,a)(j,a)}$ has rank $|A_i| + |A_j| - 1$ for all $i, j \in \boldsymbol{I}$ and $a \in A$; and

(ii) for any $i, j, \in \boldsymbol{I}$, if there is $l \neq i, j$, then $\Pi_{(i,a)(j,(a_l', a_{-l}))}$ has rank $|A_i| + |A_j|$ for all $l \neq i, j$, $a \in A$, and $a_l' \neq a_l$.

Note that we allow $i = j$ in this definition, and hence the second clause is not vacuous even in two-player games. The first clause imposes FLM's pairwise full rank condition on every action profile. The second clause implies that the statewise full rank condition holds for (i, ω) and (j, ω') if player l can distinguish the states ω and ω'.[9] The strong full rank condition is obviously satisfied for games with perfectly observable actions. It is also satisfied if the signals are isomorphic to the actions and players observe the intended action with a small noise, i.e. $Y = A$ and $\pi_y(a) < \varepsilon$ for all $a \in A$ and $y \neq a$ where ε is close to zero.

7.1 Known Monitoring Structure and Strong Full Rank

The set Q depends on the maximal score in various directions, so it is helpful to classify the directions so that the maximal score can be computed in the same way for all directions in a given class.

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[9] To see this, let $\vec{\alpha}$ be such that $\alpha^{\theta(\omega)} = a$ and $\alpha^{\theta(\omega')} = (a_l', a_{-l})$. Then this $\vec{\alpha}$ has statewise full rank for (i, ω) and (j, ω'), as the corresponding matrix has rank $|A_i| + |A_j|$.

Let Λ^1 be the set of $\lambda \in \boldsymbol{R}^{I\times|\Omega|}$ such that $(\lambda_i^\omega)_{i\in\boldsymbol{I}} \neq 0$ for some $\omega \in \Omega$ and $(\lambda_i^{\omega'}))_{i\in\boldsymbol{I}} = 0$ for all $\omega' \neq \omega$. Since these directions consider only a single state, Lemmas 5.2 and 5.4 of FLM show that the maximum score is the maximum feasible score.

Lemma 8. *Suppose that monitoring structure is known and has strong full rank. Then for each $\lambda \in \Lambda^1$, $k^*(\lambda) = \max_{v\in V^*} \lambda \cdot v$.*

Proof. The same as in FLM. *Q.E.D.*

Let Λ^2 be the set of λ such that there are $i \in \boldsymbol{I}$, $j \in \boldsymbol{I}$, $l \neq i, j$, $\omega \in \Omega$, and $\omega' \in \Omega$ such that $\lambda_i^\omega \neq 0$, $\lambda_j^{\omega'} \neq 0$, and $\theta_l(\omega) \neq \theta_l(\omega')$. Here player l can distinguish between ω and ω', and the strong full rank condition implies that if player l takes different actions in these states, the statewise full rank condition is satisfied for (i,ω) and (j,ω'). Consequently, the maximal scores for these directions are infinity, as the following lemma shows.

Lemma 9. *Suppose that monitoring structure is known and has strong full rank. Then for each $\lambda \in \Lambda^2$, $k^*(\lambda) = \infty$.*

Proof. Let $\lambda \in \Lambda^2$, and let $i \in \boldsymbol{I}$, $j \in \boldsymbol{I}$, $l \neq i, j$, $\omega \in \Omega$, and $\omega' \in \Omega$ be such that $\lambda_i^\omega \neq 0$, $\lambda_j^{\omega'} \neq 0$, and $\theta_l(\omega) \neq \theta_l(\omega')$. Let $\vec{\alpha}$ be such that $\alpha^{\theta(\omega)} = a$ for some $a \in A$ and $\alpha^{\theta(\omega')} = (a_l', a_{-l})$ for some $a_l' \neq a_l$. Since monitoring structure has strong full rank, $\vec{\alpha}$ has statewise full rank for (i,ω) and (j,ω'). Thus, $k^*(\vec{\alpha},\lambda) = \infty$. *Q.E.D.*

Let Λ^3 be the set of λ such that there are $i \in \boldsymbol{I}$, $j \neq i$, $\omega \in \Omega$, and $\omega' \neq \omega$ such that $\lambda_i^\omega > 0$, $\lambda_j^{\omega'} \neq 0$, and $\theta_i(\omega) \neq \theta_i(\omega')$. Here player i can distinguish between ω and ω', and the score is increasing in player i's payoff in state ω'; the next lemma shows that the maximal scores for these directions are infinity as well, since statewise distinguishability is satisfied for (i,ω) and (j,ω'). Note that the intersection of Λ^2 and Λ^3 might be nonempty but this is irrelevant as the maximal score is infinity for either case.

Lemma 10. *Suppose that monitoring structure is known and has strong full rank. Then for each $\lambda \in \Lambda^3$, $k^*(\lambda) = \infty$.*

Proof. Let $\lambda \in \Lambda^3$, and let $i \in \boldsymbol{I}$, $j \neq i$, $\omega \in \Omega$, and $\omega' \in \Omega$ be such that $\lambda_i^{\omega} > 0$, $\lambda_j^{\omega'} \neq 0$, and $\theta_i(\omega) \neq \theta_i(\omega')$. Let $\vec{\alpha}$ be such that $\alpha^{\theta(\omega)} = a$ for some $a \in A$ and $\alpha^{\theta(\omega')} = (a_i', a_{-i})$ for some $a_i' \neq a_i$. Then $\vec{\alpha}$ statewise distinguishes (i, ω) from (j, ω'). Since $\lambda_i^{\omega} > 0$, Lemmas A9 apply. *Q.E.D.*

Let Λ^4 be the set of λ such that there are $i \in \boldsymbol{I}$, $\omega' \in \Omega$, and $\omega'' \neq \omega'$ such that $\lambda_i^{\omega'} > 0$, $\lambda_i^{\omega''} > 0$, $(\lambda_j^{\omega})_{\omega \in \Omega} = 0$ for all $j \neq i$, and $\theta_i(\omega') \neq \theta_i(\omega'')$. Here only player i's payoffs matter, the score is increasing in i's payoff in ω and ω', and player i can distinguish between these two states. Once again, the maximal scores for these directions are infinity, as p-statewise distinguishability is satisfied for (i, ω') and (i, ω''). Note that the intersection of Λ^2 and Λ^4 might be nonempty, but that the maximal score is infinity for either case.

Lemma 11. *Suppose that monitoring structure is known and has strong full rank. Then for each $\lambda \in \Lambda^4$, $k^*(\lambda) = \infty$.*

Proof. Let $\lambda \in \Lambda^4$, and let $i \in \boldsymbol{I}$, $\omega' \in \Omega$, and $\omega'' \in \Omega$ be such that $\lambda_i^{\omega'} > 0$, $\lambda_i^{\omega''} > 0$, and $\theta_i(\omega') \neq \theta_i(\omega'')$. Let $\vec{\alpha}$ be such that $\alpha^{\theta(\omega')} = a$ for some $a \in A$ and $\alpha^{\theta(\omega'')} = (a_i', a_{-i})$ for some $a_i' \neq a_i$. Then $\vec{\alpha}$ p-statewise distinguishes (i, ω') from (i, ω''). Since $\lambda_i^{\omega'} > 0$ and $\lambda_i^{\omega''} > 0$, Lemma A9(b) applies. *Q.E.D.*

Let $\Lambda^5(i)$ be the set of λ such that $(\lambda_i^{\omega})_{\omega \in \Omega} \leq 0$, $(\lambda_i^{\omega})_{\omega \in \Omega} \neq 0$, $(\lambda_j^{\omega})_{\omega \in \Omega} = 0$ for all $j \neq i$, and $\theta_j(\omega) = \theta_j(\omega')$ for all $j \neq i$, $\omega \in \Omega$, and $\omega' \neq \omega$ satisfying $\lambda_i^{\omega} \neq 0$ and $\lambda_i^{\omega'} \neq 0$. Here only player i's payoffs matter, the score is decreasing in i's payoff, and no other player can distinguish between the states; these directions determine the minmax payoff for player i, taking into account a trade-off between the minmax level in one state and the payoffs in other states. Let $\Lambda^5 = \bigcup_{i \in \boldsymbol{I}} \Lambda^5(i)$.

Lemma 12. *Suppose that monitoring structure is known and has strong full rank. Then for each i and $\lambda \in \Lambda^5(i)$, $k^*(\lambda) = \max_{\alpha_{-i}} \min_{a_i} \sum_{\omega \in \Omega} \lambda_i^{\omega}(\omega) g_i^{\omega}(a_i, \alpha_{-i})$, that is, $k^*(\lambda) = -\min_{\alpha_{-i}} \max_{a_i} \sum_{\omega \in \Omega} -\lambda_i^{\omega} g_i^{\omega}(a_i, \alpha_{-i})$.*

Proof. See the Appendix. The intuition is as follows: Strong full rank implies that constraints (i) and (ii) can be satisfied for all $j \neq i$, and because $(\lambda_j^{\omega})_{\omega \in \Omega} = 0$ the continuation payoffs assigned to $j \neq i$ are irrelevant. Thus we only need to consider continuation payoffs for player i that satisfy (i) and (ii) for ω such that

$\lambda_i^\omega \neq 0$, and the feasibility constraint (iii). Note also that player $j \neq i$ has to use the same action α_j for all states ω with $\lambda_i^\omega \neq 0$, as he cannot distinguish these states by definition of $\Lambda^5(i)$. Summing the incentive-compatibility constraints over the states ω (taking into account that $\lambda_i^\omega \leq 0$) yields a weaker aggregate incentive condition:

$$\sum_{\omega\in\Omega} -\lambda_i^\omega v_i^\omega \geq -(1-\delta) \sum_{\omega\in\Omega} \lambda_i^\omega g_i^\omega(a_i, \alpha_{-i}) - \delta \sum_{\omega\in\Omega} \sum_{y\in Y} \pi_y(a_i, \alpha_{-i}) \lambda_i^\omega w_i^\omega(y)$$

This constraint corresponds to a game with a known state where player i's payoff is $\sum_{\omega\in\Omega} -\lambda_i^\omega g_i^\omega(a)$. Using this analogy, we can show that the maximal score in the direction of minimizing this payoff (that is, maximizing $-\sum_{\omega\in\Omega} -\lambda_i^\omega v_i^\omega$) is at most the corresponding minmax payoff, namely $-\min_{\alpha_{-i}} \max_{a_i} \sum_{\omega\in\Omega} -\lambda_i^\omega g_i^\omega(a_i, \alpha_{-i})$. We then use the strong full rank assumption to show that this bound is attained. *Q.E.D.*

Let Λ^6 be the set of λ such that there is $i \in \boldsymbol{I}$ such that $\lambda_i^\omega > 0$ for some $\omega \in \Omega$, $(\lambda_j^\omega)_{\omega\in\Omega} = 0$ for all $j \neq i$, $\theta_i(\omega') = \theta_i(\omega'')$ for all $\omega' \in \Omega$ and $\omega'' \neq \omega'$ satisfying $\lambda_i^{\omega'} > 0$ and $\lambda_i^{\omega''} > 0$, and $\theta_j(\omega') = \theta_j(\omega'')$ for all $j \neq i$, $\omega' \in \Omega$, and $\omega'' \neq \omega$ satisfying $\lambda_i^{\omega'} \neq 0$ and $\lambda_i^{\omega''} \neq 0$. In words, this says that only player i's payoff has non-zero weights, that player i cannot distinguish between any two states where his utility gets positive weight, and no other player can distinguish between any two states where player i's utility gets non-zero weight.

Finally, we construct a set Λ^7 that we show contains all directions that do not belong to one of the preceding sets. We define Λ^7 to be the set of all λ satisfying the following properties.

(i) $(\lambda_i^\omega)_{\omega\in\Omega} \neq 0$ and $(\lambda_j^\omega)_{\omega\in\Omega} \neq 0$ for some $i \in \boldsymbol{I}$ and $j \neq i$.

(ii) $(\lambda_l^{\omega'})_{l\in\boldsymbol{I}} \neq 0$ and $(\lambda_l^{\omega''})_{l\in\boldsymbol{I}} \neq 0$ for some $\omega' \in \Omega$ and $\omega'' \neq \omega'$.

(iii) $\theta_l(\omega''') = \theta_l(\omega'''')$ for $l \in \boldsymbol{I}$, $\omega''' \in \Omega$, and $\omega'''' \neq \omega'''$, if $\lambda_{l'}^{\omega'''} \neq 0$ for some $l' \neq l$ and $\lambda_{l''}^{\omega''''} \neq 0$ for some $l'' \neq l$.

(iv) $\theta_l(\omega''') = \theta_l(\omega'''')$ for $l \in \boldsymbol{I}$, $\omega''' \in \Omega$, and $\omega'''' \neq \omega'''$ if $\lambda_l^{\omega'''} > 0$ and $\lambda_{l'}^{\omega''''} \neq 0$ for some $l' \neq l$.

In words, this is the set of directions where the score depends on the payoffs of players i and j in some state ω, and where it also depends on the payoff of some player l (possibly i or j) in two other states ω' and ω'', but this player l cannot distinguish between any states ω''' and ω'''' if either (condition (iii)) in each of these states there is at least one other player whose payoff matters or (condition (iv)) the score is increasing in l's payoff in state ω''' and depends on the payoff of some l' in state ω''''.

Lemma 13. $\bigcup_{n=1}^{7} \Lambda^n = \boldsymbol{R}^{I\times|\Omega|} \setminus \{(0,\cdots,0)\}$

Proof. Let λ be such that $\lambda \neq (0,\cdots,0)$ and $\lambda \notin \Lambda^7$. It suffices to show that $\lambda \in \bigcup_{n=1}^{6} \Lambda^n$.

If λ does not satisfy the clause (ii) of the definition of Λ^7, then $\lambda \in \Lambda^1$. If λ does not satisfy (iii), then $\lambda \in \Lambda^2$. If λ does not satisfy (iv), then $\lambda \in \Lambda^3$. If λ satisfies (iii) and (iv) but not (i), then $\lambda \in \Lambda^4 \bigcup \Lambda^5 \bigcup \Lambda^6$. *Q.E.D.*

Lemma 14. *Suppose that monitoring structure is known and has strong full rank. Then for each* $\lambda \in \Lambda^6 \bigcup \Lambda^7$, $k^*(\lambda) = \max_\alpha \lambda \cdot g(\alpha)$.

Proof. See the Appendix. The first step is to show that for each $\lambda \in \Lambda^6 \bigcup \Lambda^7$, there is a single "type" θ_i^* that is relevant; we use this to show that the upper and lower bounds on the score are both $\max_\alpha \lambda \cdot g(\alpha)$. *Q.E.D.*

Combining the above lemmas yields the following characterization of the maximal scores in each direction and thus of the set Q.

Proposition 7. *Suppose that monitoring structure is known and has strong full rank. Then*

$$k^*(\lambda) = \begin{cases} \max_{v\in V^*} \lambda \cdot v & \textit{if} \quad \lambda \in \Lambda^1. \\ \infty & \textit{if} \quad \lambda \in \Lambda^2 \bigcup \Lambda^3 \bigcup \Lambda^4 \\ \max_{\alpha_{-i}} \min_{a_i} \sum_{\omega\in\Omega} \lambda_i^\omega g_i^\omega(a_i,\alpha_{-i}) & \textit{if} \quad \lambda \in \Lambda^5(i) \\ \max_\alpha \lambda \cdot g(\alpha) & \textit{if} \quad \lambda \in \Lambda^6 \bigcup \Lambda^7 \end{cases},$$

and $Q = \bigcap_{i\in\{1,\dots,7\},\lambda\in\Lambda^i} H^*(\lambda)$.

Recall that T-PPXE reduces to belief-free equilibria by Hörner and Lovo (2009) and Hörner, Lovo, and Tomala (2009) for games with perfectly observable actions.

Proposition 8. *Suppose that monitoring structure is known and has strong full rank. Suppose also that Q is full dimensional. Then the limit T-PPXE payoff set for this game, $\lim_{\delta \to 1} E(\delta) = Q$, is equal to the limit set of T-PPXE payoffs (or belief-free equilibrium payoffs) for the game that has the same information structure $(\Omega, (\Theta_i)_{i \in \boldsymbol{I}})$ and the same expected payoffs $(g_i)_{i \in \boldsymbol{I}}$ but with perfectly observable actions.*

Proof. For each direction λ, let $k^*(\lambda)$ be the maximal score in the original game, and $\tilde{k}^*(\lambda)$ be the score in the game with perfectly observable actions. Since monitoring structure is known and has strong full rank in the game with observable actions, Proposition 7 applies, and by inspection $k^*(\lambda) = \tilde{k}^*(\lambda)$ for each λ, so the set Q for the original game is equal to that for the game with observable actions. This proves the proposition. *Q.E.D.*

This shows that with a known monitoring structure and strong full rank, the analysis of the observed-action case carries over in the obvious way. When strong full rank fails, the known-monitoring-structure game can have a strictly smaller set of limit equilibrium payoffs than when actions are perfectly observable, for much the same reason that this can occur when the structure of the game- including the payoff functions- is known.

7.2 One-Sided Incomplete Information

In this subsection we consider the case where only player 1's payoff function is uncertain, and he knows his own payoff function while the other players do not. Formally, we say the game has one-sided incomplete information if $g_i^{\omega}(a) = g_i^{\omega'}(a)$ for all $i \neq 1$, $a \in A$, $\omega \in \Omega$, and $\omega' \neq \omega$, and that $\theta_1(\omega) = (\omega)$ for all ω, and $\Theta_i = \{(\Omega)\}$ for all $i \neq 1$. This is the assumption made in the Hörner and Lovo (2009, Section 4) and Hörner, Lovo and Tomala (2009, Section 6) analysis of reputations, so once again our results can be seen as extending theirs.

The main way that this extra structure simplifies the general solution of the last subsection is in the definitions of the sets Λ^i:

- Λ^1 is the same as before.
- Λ^2 is the set of λ such that there are $i \neq 1$, $j \neq 1$, $\omega \in \Omega$, and $\omega' \neq \omega$ such that $\lambda_i^{\omega} \neq 0$ and $\lambda_j^{\omega'} \neq 0$.

- Λ^3 is the set of λ such that $\lambda_1^\omega > 0$ for some $\omega \in \Omega$, $\lambda_i^{\omega'} \neq 0$ for some $i \neq 1$ and $\omega' \neq \omega$, and $\lambda_j^{\omega''} = 0$ for all $j \neq 1$ and $\omega'' \neq \omega'$.

- Λ^4 is the set of λ such that $(\lambda_i^\omega)_{\omega \in \Omega} = 0$ for all $i \neq 1$, and $\lambda_1^\omega > 0$ and $\lambda_1^{\omega'} > 0$ for some $\omega \in \Omega$ and $\omega' \neq \omega$.

- Λ^5 is the set of λ such that $(\lambda_i^\omega)_{\omega \in \Omega} = 0$ for all $i \neq 1$, $(\lambda_1^\omega)_{\omega \in \Omega} \leq 0$, and $\lambda_1^\omega < 0$ and $\lambda_1^{\omega'} < 0$ for some $\omega \in \Omega$ and $\omega' \neq \omega$.

- Λ^6 is the set of λ such that $(\lambda_i^\omega)_{\omega \in \Omega} = 0$ for all $i \neq 1$, $\lambda_1^\omega > 0$ for some $\omega \in \Omega$, $\lambda_1^{\omega'} < 0$ for some $\omega \in \Omega$, and $\lambda_1^{\omega''} \leq 0$ for all $\omega'' \neq \omega, \omega'$. As in the last case, no profile $\vec{\alpha}$ satisfies statewise conditions, and only player 1 has nonzero components.

- Λ^7 is the set of λ such that $(\lambda_1^\omega)_{\omega \in \Omega} \neq 0$, $(\lambda_1^\omega)_{\omega \in \Omega} \leq 0$, $(\lambda_i^{\omega'})_{i \neq 1} \neq 0$ for some $\omega' \in \Omega$, and $(\lambda_i^{\omega''})_{i \neq 1} = 0$ for all $\omega'' \neq \omega'$.

Recall that V^U denotes the set of feasible payoffs of the stage game with public randomization, that is, $V^U = \text{co}\{g(a) | a \in A\}$. Note that $\dim V^U$ is at most $|\Omega| + I - 1$, since $g_i^\omega(a) = g_i^{\omega'}(a)$ for all $i \neq 1$, $a \in A$, $\omega \in \Omega$, and $\omega' \neq \omega$. Let

$$V^{U*} = \text{co}\{v \in V^U | v_i^\omega \geq \underline{v}_i^\omega,\ \forall i \in \boldsymbol{I},\ \forall \omega \in \Omega\}.$$

Condition Non-E. There is a profile α_{-1}^* such that the set $V^{U**} = \{v \in V^{U*} | v_1^\omega \geq \max_{a_1} g_1^\omega(a_1, \alpha_{-1}^*),\ \forall \omega \in \Omega\}$ has dimension $|\Omega| + I - 1$.

This condition is likely to be satisfied if there is an action α_{-i} that gives low payoffs to player 1 for every state ω.

Proposition 9. *Suppose that monitoring structure is known and has strong full rank, and that there is one-sided incomplete information. Suppose also that (Non-E) holds. Then* $\dim Q = I \times |\Omega|$.

Proof. Let v be in the relative interior of V^{U**}. It suffices to show that $k^*(\lambda) > \lambda \cdot v$ for all λ.

First, consider $\lambda \in \Lambda^1$. Since $V^{U**} \subset V^*$, v is an interior point of V^*. Then $\lambda \cdot v < \max_{v' \in V^*} \lambda \cdot v' = k^*(\lambda)$ for $\lambda \in \Lambda^1$. Likewise, since $\text{int} V^{U**} \subset V^U$, we have $\lambda \cdot v < \max_{v' \in V^U} \lambda \cdot v' = \max_\alpha \lambda \cdot g(\alpha) = k^*(\lambda)$ for $\lambda \in \Lambda^6$ and $\lambda \in \Lambda^7$.

Since $k^*(\lambda) = \infty$ for $\lambda \in \Lambda^2 \bigcup \Lambda^3 \bigcup \Lambda^4$, it remains to consider $\lambda \in \Lambda^5$. Note that, by definition of Λ^5, $(\lambda_1^\omega)_{\omega\in\Omega} \neq 0$, $\lambda_1^\omega \leq 0$ for all $\omega \in \Omega$, and $(\lambda_j^\omega)_{\omega\in\Omega} = 0$ for all $j \neq 1$. Also, since v is in the relative interior of V^{U**}, $v_1^\omega > \max_{a_1} g_1^\omega(a_1, \alpha_{-1}^*)$ for all $\omega \in \Omega$. Taken together, we obtain

$$\begin{aligned}
\lambda \cdot v &= \sum_{\omega\in\Omega} \lambda_1^\omega v_1^\omega \\
&< \sum_{\omega\in\Omega} \lambda_1^\omega \max_{a_1} g_1^\omega(a_1, \alpha_{-1}^*) \\
&= \sum_{\omega\in\Omega} \min_{a_1} \lambda_1^\omega g_1^\omega(a_1, \alpha_{-1}^*) \\
&< \max_{\alpha_{-1}} \min_{a_1} \sum_{\omega\in\Omega} \lambda_1^\omega g_1^\omega(a_1, \alpha_{-1}) \\
&= k^*(\lambda),
\end{aligned}$$

as desired. Here, the equality in the third line comes from

$$\begin{aligned}
\lambda_1^\omega \max_{a_1} g_1^\omega(a_1, \alpha_{-1}^*) &= -\lambda_1^\omega \min_{a_1}(-g_1^\omega(a_1, \alpha_{-1}^*)) \\
&= |\lambda_1^\omega| \min_{a_1}(-g_1^\omega(a_1, \alpha_{-1}^*)) \\
&= \min_{a_1} |\lambda_1^\omega|(-g_1^\omega(a_1, \alpha_{-1}^*)) \\
&= \min_{a_1} \lambda_1^\omega g_1^\omega(a_1, \alpha_{-1}^*)).
\end{aligned}$$

Q.E.D.

Section 5 of Hörner, Lovo, and Tomala (2009) derives several sufficient conditions for Q (denoted by V^* in their paper) to be nonempty, which implies that there is a T-PPXE in the undiscounted case. However these conditions do not assure the existence of T-PPXE with the discounted payoff criterion used in this paper, because Q might not be full dimensional and in that case neither their existence results nor Proposition 9 might not apply. On the other hand, our Proposition 9 identifies a simple sufficient condition for Q to be full dimensional; then Proposition 3 applies, and it turns out that (Non-E) is a sufficient condition for the existence of T-PPXE.

Remark 1. If there is a "commitment type" ω^*, for which there is some $a_1^* \in A_1$ such that $g_1^{\omega^*}(a_1^*, a_{-1})$ is independent of a_{-1} and $g_1^{\omega^*}(a_1^*, a_{-1}) \geq g_1^{\omega^*}(a)$ for all $a_1 \in A_1$, the minimax payoff of this commitment type equals his best payoff $g_1^{\omega^*}(a_i^*, a_{-i})$. In this case the set Q does not have full dimension, and our results do not apply.[10] Moreover, in this case the set of T-PPXE is often empty. Suppose that there are two players, and player 2 has a unique best reply against a_1^*, and call it a_2^*. In a T-PPXE, player 1 in state ω^* always play a_1^*, so that player 2 must play a_2^* after every history, independently of the state. Then player 1's optimal strategy for state $\omega \neq \omega^*$ is to choose $a_1^\omega \in \arg\max_{a_1 \in A_1} g_1^\omega(a_1, a_2^*)$ after every history. For this strategy profile to be a T-PPXE, a_2^* must be a best reply to a_1^ω for all $\omega \neq \omega^*$, but such a condition is not satisfied in general. Thus we conclude that there is no T-PPXE for any discount factor.[11]

7.3 The Folk Theorem with Known Monitoring Structure

Our general folk theorem uses (SFR1), which requires either that all players can distinguish every pair of states, or that there are profiles $\vec{\alpha}$ that satisfy various full rank conditions. With a known monitoring structure (and strong full rank) the following simpler condition is sufficient.

Proposition 10. *Suppose that monitoring structure is known and has strong full rank. Suppose also that for each* (ω, ω') *satisfying* $\omega \neq \omega'$*, there are at least three players who can distinguish* ω *and* ω'*, i.e., there are* $i \in \boldsymbol{I}$*,* $j \neq i$*, and* $l \neq i, j$ *such that* $\theta_i(\omega) \neq \theta_i(\omega')$*,* $\theta_j(\omega) \neq \theta_j(\omega')$*, and* $\theta_l(\omega) \neq \theta_l(\omega')$*. Then, for any smooth strict subset* W *of* V^**, there exists* $\overline{\delta} \in (0,1)$ *such that* $W \subseteq E(\delta)$ *for all* $\delta \in (\overline{\delta}, 1)$*.*

Proof. Since there are at least three players who can distinguish ω and ω', any cross-state direction λ is an element of Λ^2. Then, from Proposition 7, we have $k^*(\lambda) = \infty$. Since $k^*(\lambda) = \max_{v \in V^*} \lambda \cdot v$ for any $\lambda \in \Lambda^1$, we obtain $Q = V^*$. *Q.E.D.*

Proposition 5.6 of Hörner, Lovo, and Tomala (2009) shows that Q is nonempty for games with perfect monitoring, if there are there are at least three players who can distinguish ω and ω' for each (ω, ω') satisfying $\omega \neq \omega'$; our result shows that the assumptions of that proposition are in fact sufficient for a folk theorem.

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[10] Hörner and Lovo (2009) make essentially this point on page 475.

[11] If there are observed actions, these same assumptions imply that there is not a belief-free equilibrium. Hörner and Lovo (2009) note that there is a belief-free equilibrium with a commitment type in strictly dominant action games with a unique Stackelberg type.

Proposition 11. *Suppose that monitoring structure is known satisfies strong full rank. Suppose also that for each* (ω, ω') *satisfying* $\omega \neq \omega' \neq \omega$*, there are at least two players who can distinguish* ω *and* ω'*, i.e., there are* $i \in \boldsymbol{I}$ *and* $j \neq i$ *such that* $\theta_i(\omega) \neq \theta_i(\omega')$ *and* $\theta_j(\omega) \neq \theta_j(\omega')$*. Let* $V^{**} \equiv \{v \in V^* | \exists \tilde{v} \in V^U, \ \forall i \in \boldsymbol{I}, \ \forall \omega \in \Omega, \ v_i^\omega \geq \tilde{v}_i^\omega\}$*. Then, for any smooth strict subset* W *of* V^{**}*, there is* $\overline{\delta} \in (0,1)$ *such that* $W \subseteq E(\delta)$ *for all* $\delta \in (\overline{\delta}, 1)$*.*

Note that if there is a "bad outcome" $\alpha \in \triangle A$ such that $g_i^\omega(\alpha) \leq v_i^\omega$ for all $i \in \boldsymbol{I}$ and $\omega \in \Omega$, then we have $V^{**} = V^*$, so that the folk theorem obtains. Proposition 5.9 of Hörner, Lovo, and Tomala (2009) shows that Q is nonempty for games with perfect monitoring and a bad outcome, if there are there are at least two players who can distinguish ω and ω' for each (ω, ω') satisfying $\omega \neq \omega'$. Again our result shows that the assumptions of the proposition are sufficient for a folk theorem.

Proof. It suffices to show that $V^{**} \subseteq Q$. To do so, we compute the maximal score $k^*(\lambda)$ for every direction, using Proposition 7.

First, consider λ such that $(\lambda_i^\omega)_{i \in \boldsymbol{I}} \neq 0$ for some ω and $(\lambda_i^{\omega'})_{i \in \boldsymbol{I}} = 0$ for all $\omega' \neq \omega$. By definition, $\lambda \in \Lambda^1$, and hence from Proposition 7, $k^*(\lambda) = \max_{v \in V^*} \lambda \cdot v$ for this direction.

Next, consider λ such that $\lambda_i^{\omega'} \neq 0$ and $\lambda_i^{\omega''} \neq 0$ for some $i \in \boldsymbol{I}$, $\omega' \in \Omega$, and $\omega'' \neq \omega'$, and $(\lambda_j^\omega)_{\omega \in \Omega} = 0$ for all $j \neq i$. Since there are at least two players who can distinguish ω' and ω'', there is $l \neq i$ such that $\theta_l(\omega') \neq \theta_l(\omega'')$. Thus $\lambda \in \Lambda^2$, and hence $k^*(\lambda) = \infty$ for this direction.

Consider λ such that $\lambda_i^{\omega'} \neq 0$ and $\lambda_j^{\omega''} \neq 0$ for some $i \in \boldsymbol{I}$, $j \neq i$, $\omega' \in \Omega$, and $\omega'' \neq \omega'$, and $\theta_l(\omega') \neq \theta_l(\omega'')$ for some $l \neq i, j$. Again, $\lambda \in \Lambda^2$ in this case, so that $k^*(\lambda) = \infty$.

Consider λ such that $\lambda_i^{\omega'} > 0$ and $\lambda_j^{\omega''} \neq 0$ for some $i \in \boldsymbol{I}$, $j \neq i$, $\omega' \in \Omega$, and $\omega'' \neq \omega'$, and $\theta_l(\omega') = \theta_l(\omega'')$ for all $l \neq i, j$. Since there are at least two players who can distinguish ω' and ω'', it must be that $\theta_i(\omega') \neq \theta_i(\omega'')$ and $\theta_j(\omega') \neq \theta_j(\omega'')$. This implies that $\lambda \in \Lambda^3$, and hence $k^*(\lambda) = \infty$.

Finally, consider λ such that $\lambda \leq 0$, $\lambda_i^{\omega'} < 0$ and $\lambda_j^{\omega''} < 0$ for some $i \in \boldsymbol{I}$, $j \neq i$, $\omega' \in \Omega$, and $\omega'' \neq \omega'$, and for any pair (i, ω''') and (j, ω'''') satisfying $\omega''' \neq \omega''''$, $\lambda_i^{\omega'''} < 0$, and $\lambda_j^{\omega''''} < 0$, and for any $l \neq i, j$, $\theta_l(\omega''') = \theta_l(\omega'''')$. By definition, $\lambda \in \Lambda^7$ in this case, so that $k^*(\lambda) = \max_\alpha \lambda \cdot g(\alpha) = \max_{v \in V^U} \lambda \cdot v$.

Obviously, $V^{**} \subset H^*(\lambda)$ for the first four cases. Also, $V^{**} \subset H^*(\lambda)$ for the last case, since $\lambda \leq 0$. Therefore, $V^{**} \subseteq Q$. *Q.E.D.*

8 Conclusion

This paper shows how to extend the insights and techniques of the repeated games literature to games with imperfectly observed actions, an unknown monitoring structure, and private information. Our analysis is based on the fact that the set of T-PPXE payoffs have a recursive structure, and says little about the entire set of equilibrium payoffs. When the folk theorem holds in T-PPXE, or more generally when there are asymptotically efficient T-PPXE, the restriction to T-PPXE may be of less concern, especially given their desirable robustness properties. When the set of T-PPXE is small or empty, it would be nice to know more about the entire set of sequential equilibrium payoffs; that more difficult problem is still unresolved. Another open question is to extend the analysis to games where the state evolves according to a finite Markov chain.

Appendix

A.1 Proof of Proposition 4

Lemmas A1 through A4 are straightforward generalizations of Fudenberg and Yamamoto (2009), so we omit the proofs.

Lemma A1. *Suppose that (PFR) holds. Then, there is an open and dense set of profiles $\vec{\alpha}$ each of which has pairwise full rank for all (i,ω) and (j,ω) satisfying $i \neq j$.*

Lemma A2. *Suppose that (IFR) holds. Then, for any $i \in \boldsymbol{I}$, $\omega \in \Omega$, and $\varepsilon > 0$, there is a profile $\vec{\alpha} \in \times_{i \in \boldsymbol{I}} \triangle A_i$ such that $\alpha_i^{\theta(\omega)} \in \arg\max_{\alpha_i} g_i^{\omega}(\alpha_i, \underline{\alpha}_{-i}^{\theta_{-i}(\omega)})$; $|g_i^{\omega}(\vec{\alpha}^{\theta(\omega)}) - \underline{v}_i^{\omega}| < \varepsilon$; and $\vec{\alpha}$ has individual full rank for all $(j,\omega') \neq (i,\omega)$.*

Lemma A3. *Suppose that a profile $\vec{\alpha}$ has pairwise full rank for all (i,ω) and (j,ω) satisfying $i \neq j$. Then, $k^*(\vec{\alpha},\lambda) = \lambda \cdot g(\vec{\alpha})$ for direction λ such that $(\lambda_i^{\omega})_{i \in \boldsymbol{I}}$ has at least two non-zero components for some ω while $(\lambda_j^{\omega'})_{i \in \boldsymbol{I}} = 0$ for all $\omega' \neq \omega$.*

Lemma A4. *Suppose that* $\vec{\alpha}$ *has individual full rank for all* $(j,\omega') \neq (i,\omega)$ *and has the best-response property for player* i *and for state* ω*. Then,* $k^*(\vec{\alpha},\lambda) = \lambda \cdot g(\vec{\alpha})$ *for direction* λ *such that* $\lambda_i^{\omega} \neq 0$ *and* $\lambda_j^{\omega'} = 0$ *for all* $(j,\omega') \neq (i,\omega)$*.*

Lemma A5. *Suppose that a profile* $\vec{\alpha}$ *is ex-post enforceable and has statewise full rank for* (i,ω) *and* (j,ω') *satisfying* $\omega \neq \omega'$*. Then,* $k^*(\vec{\alpha},\lambda) = \infty$ *for direction* λ *such that* $\lambda_i^{\omega} \neq 0$ *and* $\lambda_j^{\omega'} \neq 0$*.*

The proof of this lemma requires a minor adaptation of earlier results so we include it in the supplementary materials.

Lemma A6. *Suppose that (IFR) and (PFR) hold. Let* λ *be such that* $\theta_i(\omega) \neq \theta_i(\omega')$ *for all* $i \in \boldsymbol{I}$*,* $\omega \in \Omega$ *and* $\omega' \neq \omega$ *satisfying* $(\lambda_j^{\omega})_{j\in\boldsymbol{I}} \neq 0$ *and* $(\lambda_j^{\omega'})_{j\in\boldsymbol{I}} \neq 0$*. Then,* $k^*(\lambda) \geq \max_{v\in V^*} \lambda \cdot v$*.*

Proof. For each $\omega \in \Omega$, let $\lambda(\omega) = (\lambda_i^{\omega'}(\omega))_{(i,\omega')}$ be such that $(\lambda_i^{\omega}(\omega))_{i\in\boldsymbol{I}} = (\lambda_i^{\omega})_{i\in\boldsymbol{I}}$ and $(\lambda_i^{\omega'}(\omega))_{i\in\boldsymbol{I}} = 0$ for all $\omega' \neq \omega$. Let Ω^* be the set of all ω such that $\lambda(\omega) \neq 0$.

We claim

$$k^*(\vec{\alpha},\lambda) \geq \sum_{\omega\in\Omega^*} k^*(\vec{\alpha},\lambda(\omega)) \tag{1}$$

for each $\vec{\alpha}$. In words, $k^*(\vec{\alpha},\lambda)$ is at least the sum of the maximal scores when we solve the LP problem for each state ω in isolation. To prove this, consider the LP problem for $(\vec{\alpha},\lambda)$ but constraint (iii) is replaced with a more restrictive condition

$$\text{(iii}'\text{)} \quad \sum_{i\in\boldsymbol{I}} \lambda_i^{\omega} v_i^{\omega} \geq \sum_{i\in\boldsymbol{I}} \lambda_i^{\omega} w_i^{\omega}(y) \qquad \text{for all } \omega \in \Omega \text{ and } y \in Y.$$

Let $k^U(\vec{\alpha},\lambda)$ denote the solution to this new problem. Since condition (iii′) does not allow utility transfer across different states, considering this new LP problem is equivalent to solving a separate LP problem for each state $\omega \in \Omega^*$ in isolation. Thus we have $k^U(\vec{\alpha},\lambda) = \sum_{\omega\in\Omega^*} k^*(\vec{\alpha},\lambda(\omega))$. Since $k^*(\vec{\alpha},\lambda) \geq k^U(\vec{\alpha},\lambda)$, (1) follows.

Recall that $\lambda(\omega)$ considers only a single state ω. Thus the maximal score $k^*(\vec{\alpha},\lambda(\omega))$ depends on $\alpha^{\theta(\omega)}$ but not on $\alpha^{\theta'}$ for other θ'. This observation, together with the fact that all players can distinguish any state in the set Ω^*, implies that

$$\sup_{\vec{\alpha}} \sum_{\omega\in\Omega^*} k^*(\vec{\alpha},\lambda(\omega)) = \sum_{\omega\in\Omega^*} \sup_{\vec{\alpha}} k^*(\vec{\alpha},\lambda(\omega)).$$

It follows from Lemmas A1 through A4 that $\sup_{\vec{\alpha}} k^*(\vec{\alpha},\lambda(\omega)) = \max_{v\in V^*}\lambda(\omega)\cdot v$. Therefore,

$$\sup_{\vec{\alpha}} \sum_{\omega\in\Omega^*} k^*(\vec{\alpha},\lambda(\omega)) = \sum_{\omega\in\Omega^*} \max_{v\in V^*}\lambda(\omega)\cdot v = \max_{v\in V^*}\lambda\cdot v.$$

Using (1), we obtain the desired result. *Q.E.D.*

Using these lemmas, we can prove Proposition 4, by adapting the arguments in Fudenberg and Yamamoto (2009). One difference is that $k^*(\lambda)$ might not be infinitely large for cross-state directions λ such that $\theta_i(\omega)\neq\theta_i(\omega')$ for all $i\in\boldsymbol{I}$, $\omega\in\Omega$ and $\omega'\neq\omega$ satisfying $(\lambda_j^{\omega})_{j\in\boldsymbol{I}}\neq 0$ and $(\lambda_j^{\omega'})_{j\in\boldsymbol{I}}\neq 0$, since (SFR1) does not assure that there be a type-contingent profile $\vec{\alpha}$ that has statewise full rank. However, Lemma A6 shows that $k^*(\lambda)\geq\max_{v\in V^*}\lambda\cdot v$ for these directions, so that the maximal score is high enough to establish the folk theorem.

A.2 Proof of Proposition 5

The following lemmas prove Proposition 5. Again, the details are similar to Fudenberg and Yamamoto (2009) so we omit them.

Lemma A7. *Suppose that there is a static ex-post equilibrium $\vec{\alpha}^0$. Then, for any direction λ, $k^*(\vec{\alpha}^0,\lambda)\geq\lambda\cdot g(\vec{\alpha}^0)$.*

Lemma A8.

- **(a)** *Suppose that (X-Eff) holds. Then, $k^*(\lambda)=\lambda\cdot g(a)$ for direction λ such that $(\lambda_i^{\omega})_{i\in\boldsymbol{I}}$ has at least two non-zero components for some $\omega\in\Omega$ while $\lambda_j^{\omega'}=0$ for all $j\in\boldsymbol{I}$ and $\omega'\neq\omega$.*
- **(b)** *Suppose that (X-Eff) holds. Then, $k^*(\lambda)=\max_{v\in V}\lambda\cdot v$ for direction λ such that $\lambda_i^{\omega}>0$ and $\lambda_j^{\omega'}=0$ for all $(j,\omega')\neq(i,\omega)$.*

Lemma A9.

- **(a)** *Suppose that $\vec{\alpha}$ is ex-post enforceable and n-statewise distinguishes (i,ω) from (j,ω'). Then $k^*(\alpha,\lambda)=\infty$ for direction λ such that $\lambda_i^{\omega}>0$ and $\lambda_j^{\omega'}<0$.*

(b) *Suppose that $\vec{\alpha}$ is ex-post enforceable and p-statewise distinguishes (i,ω) from (j,ω'). Then $k^*(\alpha,\lambda)=\infty$ for direction λ such that $\lambda_i^\omega>0$ and $\lambda_j^{\omega'}>0$.*

A.3 Proof of Lemma 12

Recall that $\Lambda^5(i)$ is the set of λ such that $(\lambda_i^\omega)_{\omega\in\Omega}\leq 0$, $(\lambda_i^\omega)_{\omega\in\Omega}\neq 0$, $(\lambda_j^\omega)_{\omega\in\Omega}=0$ for all $j\neq i$, and $\theta_j(\omega)=\theta_j(\omega')$ for all $j\neq i$, $\omega\in\Omega$, and $\omega'\neq\omega$ satisfying $\lambda_i^\omega\neq 0$ and $\lambda_i^{\omega'}\neq 0$.

Lemma 12. *Suppose that monitoring structure is known and has strong full rank. Then for each i and $\lambda\in\Lambda^5(i)$, $k^*(\lambda)=\max_{\alpha_{-i}}\min_{a_i}\sum_{\omega\in\Omega}\lambda_i^\omega(\omega)g_i^\omega(a_i,\alpha_{-i})$, that is, $k^*(\lambda)=-\min_{\alpha_{-i}}\max_{a_i}\sum_{\omega\in\Omega}-\lambda_i^\omega g_i^\omega(a_i,\alpha_{-i})$.*

To prove this lemma, we use the following claims.

Claim 2. *Let $\lambda\in\Lambda^5(i)$. Then for each $j\neq i$, there is $\theta_j^*\in\Theta_j$ that contains all ω such that $\lambda_i^\omega\neq 0$.*

Proof. Suppose not, so that there are $\omega\in\Omega$ and $\omega'\neq\omega$ such that such that $\theta_j(\omega)\neq\theta_j(\omega')$, $\lambda_i^\omega\neq 0$, and $\lambda_i^{\omega'}\neq 0$. Then $\lambda\notin\Lambda^5(i)$, since it does not satisfy the last condition of the definition of $\Lambda^5(i)$. A contradiction. *Q.E.D.*

Claim 3. *Suppose that monitoring structure is known. Let $\lambda\in\Lambda^5(i)$. Then for each $\vec{\alpha}=((\alpha_i^{\theta_i})_{\theta_i\in\Theta_i})_{i\in\boldsymbol{I}}$, $k^*(\vec{\alpha},\lambda)\leq\min_{a_i}\lambda\cdot g(a_i,\alpha_{-i}^{\theta_{-i}^*})$ where θ_{-i}^* is chosen as in Claim 2 and $\alpha_{-i}^{\theta_{-i}^*}=(\alpha_j^{\theta_j^*})_{j\neq i}$.*

Proof. Let $a_i'\in\arg\min_{a_i}\lambda\cdot g(a_i,\alpha_{-i}^{\theta_{-i}^*})$. If $k^*(\vec{\alpha},\lambda)=-\infty$ then the result is obvious. If $k^*(\vec{\alpha},\lambda)>-\infty$, we can choose (v,w) to satisfy constraints (i) through (iii) in the LP problem associated with $(\vec{\alpha},\lambda,\delta)$ for some $\delta\in(0,1)$. It follows from constraint (ii) that

$$v_i^\omega\geq(1-\delta)g_i^\omega(a_i,\alpha_{-i}^{\theta_{-i}^*})+\delta\pi(a_i,\alpha_{-i}^{\theta_{-i}^*})\cdot w_i^\omega$$

for all i,a_i, and all $\omega\in\Omega$ such that $\lambda_i^\omega\neq 0$, since $\theta_j(\omega)=\theta_j^*$ for $j\neq i$ for such ω. Multiplying both sides by λ_i^ω, summing over all ω, and using the fact that $\lambda_j^\omega=0$

for all $j \neq i$, we have

$$\begin{aligned}\lambda \cdot v = \sum_{\omega\in\Omega} \lambda_i^\omega v_i^\omega &\leq (1-\delta)\sum_{\omega\in\Omega} \lambda_i^\omega g_i^\omega(a_i', \alpha_{-i}^{\theta^*_{-i}}) + \delta \sum_{\omega\in\Omega}\sum_{y\in Y} \pi_y(a_i', \alpha_{-i}^{\theta^*_{-i}})\lambda_i^\omega w_i^\omega(y) \\ &= (1-\delta)\lambda \cdot g(a_i', \alpha_{-i}^{\theta^*_{-i}}) + \delta \sum_{y\in Y} \pi_y(a_i', \alpha_{-i}^{\theta^*_{-i}})\lambda \cdot w(y),\end{aligned}$$

so from (iii),

$$\lambda \cdot v \leq (1-\delta)\lambda \cdot g(a_i', \alpha_{-i}^{\theta^*_{-i}}) + \delta\lambda \cdot v.$$

Subtracting $\delta\lambda \cdot v$ from both sides and dividing by $(1-\delta)$, we get $\lambda \cdot v \leq \lambda \cdot g(a_i', \alpha_{-i}^{\theta^*_{-i}})$. Therefore, $k^*(\vec{\alpha}, \lambda, \delta) \leq g(a_i', \alpha_{-i}^{\theta^*_{-i}})$. *Q.E.D.*

Let $\tilde{g}_j(a) = -\sum_{\omega\in\Omega} \lambda_i^\omega g_j^\omega(a)$. Let $\tilde{\lambda} \in \boldsymbol{R}^I$ be such that $\tilde{\lambda}_i = -1$ and $\tilde{\lambda}_j = 0$ for all $j \neq i$. Consider the following LP problem:

$$\begin{aligned} \tilde{k}^*(\alpha, \tilde{\lambda}, \delta) = \max_{\substack{\tilde{v}\in\boldsymbol{R}^I \\ \tilde{w}:Y\to\boldsymbol{R}^I}} \tilde{\lambda} \cdot \tilde{v} \qquad &\text{subject to} \\ \text{(i)} \quad \tilde{v}_j = (1-\delta)\tilde{g}_j(\alpha) + \delta\pi(\alpha) \cdot \tilde{w}_j & \\ \text{for all } j \in \boldsymbol{I}, & \\ \text{(ii)} \quad \tilde{v}_j = (1-\delta)\tilde{g}_j(a_j, \alpha_{-j}) + \delta\pi(a_j, \alpha_{-j}) \cdot \tilde{w}_j & \\ \text{for all } j \in \boldsymbol{I} \text{ and } a_i \in A_i, & \\ \text{(iii)} \quad \tilde{\lambda} \cdot \tilde{v} \geq \tilde{\lambda} \cdot \tilde{w}(y) \qquad \text{for all } y \in Y. & \end{aligned}$$

This is the problem of finding the maximum score for a known-state game (i.e., $|\Omega| = 1$) for direction $\tilde{\lambda}$, so its value (which does not depend on δ) follows from past work:

Claim 4. *Suppose that monitoring structure is known and has strong full rank. Then* $\sup_\alpha \tilde{k}^*(\alpha, \tilde{\lambda}) = -\min_{\alpha_{-i}} \max_{a_i} \tilde{g}_i(a_i, \alpha_{-i})$

Proof. Strong full rank implies that every pure action profile has individual full rank. Then from FLM Lemma 6.3, the maximal score for direction $\tilde{\lambda}$ is given by player i's minimax score. Therefore, $\tilde{k}^*(\alpha, \tilde{\lambda}) = -\min_{\alpha_{-i}} \max_{a_i} \tilde{g}_i(a_i, \alpha_{-i})$. *Q.E.D.*

Claim 5. *Suppose that the monitoring structure is known and has strong full rank. Let* $\lambda \in \Lambda^5(i)$*. Then* $k^*(\vec{\alpha}, \lambda) = \tilde{k}^*(\alpha, \tilde{\lambda})$ *if* $\vec{\alpha}$ *is a state-independent action* α*.*

Proof. First, we show $k^*(\vec{\alpha},\lambda) \leq \tilde{k}^*(\alpha,\tilde{\lambda})$. When $k^*(\vec{\alpha},\lambda) = -\infty$, then this inequality obviously follows. So assume $k^*(\vec{\alpha},\lambda) > -\infty$. Choose (v,w) to satisfy constraints (i) through (iii) in the LP problem for $(\vec{\alpha},\lambda,\delta)$, and let $\tilde{v}_j = -\sum_{\omega\in\Omega}\lambda_j^\omega v_j^\omega$ and $\tilde{w}_j = -\sum_{\omega\in\Omega}\lambda_j^\omega w_j^\omega(y)$ for all $j \in \boldsymbol{I}$ and $y \in Y$. Then this $(\tilde{v},\tilde{w})$ satisfies all the constraints of the LP problem for $(\alpha,\tilde{\lambda},\delta)$, and $\lambda \cdot v = \tilde{\lambda}\cdot\tilde{v}$. This shows that $k^*(\vec{\alpha},\lambda) \leq \tilde{k}^*(\alpha,\tilde{\lambda})$.

Next, we show $k^*(\vec{\alpha},\lambda) \geq \tilde{k}^*(\alpha,\tilde{\lambda})$. As before we restrict attention to the case of $\tilde{k}^*(\alpha,\tilde{\lambda}) > -\infty$.

We claim there are $(z_i^\omega(y))_{(\omega,y)}$ such that

$$(1-\delta)\left(-\frac{\tilde{g}_i(a_i,\alpha_{-i})}{\sum_{\omega\in\Omega}\lambda_i^\omega} - g_i^\omega(a_i,\alpha_{-i})\right) = \delta\pi(a_i,\alpha_{-i})\cdot z_i^\omega \tag{2}$$

for all $\omega \in \Omega$ and $a_i \in A_i$, and

$$\sum_{\omega\in\Omega}\lambda_i^\omega z_i^\omega(y) = 0 \tag{3}$$

for all $y \in Y$. To see that this system has a solution, choose ω' such that $\lambda_i^{\omega'} \neq 0$, and eliminate $z_i^{\omega'}$ using (3). Then we can check that (2) for ω' are redundant equations; that is, (2) for ω' automatically holds if (2) holds for all $\omega \neq \omega'$. This leaves $(|\Omega|-1)\times|A_i|$ equations and $(|\Omega|-1)\times|A_i|$ unknowns, and strong full rank assures that the coefficient matrix has full rank. Therefore, the system has a solution.

Choose $(\tilde{v},\tilde{w})$ to satisfy all the constraints of the LP problem for $(\alpha,\tilde{\lambda},\delta)$, let $v_i^\omega = -\frac{\tilde{v}_i}{\sum_{\omega\in\Omega}\lambda_i^\omega}$, and $w_i^\omega(y) = -\frac{\tilde{w}_i(y)}{\sum_{\omega\in\Omega}\lambda_i^\omega} + z_i^\omega(y)$. Since $\lambda\cdot v = \tilde{\lambda}\cdot\tilde{v}$, it suffices to show that this (v,w) satisfies all the constraints of the LP problem for $(\vec{\alpha},\lambda,\delta)$. (We can ignore the adding-up constraint and the incentive compatibility constraint for player $j \neq i$, as strong full rank holds.) Note that

$$\begin{aligned}
&(1-\delta)g_i^\omega(a_i,\alpha_{-i})+\delta\pi(a_i,\alpha_{-i})\cdot w_i^\omega\\
=&(1-\delta)g_i^\omega(a_i,\alpha_{-i})+\delta\pi(a_i,\alpha_{-i})\cdot\left(z_i^\omega-\frac{1}{\sum_{\omega\in\Omega}\lambda_i^\omega}\tilde{w}_i(y)\right)\\
=&(1-\delta)g_i^\omega(a_i,\alpha_{-i})+(1-\delta)\left(-\frac{\tilde{g}_i(a_i,\alpha_{-i})}{\sum_{\omega\in\Omega}\lambda_i^\omega}-g_i^\omega(a_i,\alpha_{-i})\right)-\frac{\delta\pi(a_i,\alpha_{-i})\cdot\tilde{w}_i}{\sum_{\omega\in\Omega}\lambda_i^\omega}\\
=&-\frac{(1-\delta)\tilde{g}_i(a_i,\alpha_{-i})+\delta\pi(a_i,\alpha_{-i})\cdot\tilde{w}_i}{\sum_{\omega\in\Omega}\lambda_i^\omega}\\
\leq&-\frac{\tilde{v}_i}{\sum_{\omega\in\Omega}\lambda_i^\omega}\\
=&v_i^\omega
\end{aligned}$$

for all $a_i \in A_i$ with equality if $a_i \in \operatorname{supp}\alpha_i$. Here, the second equality comes from (2), and the inequality comes from the fact that $(\tilde{v},\tilde{w})$ satisfies the constraints of the LP problem for $(\alpha,\tilde{\lambda},\delta)$. Therefore, this (v,w) satisfies constraints (i) and (ii). Also,

$$\begin{aligned}
\lambda\cdot w(y) &= \sum_{\omega\in\Omega}\lambda_i^\omega w_i^\omega(y)\\
&= \sum_{\omega\in\Omega}\lambda_i^\omega\left(z_i^\omega(y)-\frac{\tilde{w}_i(y)}{\sum_{\omega\in\Omega}\lambda_i^\omega}\right)\\
&= -\tilde{w}_i(y)\\
&\leq -\tilde{v}_i = \lambda\cdot v.
\end{aligned}$$

Here, the third equality comes from (3) and the inequality comes from the fact that $(\tilde{v},\tilde{w})$ satisfies the constraints of the LP problem for $(\alpha,\tilde{\lambda},\delta)$. Therefore, this (v,w) satisfies constraint (iii). *Q.E.D.*

It follows from Claims 4 and 5 and $\tilde{g}_j(a) = -\sum_{\omega\in\Omega}\lambda_i^\omega g_j^\omega(a)$ that

$$\begin{aligned}
k^*(\lambda) &\geq \sup_\alpha \tilde{k}^*(\alpha,\tilde{\lambda})\\
&= -\min_{\alpha_{-i}}\max_{a_i}\tilde{g}_i(a_i,\alpha_{-i})\\
&= -\min_{\alpha_{-i}}\max_{a_i} -\lambda\cdot g(a_i,\alpha_{-i})\\
&= \max_{\alpha_{-i}}\min_{a_i}\lambda\cdot g(a_i,\alpha_{-i}).
\end{aligned}$$

On the other hand, Claim 3 shows that $k^*(\lambda) \leq \max_{\alpha_{-i}} \min_{a_i} \lambda \cdot g(a_i, \alpha_{-i})$. Therefore, $k^*(\lambda) = \max_{\alpha_{-i}} \min_{a_i} \lambda \cdot g(a_i, \alpha_{-i})$.

A.4 Proof of Lemma 14

Recall that Λ^6 is the set of λ such that there is $i \in \boldsymbol{I}$ such that $\lambda_i^{\omega} > 0$ for some $\omega \in \Omega$, $(\lambda_j^{\omega})_{\omega \in \Omega} = 0$ for all $j \neq i$, $\theta_i(\omega') = \theta_i(\omega'')$ for all $\omega' \in \Omega$ and $\omega'' \neq \omega'$ satisfying $\lambda_i^{\omega'} > 0$ and $\lambda_i^{\omega''} > 0$, and $\theta_j(\omega') = \theta_j(\omega'')$ for all $j \neq i$, $\omega' \in \Omega$, and $\omega'' \neq \omega$ satisfying $\lambda_i^{\omega'} \neq 0$ and $\lambda_i^{\omega''} \neq 0$. Also, Λ^7 is the set of λ satisfying the following properties.

(i) $(\lambda_i^{\omega})_{\omega \in \Omega} \neq 0$ and $(\lambda_j^{\omega})_{\omega \in \Omega} \neq 0$ for some $i \in \boldsymbol{I}$ and $j \neq i$.

(ii) $(\lambda_l^{\omega'})_{l \in \boldsymbol{I}} \neq 0$ and $(\lambda_l^{\omega''})_{l \in \boldsymbol{I}} \neq 0$ for some $\omega' \in \Omega$ and $\omega'' \neq \omega'$.

(iii) $\theta_l(\omega''') = \theta_l(\omega'''')$ for $l \in \boldsymbol{I}$, $\omega''' \in \Omega$, and $\omega'''' \neq \omega'''$, if $\lambda_{l'}^{\omega'''} \neq 0$ for some $l' \neq l$ and $\lambda_{l''}^{\omega''''} \neq 0$ for some $l'' \neq l$.

(iv) $\theta_l(\omega''') = \theta_l(\omega'''')$ for $l \in \boldsymbol{I}$, $\omega''' \in \Omega$, and $\omega'''' \neq \omega'''$ if $\lambda_l^{\omega'''} > 0$ and $\lambda_{l'}^{\omega''''} \neq 0$ for some $l' \neq l$.

Lemma 14. *Suppose that monitoring structure is known and has strong full rank. Then for each* $\lambda \in \Lambda^6 \bigcup \Lambda^7$, $k^*(\lambda) = \max_{\alpha} \lambda \cdot g(\alpha)$.

The proof consists of a series of claims.

Claim 6. *Let* $\lambda \in \Lambda^6$*, and let* $i \in \boldsymbol{I}$ *be such that* $(\lambda_i^{\omega})_{\omega \in \Omega} \neq 0$. *Then*

(a) *there is* $\theta_i^* \in \Theta_i$ *such that* $\theta_i(\omega) = \theta_i^*$ *for all* ω *such that* $\lambda_i^{\omega} > 0$*; and*

(b) *for each* $j \neq i$*, there is* $\theta_j^* \in \Theta_j$ *that contains all* ω *such that* $\lambda_i^{\omega} \neq 0$.

Let $\lambda \in \Lambda^7$. *Then*

(c) *for each* $i \in \boldsymbol{I}$*, there is* $\theta_i^* \in \Theta_i$ *that contains all* ω *such that* $\lambda_j^{\omega} \neq 0$ *for some* $j \neq i$*; and*

(d) *this* θ_i^* *contains all* ω *such that* $\lambda_i^{\omega} > 0$.

Proof. For part (a), suppose not, so that there are $\omega' \in \Omega$ and $\omega'' \neq \omega$ such that $\theta_i(\omega') \neq \theta_i(\omega'')$, $\lambda_i^{\omega'} > 0$, and $\lambda_i^{\omega''} > 0$. Then $\lambda \notin \Lambda^6$, as for λ to be in Λ^6, $\theta_i(\omega') = \theta_i(\omega'')$ for all $\omega' \in \Omega$ and $\omega'' \neq \omega$ satisfying $\lambda_i^{\omega'} > 0$ and $\lambda_i^{\omega''} > 0$. A contradiction.

For part (b), suppose that there are $\omega' \in \Omega$ and $\omega'' \neq \omega'$ such that $\theta_j(\omega') \neq \theta_j(\omega'')$, $\lambda_i^{\omega'} \neq 0$, and $\lambda_i^{\omega''} \neq 0$. Then $\lambda \notin \Lambda^6$, as for λ to be in Λ^6, $\theta_j(\omega') = \theta_j(\omega'')$ for all $j \neq i$, $\omega' \in \Omega$, and $\omega'' \neq \omega$ satisfying $\lambda_i^{\omega'} \neq 0$ and $\lambda_i^{\omega''} \neq 0$. A contradiction.

For part (c), suppose that there are (j, ω') and (l, ω'') such that $j \neq i$, $l \neq i$, $\theta_i(\omega') \neq \theta_i(\omega'')$, $\lambda_j^{\omega'} \neq 0$, and $\lambda_l^{\omega''} \neq 0$. Then $\lambda \notin \Lambda^7$, as the last condition of the definition of Λ^7 requires that $\theta_i(\omega') = \theta_i(\omega'')$ for all $i \in \boldsymbol{I}$, $\omega' \in \Omega$, and $\omega'' \neq \omega'$ such that $\lambda_j^{\omega'} \neq 0$ for some $j \neq i$ and $\lambda_l^{\omega''} \neq 0$ for some $l \neq i$. A contradiction.

For part (d), suppose that there are $i \in \boldsymbol{I}$ and $\omega' \in \Omega$ such that $\lambda_i^{\omega'} > 0$ and $\omega \notin \theta_i^*$. Let (j, ω'') be such that $j \neq i$ and $\lambda_j^{\omega''} \neq 0$. Then from part (c), $\omega'' \in \theta_i^*$, so that $\theta_i(\omega'') = \theta_i^* \neq \theta_i(\omega')$. This implies that $\lambda \notin \Lambda^7$, as the last condition of the definition of Λ^7 requires that $\theta_i(\omega') = \theta_i(\omega'')$ for all $i \in \boldsymbol{I}$, $\omega' \in \Omega$, and $\omega'' \neq \omega'$ such that $\lambda_i^{\omega'} > 0$ and $\lambda_j^{\omega''} \neq 0$ for some $j \neq i$. A contradiction. *Q.E.D.*

Claim 7. *Suppose that monitoring structure is known, and let* $\lambda \in \Lambda^6 \bigcup \Lambda^7$. *Then for each* $\vec{\alpha} = ((\alpha_i^{\theta_i})_{\theta_i \in \Theta_i})_{i \in \boldsymbol{I}}$, $k^*(\vec{\alpha}, \lambda) \leq \lambda \cdot g(\alpha^{\theta^*})$ *where* θ^* *is chosen as in Claim 6 and* $\alpha^{\theta^*} = (\alpha_i^{\theta_i^*})_{i \in \boldsymbol{I}}$.

Proof. Choose (v, w) to satisfy constraints (i) through (iii) in the LP problem associated with $(\vec{\alpha}, \lambda, \delta)$ for some $\delta \in (0,1)$. It follows from constraint (ii) that

$$v_i^{\omega} \geq (1-\delta) g_i^{\omega}(a_i, \alpha_{-i}^{\theta_{-i}^*}) + \delta \pi(a_i, \alpha_{-i}^{\theta_{-i}^*}) \cdot w_i^{\omega}$$

for all $i \in \boldsymbol{I}$, $a_i \in A_i$, and $\omega \in \Omega$ such that $\lambda_i^{\omega} \neq 0$, since $\theta_j(\omega) = \theta_j^*$ for $j \neq i$ for such ω. In particular, we have

$$v_i^{\omega} \geq (1-\delta) g_i^{\omega}(\alpha^{\theta^*}) + \delta \pi(\alpha^{\theta^*}) \cdot w_i^{\omega} \tag{4}$$

for all $i \in \boldsymbol{I}$ and $\omega \in \Omega$ such that $\lambda_i^{\omega} \neq 0$. Also, from constraint (i), we obtain

$$v_i^{\omega} = (1-\delta) g_i^{\omega}(\alpha^{\theta^*}) + \delta \pi(\alpha^{\theta^*}) \cdot w_i^{\omega} \tag{5}$$

for all $i \in \boldsymbol{I}$ and $\omega \in \Omega$ such that $\lambda_i^{\omega} > 0$, since $\theta(\omega) = \theta^*$ for such $\omega \in \Omega$. It follows from (4) and (5) that

$$\begin{aligned}\lambda\cdot v \leq& \sum_{i\in \boldsymbol{I}}\sum_{\omega\in\Omega}\lambda_i^{\omega}\left[(1-\delta)g_i^{\omega}(\alpha^{\theta^*})+\delta\pi(\alpha^{\theta^*})\cdot w_i^{\omega}\right]\\ =&(1-\delta)\lambda\cdot g(\alpha^{\theta^*})+\delta\sum_{y\in Y}\pi_y(\alpha^{\theta^*})\lambda\cdot w(y).\end{aligned}$$

Using constraint (iii),

$$\begin{aligned}\lambda\cdot v \leq&(1-\delta)\lambda\cdot g(\alpha^{\theta^*})+\delta\sum_{y\in Y}\pi_y(\alpha^{\theta^*})\lambda\cdot v\\ =&(1-\delta)\lambda\cdot g(\alpha^{\theta^*})+\delta\lambda\cdot v.\end{aligned}$$

Subtracting $\delta\lambda\cdot v$ from both sides and dividing by $(1-\delta)$, we get $\lambda\cdot v\leq\lambda\cdot g(\alpha^{\theta^*})$. Therefore, $k^*(\vec{\alpha},\lambda,\delta)\leq g(\alpha^{\theta^*})$. *Q.E.D.*

Claim 8. *Suppose that monitoring structure is known and has strong full rank. Let $\lambda\in\Lambda^6$. Then $k^*(\lambda)\geq\max_{\alpha}\lambda\cdot g(\alpha)$.*

Proof. Let $\alpha\in\arg\max_{\alpha'}\lambda\cdot g(\alpha')$. Without loss of generality we can assume that α is a pure action profile, so that we denote it by a. In what follows, we show that $k^*(a,\lambda)\geq\lambda\cdot g(a)$.

Let $\lambda\in\Lambda^6$, and let (i,ω') be such that $\lambda_i^{\omega'}>0$. Consider the LP problem associated with (a,λ,δ). Note that we can ignore constraints (i) and (ii) for $j\neq i$, as $(\lambda_j^{\omega})_{\omega\in\Omega}=0$.

Let $v_i^{\omega}=g_i^{\omega}(a)$ for each $\omega\in\Omega$. For $\omega\neq\omega'$, let $(w_i^{\omega}(y))_{y\in Y}$ be such that

$$g_i^{\omega}(a)=(1-\delta)g_i^{\omega}(a_i',a_{-i})+\delta\pi(a_i',a_{-i})\cdot w_i^{\omega}(y) \tag{6}$$

for all $a_i'\in A_i$. Also, let

$$w_i^{\omega'}(y)=\frac{1}{\lambda_i^{\omega'}}\left(\lambda\cdot g(a)-\sum_{\omega\neq\omega'}\lambda_i^{\omega}w_i^{\omega}(y)\right) \tag{7}$$

for all $y\in Y$.

We claim that this (v,w) satisfies constraints (i) through (iii) in the LP problem. First, constraints (i) and (ii) hold for $\omega\neq\omega'$, since (6) holds. Also, as in the proof of Claim 5, we have

$$\begin{aligned}&(1-\delta)g_i^{\omega'}(a_i',a_{-i})+\delta\pi(a_i',a_{-i})\cdot w_i^{\omega'}(y)\\=&g_i^{\omega'}(a)+(1-\delta)\frac{\lambda\cdot g(a_i',a_{-i})-\lambda\cdot g(a)}{\lambda_i^{\omega'}}\\\leq&g_i^{\omega'}(a)\end{aligned}$$

for all $a_i'\in A_i$ with equality if $a_i'=a_i$. Here, the inequality is from the fact that a maximizes $\arg\max\lambda\cdot g(a')$ and $\lambda_i^{\omega'}>0$. This shows that constraints (i) and (ii) hold for ω'. Finally, constraint (iii) follows from (7). Thus we conclude $k^*(a,\lambda)\geq\sum_{\omega\in\Omega}\lambda_i^\omega v_i^\omega=\lambda\cdot g(a)$, as desired. *Q.E.D.*

Claim 9. *Suppose that monitoring structure is known and has strong full rank. Let $\lambda\in\Lambda^7$. Then for each α, $k^*(\alpha,\lambda)\geq\lambda\cdot g(\alpha)$.*

Proof. Let $\lambda\in\Lambda^7$, and given this λ, let $\lambda_{(i,\omega)(j,\omega')}$ be a direction such that the components for (i,ω) and (j,ω') are equal to those of λ and the remaining components are zero. (Thus the direction $\lambda_{(i,\omega)(j,\omega')}$ has at most two non-zero components.) In order to prove the claim, it suffices to show that α is enforceable with respect to the hyperplane orthogonal to λ at $g(\alpha)$. This enforceability follows from the following two facts: (i) If monitoring structure has strong full rank, then α is enforceable with respect to the hyperplane orthogonal to $\lambda_{(i,\omega)(j,\omega')}$ at $g(\alpha)$ for each (i,ω) and (j,ω') such that $i\neq j$ (but possibly $\omega=\omega'$), $\lambda_i^\omega\neq 0$, and $\lambda_j^{\omega'}\neq 0$. (ii) α is enforceable with respect to the hyperplane orthogonal to λ at $g(\alpha)$ if α is enforceable with respect to the hyperplane orthogonal to $\lambda_{(i,\omega)(j,\omega')}$ at $g(\alpha)$ for each (i,ω) and (j,ω') such that $i\neq j$, $\lambda_i^\omega\neq 0$, and $\lambda_j^{\omega'}\neq 0$. Note that (i) follows from Lemma 5.4 of FLM, since here we assume that monitoring structure does not depend on ω. Likewise, (ii) follows from Lemma 5.3 of FLM, since $\lambda\in\Lambda^7$ implies that for each (i,ω) such that $\lambda_i^\omega\neq 0$, there is (j,ω') such that $i\neq j$ and $\lambda_j^{\omega'}\neq 0$. *Q.E.D.*

Proof of Lemma 14. The statement follows from Claims 7, 8, and 9.

A.5 Proof of the Claims in Section 6.4

The first part of this section deals solely with the distinguishability conditions; the second part adds assumptions on the payoff structure of the game to compute bounds on the limit payoffs.

A.5.1 Distinguishability

To begin we restate the distinguishability conditions:

Definition 9. Profile $\vec{\alpha}$ *n-statewise distinguishes* (i,ω) from (j,ω') if there is a vector $\xi = (\xi(y))_{y\in Y} \in \boldsymbol{R}^{|Y|}$ such that

(i) $\xi \cdot \pi^{\omega}(\alpha^{\theta(\omega)}) > \xi \cdot \pi^{\omega'}(\alpha^{\theta(\omega')})$,

(ii) $\xi \cdot \pi^{\omega}(\alpha^{\theta(\omega)}) = \xi \cdot \pi^{\omega}(a_i, \alpha_{-i}^{\theta_{-i}(\omega)}) \geq \xi \cdot \pi^{\omega}(a_i', \alpha_{-i}^{\theta_{-i}(\omega)})$ for all $a_i \in \text{supp}\alpha_i^{\theta_i(\omega)}$ and $a_i' \in A_i$,

(iii) $\xi \cdot \pi^{\omega'}(\alpha^{\theta(\omega')}) = \xi \cdot \pi^{\omega'}(a_j, \alpha_{-j}^{\theta_{-j}(\omega')}) \geq \xi \cdot \pi^{\omega'}(a_j', \alpha_{-j}^{\theta_{-j}(\omega')})$ for all $a_j \in \text{supp}\alpha_j^{\theta_j(\omega')}$ and $a_j' \in A_j$.

Definition 10. A profile $\vec{\alpha}$ *p-statewise distinguishes* (i,ω) from (j,ω') if there is a vector $\xi = (\xi(y))_{y\in Y} \in \boldsymbol{R}^{|Y|}$ such that

(i) $\xi \cdot \pi^{\omega}(\alpha^{\theta(\omega)}) > \xi \cdot \pi^{\omega'}(\alpha^{\theta(\omega')})$,

(ii) $\xi \cdot \pi^{\omega}(\alpha^{\theta(\omega)}) = \xi \cdot \pi^{\omega}(a_i, \alpha_{-i}^{\theta_{-i}(\omega)}) \geq \xi \cdot \pi^{\omega}(a_i', \alpha_{-i}^{\theta_{-i}(\omega)})$ for all $a_i \in \text{supp}\alpha_i^{\theta_i(\omega)}$ and $a_i' \in A_i$,

(iii) $\xi \cdot \pi^{\omega'}(\alpha^{\theta(\omega')}) = \xi \cdot \pi^{\omega'}(a_j, \alpha_{-j}^{\theta_{-j}(\omega')}) \leq \xi \cdot \pi^{\omega'}(a_j', \alpha_{-j}^{\theta_{-j}(\omega')})$ for all $a_j \in \text{supp}\alpha_j^{\theta_j(\omega')}$ and $a_j' \in A_j$.

Claim 10.

(a) *The profile* (C_1, C_2) *p-statewise distinguishes* (i,ω) *from* (j,ω') *for* $((i,\omega),(j,\omega')) = ((1,\omega_1),(1,\omega_2))$, $((i,\omega),(j,\omega')) = ((1,\omega_2),(1,\omega_1))$, $((i,\omega),(j,\omega')) = ((2,\omega_1),(1,\omega_2))$, *or* $((i,\omega),(j,\omega')) = ((1,\omega_2),(2,\omega_1))$.

(b) *No type-independent profile p-statewise distinguishes* (i,ω) *from* (j,ω') *for* $((i,\omega),(j,\omega')) = ((2,\omega_1),(2,\omega_2))$, $((i,\omega),(j,\omega')) = ((2,\omega_2),(2,\omega_1))$, $((i,\omega),(j,\omega')) = ((1,\omega_1),(2,\omega_2))$, *or* $((i,\omega),(j,\omega')) = ((2,\omega_2),(1,\omega_1))$.

(c) *The profile* (C_1, C_2) *n-statewise distinguishes* (i,ω) *from* (j,ω') *for* $((i,\omega),(j,\omega')) = ((1,\omega_1),(1,\omega_2))$, $((i,\omega),(j,\omega')) = ((1,\omega_2),(1,\omega_1))$, $((i,\omega),(j,\omega')) = ((2,\omega_1),(2,\omega_2))$, $((i,\omega),(j,\omega')) = ((1,\omega_1),(2,\omega_2))$, *or* $((i,\omega),(j,\omega')) = ((2,\omega_1),(1,\omega_2))$.

(d) *No type-independent profile n-statewise distinguishes* (i,ω) *from* (j,ω') *for* $((i,\omega),(j,\omega')) = ((2,\omega_2),(2,\omega_1))$*,* $((i,\omega),(j,\omega')) = ((2,\omega_2),(1,\omega_1))$*, or* $((i,\omega),(j,\omega')) = ((1,\omega_2),(2,\omega_1))$*.*

Proof. Part (a) follows from Lemmas 6(b) and 7(a). Part (b) follows from Lemma 15 of Fudenberg and Yamamoto (2009). In the terminology of that paper, all type-independent profiles "entangle" ω_1 and ω_2 for player 2, so that no type-independent profile p-statewise distinguishes (i,ω) from (j,ω') if $((i,\omega),(j,\omega')) = ((2,\omega_1),(2,\omega_2))$, or $((i,\omega),(j,\omega')) = ((1,\omega_1),(2,\omega_2))$. Then Lemma 3 applies.

Part (c). For notational convenience, given a $\xi = (\xi(y))_{y\in Y}$, let $\xi \cdot p = p_H\xi(H) + p_M\xi(M) + (1-p_H-p_M)\xi(L)$ and $\xi\cdot q = q_H\xi(H)+q_M\xi(M)+(1-q_H-q_M)\xi(L)$. Let $\kappa > 0$. Because p and q are linearly independent, there is a ξ such that $\xi \cdot p = 0$ and $\xi \cdot q = -\kappa$. We claim that this ξ satisfies all the conditions of n-statewise distinguishability for $((i,\omega),(j,\omega')) = ((1,\omega_2),(1,\omega_1))$: Condition (i) holds since $\xi\cdot\pi^{\omega_2}(C_1,C_2) - \xi\cdot\pi^{\omega_1}(C_1,C_2) = -(1-\beta)\xi\cdot q > 0$; condition (ii) follows from $\xi\cdot\pi^{\omega_2}(C_1,C_2) - \xi\cdot\pi^{\omega_2}(D_1,C_2) = \xi\cdot p = 0$, and (iii) holds from $\xi\cdot\pi^{\omega_1}(C_1,C_2) - \xi\cdot\pi^{\omega_1}(D_1,C_2) = \xi\cdot p = 0$. For the remaining cases, use $\xi' = -\xi$; this gets the sign right in condition (i) and as above conditions (ii) and (iii) hold with equality because $\xi\cdot p = 0$.

Part (d). If profile α n-statewise distinguishes (i,ω) from (j,ω'), clause (i) of the definition implies α_2 chooses C_2 with positive probability. (Otherwise $\pi^{\omega_1}(\alpha) = \pi^{\omega_2}(\alpha)$ so that we have $\xi\cdot\pi^{\omega_1}(\alpha) = \xi\cdot\pi^{\omega_2}(\alpha)$ for any ξ.) Also, since we consider $((i,\omega),(j,\omega')) = ((2,\omega_2),(2,\omega_1))$, $((i,\omega),(j,\omega')) = ((2,\omega_2),(1,\omega_1))$, or $((i,\omega),(j,\omega')) = ((1,\omega_2),(2,\omega_1))$, we have $\xi\cdot\pi^{\omega}(\alpha) - \xi\cdot\pi^{\omega'}(\alpha) = -\gamma\xi\cdot q$ for some $\gamma > 0$, and hence clause (i) requires that the corresponding vector ξ satisfy $\xi\cdot q < 0$. But this implies that $\xi\cdot\pi^{\omega}(\alpha) < \xi\cdot\pi^{\omega}(\alpha_1,D_2)$ for each $\omega\in\Omega$, so that clauses (ii) and (iii) could not hold. *Q.E.D.*

Claim 11. *Suppose that player* 1 *knows the state but player* 2 *does not. Then,*

(a) *The type-contingent profile* $((C_1,D_1),C_2)$ *p-statewise distinguishes* (i,ω) *from* (j,ω') *for* $((i,\omega),(j,\omega')) = ((2,\omega_1),(2,\omega_2))$*,* $((i,\omega),(j,\omega')) = ((2,\omega_2),(2,\omega_1))$*,* $((i,\omega),(j,\omega')) = ((1,\omega_1),(2,\omega_2))$*, or* $((i,\omega),(j,\omega')) = ((2,\omega_2),(1,\omega_1))$*.*

(b) *The type-contingent profile* $((C_1,D_1),C_2)$ *n-statewise distinguishes* (i,ω) *from* (j,ω') *for* $((i,\omega),(j,\omega')) = ((2,\omega_2),(2,\omega_1))$ *or* $((i,\omega),(j,\omega')) = ((1,\omega_2),(2,\omega_1))$*.*

(c) *No type-contingent profile $\vec{\alpha}$ n-statewise distinguishes (i,ω) from (j,ω') for $((i,\omega),(j,\omega')) = ((2,\omega_2),(1,\omega_1))$.*

Proof. Part (a). Let $\kappa > 0$. For $((i,\omega),(j,\omega')) = ((2,\omega_1),(2,\omega_2))$ or $((i,\omega),(j,\omega')) = ((1,\omega_1),(2,\omega_2))$, let ξ be such that $\xi \cdot p = \kappa$ and $\xi \cdot q = 0$. Then this ξ satisfies all the conditions for p-statewise distinguishability. Indeed, conditions (i) through (iii) hold for $((i,\omega),(j,\omega')) = ((2,\omega_1),(2,\omega_2))$, since $\xi \cdot \pi^{\omega_1}(C_1,C_2) - \xi \cdot \pi^{\omega_2}(D_1,C_2) = \xi \cdot p + (1-\beta)\xi \cdot q > 0$, $\xi \cdot \pi^{\omega_1}(C_1,C_2) - \xi \cdot \pi^{\omega_1}(C_1,D_2) = \xi \cdot q = 0$, and $\xi \cdot \pi^{\omega_2}(D_1,C_2) - \xi \cdot \pi^{\omega_2}(D_1,D_2) = \beta\xi \cdot q = 0$. Likewise, conditions (i) through (iii) hold for $((i,\omega),(j,\omega')) = ((1,\omega_1),(2,\omega_2))$, since $\xi \cdot \pi^{\omega_1}(C_1,C_2) - \xi \cdot \pi^{\omega_2}(D_1,C_2) = \xi \cdot p + (1-\beta)\xi \cdot q > 0$, $\xi \cdot \pi^{\omega_1}(C_1,C_2) - \xi \cdot \pi^{\omega_1}(D_1,C_2) = \xi \cdot p > 0$, and $\xi \cdot \pi^{\omega_2}(D_1,C_2) - \xi \cdot \pi^{\omega_2}(D_1,D_2) = \beta\xi \cdot q = 0$.

For $((i,\omega),(j,\omega')) = ((2,\omega_2),(2,\omega_1))$ or $((i,\omega),(j,\omega')) = ((2,\omega_2),(1,\omega_1))$, use ξ such that $\xi \cdot p = -\kappa$ and $\xi \cdot q = 0$.

Part (b). Let $\kappa > 0$, and let ξ be such that $\xi \cdot p = -\kappa$ and $\xi \cdot q = 0$. Then this *xi* satisfies all the conditions for n-statewise distinguishability. Indeed, for $((i,\omega),(j,\omega')) = ((2,\omega_2),(2,\omega_1))$, we have $\xi \cdot \pi^{\omega_2}(D_1,C_2) - \xi \cdot \pi^{\omega_1}(C_1,C_2) = -\xi \cdot p - (1-\beta)\xi \cdot q > 0$, $\xi \cdot \pi^{\omega_2}(D_1,C_2) - \xi \cdot \pi^{\omega_2}(D_1,D_2) = \beta\xi \cdot q = 0$, and $\xi \cdot \pi^{\omega_1}(C_1,C_2) - \xi \cdot \pi^{\omega_1}(C_1,D_2) = \xi \cdot q = 0$, so that conditions (i) through (iii) for n-statewise distinguishability holds. Likewise, for $((i,\omega),(j,\omega')) = ((1,\omega_2),(2,\omega_1))$, we have $\xi \cdot \pi^{\omega_2}(D_1,C_2) - \xi \cdot \pi^{\omega_1}(C_1,C_2) = -\xi \cdot p - (1-\beta)\xi \cdot q > 0$, $\xi \cdot \pi^{\omega_2}(D_1,C_2) - \xi \cdot \pi^{\omega_2}(C_1,C_2) = -\xi \cdot p > 0$, and $\xi \cdot \pi^{\omega_1}(C_1,C_2) - \xi \cdot \pi^{\omega_1}(C_1,D_2) = \xi \cdot q = 0$, and hence conditions (i) through (iii) hold.

Part (c). Let $\vec{\alpha}$ be such that player 1 chooses α_1 for state ω_1 and α_1' for state ω_2, while player 2 chooses α_2 for both states. Suppose that $\vec{\alpha}$ n-statewise distinguishes (i,ω) from (j,ω'). From clause (i) of the definition of n-statewise distinguishability, we have $(\alpha_1'(C_1) - \alpha_1(C_1))\xi \cdot p - \alpha_2(C_2)(1-\beta)\xi \cdot q > 0$. Also, clause (ii) implies that $\xi \cdot q \geq 0$ for $\alpha_2(C_2) > 0$, and hence $\alpha_2(C_2)\xi \cdot q \geq 0$. Taken together, it must be that $(\alpha_1'(C_1) - \alpha_1(C_1))\xi \cdot p > 0$. However, clause (iii) requires that $\alpha_1(C_1) = 1$ if $\xi \cdot p > 0$, and $\alpha_1(C_1) = 0$ if $\xi \cdot p < 0$; this implies that $(\alpha_1'(C_1) - \alpha_1(C_1))\xi \cdot p \leq 0$, a contradiction. *Q.E.D.*

Claim 12. *Suppose that player 2 knows the state but player 1 does not. Then*

(a) *The type-contingent profile $(C_1,(C_2,D_2))$ p-statewise distinguishes (i,ω)*

from (j,ω') *for* $((i,\omega),(j,\omega')) = ((2,\omega_1),(2,\omega_2))$*,* $((i,\omega),(j,\omega')) = ((2,\omega_2),(2,\omega_1))$*,* $((i,\omega),(j,\omega')) = ((1,\omega_1),(2,\omega_2))$*, or* $((i,\omega),(j,\omega')) = ((2,\omega_2),(1,\omega_1))$*.*

(b) *The type-contingent profile* $(C_1,(C_2,D_2))$ *n-statewise distinguishes* (i,ω) *from* (j,ω') *for* $((i,\omega),(j,\omega')) = ((2,\omega_2),(1,\omega_1))$*.*

(c) *No type-contingent profile* $\vec{\alpha}$ *n-statewise distinguishes* (i,ω) *from* (j,ω') *for* $((i,\omega),(j,\omega')) = ((2,\omega_2),(2,\omega_1))$*, or* $((i,\omega),(j,\omega')) = ((1,\omega_2),(2,\omega_1))$*.*

Proof. Part (a) follows from Lemmas 6(a) and 7(b).

Part (b). Let $\kappa > 0$, and let ξ be such that $\xi \cdot p = 0$ and $\xi \cdot q = -\kappa$. Then $\xi \cdot \pi^{\omega_2}(C_1,D_2) - \xi \cdot \pi^{\omega_1}(C_1,C_2) = -\xi \cdot q > 0$, $\xi \cdot \pi^{\omega_2}(C_1,D_2) - \xi \cdot \pi^{\omega_2}(C_1,C_2) = -\beta\xi \cdot q > 0$, and $\xi \cdot \pi^{\omega_1}(C_1,C_2) - \xi \cdot \pi^{\omega_1}(D_1,C_2) = \xi \cdot p = 0$, so that conditions (i) through (iii) for *n*-statewise distinguishability hold.

Part (c). Consider $((i,\omega),(j,\omega')) = ((2,\omega_2),(2,\omega_1))$, and let $\vec{\alpha}$ be such that player 2 chooses α_2 for state ω_1 and α_2' for state ω_2, while player 1 chooses α_1 for both states. Suppose that $\vec{\alpha}$ *n*-statewise distinguishes (i,ω) from (j,ω'). Then, from clause (i) of the definition, we have $(\beta\alpha_2'(C_2) - \alpha_2(C_2))\xi \cdot q > 0$. If $\xi \cdot q > 0$, then clause (iii) implies that $\alpha_2(C_2) = 1$, and hence $(\beta\alpha_2'(C_2) - \alpha_2(C_2))\xi \cdot q < 0$. Therefore $\xi \cdot q < 0$. But then clauses (iii) implies that $\alpha_2(C_2) = 0$, and thus $(\beta\alpha_2'(C_2) - \alpha_2(C_2))\xi \cdot q \leq 0$, a contradiction.

A similar argument shows that no profile *n*-statewise distinguishes $(1,\omega_2),(2,\omega_1))$.

Q.E.D.

A.5.2 Limit Equilibrium Payoffs

In what follows, we assume that the payoffs are

$$u_i(C_i,y) = r_i(y) - e_i \quad \text{and} \quad u_i(D_i,y) = r_i(y)$$

for each $i \in \boldsymbol{I}$ and $y \in Y$, where the function r_i satisfies $r_i(H) > r_i(M) > r_i(L)$; $e_1 > p_H(r_1(H) - r_1(L)) + p_M(r_1(M) - r_1(L))$; $e_2 > q_H(r_2(H) - r_2(L)) + q_M(r_2(M) - r_2(L))$; $e_1 < p_H(r_1(H) + r_2(H) - r_1(L) - r_2(L)) + p_M(r_1(M) + r_2(M) - r_1(L) - r_2(L))$; and $e_2 < q_H(r_1(H) + r_2(H) - r_1(L) - r_2(L)) + q_M(r_1(M) + r_2(M) - r_1(L) - r_2(L))$. This implies that the stage game payoffs in each state correspond to a prisoner's dilemma when β exceeds some critical level $\overline{\beta} < 1$. (That is, for any

$\beta \in (\overline{\beta}, 1)$, in each state, D_i is a strictly dominant strategy for each player i, and the sum of the payoffs is maximized at the profile (C_1, C_2).)

Proposition 12. *Suppose that player* 1 *knows the state. Then* $Q = V^*$ *for any* $\beta \in (\overline{\beta}, 1)$.

Proof. Notice that p-statewise distinguishability holds for all $((i,\omega),(j,\omega'))$, and n-statewise distinguishability holds for all $((i,\omega),(j,\omega')) \neq ((2,\omega_2),(1,\omega_1))$. Therefore, it suffices to show that $V^* \subseteq H^*(\lambda)$ for all $\lambda \in \Lambda^*$, where Λ^* is the set of all λ such that $\lambda_2^{\omega_2} > 0$, $\lambda_1^{\omega_1} < 0$, $\lambda_1^{\omega_2} \leq 0$, and $\lambda_2^{\omega_1} = 0$. (For the other cross-state directions, we have $k^*(\lambda) = \infty$ thanks to statewise distinguishability.)

First we focus on λ such that $\lambda_2^{\omega_2} > 0$, $\lambda_1^{\omega_1} < 0$, and $\lambda_1^{\omega_2} = \lambda_2^{\omega_1} = 0$. Consider the LP problem associated with such a direction λ and the type-independent profile (C_1, D_2). Since we can ignore constraints (i) and (ii) for $(l, \omega'') \neq (1,\omega_1),(2,\omega_2)$, the maximal score $k^*((C_1,D_2),\lambda)$ is defined to be a solution to

$$\begin{aligned}
&\max_{v,w} \lambda_1^{\omega_1} v_1^{\omega_1} + \lambda_2^{\omega_2} v_2^{\omega_2} \\
&\text{s.t. } v_1^{\omega_1} = (1-\delta) g_1^{\omega_1}(C_1,D_2) + \delta \pi^{\omega_1}(C_1,D_2) \cdot w_1^{\omega_1}, \\
&\quad v_2^{\omega_2} = (1-\delta) g_2^{\omega_2}(C_1,D_2) + \delta \pi^{\omega_2}(C_1,D_2) \cdot w_2^{\omega_2}, \\
&\quad v_1^{\omega_1} \geq (1-\delta) g_1^{\omega_1}(D_1,D_2) + \delta \pi^{\omega_1}(D_1,D_2) \cdot w_1^{\omega_1}, \\
&\quad v_2^{\omega_2} \geq (1-\delta) g_2^{\omega_2}(C_1,C_2) + \delta \pi^{\omega_2}(C_1,C_2) \cdot w_2^{\omega_2}, \\
&\quad \lambda_1^{\omega_1} v_1^{\omega_1} + \lambda_2^{\omega_2} v_2^{\omega_2} \geq \lambda_1^{\omega_1} w_1^{\omega_1}(y) + \lambda_2^{\omega_2} w_2^{\omega_2}(y) \qquad \text{for all } y \in Y.
\end{aligned}$$

Let $v_1^{\omega_1} = g_1^{\omega_1}(C_1,D_2)$ and $v_2^{\omega_2} = g_2^{\omega_2}(C_1,D_2)$. Also, let w be such that

$$\begin{aligned}
g_1^{\omega_1}(C_1,D_2) &= (1-\delta) g_1^{\omega_1}(C_1,D_2) + \delta \pi^{\omega_1}(C_1,D_2) \cdot w_1^{\omega_1}, \\
g_2^{\omega_2}(C_1,D_2) &= (1-\delta) g_2^{\omega_2}(C_1,D_2) + \delta \pi^{\omega_2}(C_1,D_2) \cdot w_2^{\omega_2}, \\
g_1^{\omega_1}(C_1,D_2) &= (1-\delta) g_1^{\omega_1}(D_1,D_2) + \delta \pi^{\omega_1}(D_1,D_2) \cdot w_1^{\omega_1}, \\
g_2^{\omega_2}(C_1,D_2) &= (1-\delta) g_2^{\omega_2}(C_1,C_2) + \delta \pi^{\omega_2}(C_1,C_2) \cdot w_2^{\omega_2}, \\
\lambda_1^{\omega_1} g_1^{\omega_1}(C_1,D_2) + \lambda_2^{\omega_2} g_2^{\omega_2}(C_1,D_2) &= \lambda_1^{\omega_1} w_1^{\omega_1}(y) + \lambda_2^{\omega_2} w_2^{\omega_2}(y) \qquad \text{for all } y \in Y.
\end{aligned}$$

To see that there exists such a w, note that the second equation is automatically satisfied if the first and last equations hold, as $\pi^{\omega_1}(C_1,D_2) = \pi^{\omega_2}(C_1,D_2)$. Eliminate $w_2^{\omega_2}(y)$ using the last equation. Then there remain three linearly independent equations and three unknowns, so that we can solve the system of equations.[12] Obviously this (v,w) satisfies all the constraints of the above LP problem, and hence $k^*((C_1,D_2),\lambda) \geq \lambda \cdot v = \lambda \cdot g(C_1,D_2)$. Since the stage game corresponds to a prisoner's dilemma for both states, $\lambda \cdot g(C_1,D_2) = \max_{v' \in V} \lambda \cdot v'$, and thus $k^*((C_1,D_2),\lambda) \geq \max_{v' \in V} \lambda \cdot v'$. This shows that $V^* \subseteq H^*(\lambda)$.

For $\lambda \in \Lambda^*$ such that $\lambda_2^{\omega_2} > 0$, $\lambda_1^{\omega_1} < 0$, $\lambda_1^{\omega_2} < 0$, and $\lambda_2^{\omega_1} = 0$, we can show that $k^*((C_1,D_2),\lambda) \geq \lambda \cdot g(C_1,D_2)$ in a similar way, and hence $V^* \subseteq H^*(\lambda)$. *Q.E.D.*

Proposition 13. *If only player 2 knows the state (i.e., $\Theta_1 = \{\Omega\}$ and $\Theta_2 = \{(\omega_1),(\omega_2)\}$), the folk theorem fails, because the maximal score for direction $\lambda' = ((0,-1),(0,1))$ is less than $\max_{v \in V^*} \lambda' \cdot v$.*

The proof of this result relies on the following claims, which are verified in the supplementary files:

Claim 13.

(a) *Let $\vec{\alpha}$ denote the type-independent profile (a_1,D_2) for any $a_1 \in A_1$. Then, $k^*(\vec{\alpha},\lambda') \leq \lambda' \cdot g(a_1,D_2)$.*

(b) *Let $\vec{\alpha}$ denote the type-independent profile (a_1,C_2) for any $a_1 \in A_1$. Then $k^*(\vec{\alpha},\lambda') \leq \lambda' \cdot g(a_1,C_2) - \frac{1-\beta}{\beta}(g_2^{\omega_2}(a_1,D_2) - g_2^{\omega_2}(a_1,C_2))$.*

Claim 14. *Suppose that $\Theta_2 = \{(\omega_1),(\omega_2)\}$.*

(a) *Let $\vec{\alpha}$ denote the type-contingent profile such that player 1 chooses $a_1 \in A_1$ and player 2 chooses C_2 for state ω_1 while D_2 for state ω_2. Then $k^*(\vec{\alpha},\lambda') \leq \lambda' \cdot g(\vec{\alpha}) - (g_2^{\omega_1}(a_1,D_2) - g_2^{\omega_1}(a_1,C_2))$.*

(b) *Let $\vec{\alpha}$ denote the type-contingent profile such that player 1 chooses $a_1 \in A_1$ and player 2 chooses D_2 for state ω_1 while C_2 for state ω_2. Then $k^*(\vec{\alpha},\lambda') \leq \lambda' \cdot g(a_1,C_2) - \frac{1-\beta}{\beta}(g_2^{\omega_2}(a_1,D_2) - g_2^{\omega_2}(a_1,C_2))$.*

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[12] Recall that $\pi^{\omega_2}(C_1,C_2) = \pi^{\omega_1}(D_1,D_2) + p + \beta q$ and $\pi^{\omega_1}(C_1,D_2) = \pi^{\omega_1}(D_1,D_2) + p$ where $p = (p_H, p_M, -p_H - p_M)$ and $q = (q_H, q_M, -q_H - q_M)$. Since $(1,1,1) \cdot p = 0$, $(1,1,1) \cdot q = 0$, $(1,1,1) \cdot \pi^{\omega_1}(D_1,D_2) = 1$, and p and q are linearly independent, the vectors $\pi^{\omega_1}(D_1,D_2)$, p, and q are linearly independent. This implies that $\pi^{\omega_2}(C_1,C_2)$, $\pi^{\omega_1}(D_1,D_2)$, and $\pi^{\omega_1}(C_1,D_2)$ are linearly independent.

Now we can complete the proof of Proposition 13: In order to have $Q = V^*$, we need $k^*(\lambda') \geq \max_{v \in V^*} \lambda' \cdot v > g_2^{\omega_2}(C_1, C_2) - g_2^{\omega_1}(D_1, D_2)$. However, the above claims imply that $k^*(\lambda') < g_2^{\omega_2}(C_1, C_2) - g_2^{\omega_1}(D_1, D_2)$, since

$$\begin{aligned} g_2^{\omega_2}(C_1,C_2) - g_2^{\omega_1}(D_1,D_2) &> g_2^{\omega_2}(D_1,D_2) - g_2^{\omega_1}(D_1,D_2) \\ &= g_2^{\omega_2}(a_1,D_2) - g_2^{\omega_1}(a_1,D_2) \quad \forall a_1 \in A_1, \end{aligned}$$

$$\begin{aligned} g_2^{\omega_2}(C_1,C_2) - g_2^{\omega_1}(D_1,D_2) &> g_2^{\omega_2}(C_1,C_2) - g_2^{\omega_1}(C_1,C_2) \\ &= g_2^{\omega_2}(a_1,C_2) - g_2^{\omega_1}(a_1,C_2) \quad \forall a_1 \in A_1, \end{aligned}$$

and

$$\begin{aligned} g_2^{\omega_2}(C_1,C_2) - g_2^{\omega_1}(D_1,D_2) &> g_2^{\omega_2}(C_1,C_2) - g_2^{\omega_1}(C_1,D_2) \\ &= g_2^{\omega_2}(a_1,C_2) - g_2^{\omega_1}(a_1,D_2) \quad \forall a_1 \in A_1. \end{aligned}$$

References

Abreu, D., D. Pearce, and E. Stacchetti (1986): "Optimal Cartel Equilibria with Imperfect Monitoring," *Journal of Economic Theory* 39, 251-269.

Abreu, D., D. Pearce, and E. Stacchetti (1990): "Toward a Theory of Discounted Repeated Games with Imperfect Monitoring," *Econometrica* 58, 1041-1063.

Athey, S., and K. Bagwell (2001): "Optimal Collusion with Private Information," *RAND Journal of Economics* 32, 428-465.

Aumann, R., and M. Maschler (1995): *Repeated Games with Incomplete Information*. MIT Press, Cambridge, MA. With the collaboration of R.E. Stearns.

Bhaskar, V., G.J. Mailath, and S. Morris (2008): "Purification in the Infinitely Repeated Prisoner's Dilemma," *Review of Economic Dynamics* 11, 515-528.

Bergemann, D., and S. Morris (2007): "Belief-Free Incomplete Information Games," mimeo.

Compte, O. (1998): "Communication in repeated games with imperfect private monitoring," *Econometrica* 66, 597-626.

Cripps, M., and J. Thomas (2003): "Some Asymptotic Results in Discounted Repeated Games of One-Side Incomplete Information," *Mathematics of Operations Research* 28, 433-462.

Deb, J. (2008): "Cooperation and Community Responsibility: A Folk Theorem for Repeated Random Matching Games," mimeo.

Dekel, E., D. Fudenberg, and D.K. Levine (2004): "Learning to Play Bayesian Games," *Games and Economic Behavior* 46, 282-303.

Ely, J., J. Hörner, and W. Olszewski (2003): "Belief-Free Equilibria in Repeated Games," *Econometrica* 73, 377-415.

Ely, J., and J. Välimäki (2002): "A Robust Folk Theorem for the Prisoner's Dilemma," *Journal of Economic Theory* 102, 84-105.

Ellison, G. (1993): "Cooperation in the Prisoner's Dilemma with Anonymous Random Matching," *Review of Economic Studies* 61, 567-588.

Forges, F. (1984): "Note on Nash Equilibria in Infinitely Repeated Games with Incomplete Information," *International Journal of Game Theory* 13, 179-187.

Fudenberg, D., D, Kreps, and E. Maskin (1990) "Repeated Games with Long-run and Short-run Players," *Review of Economic Studies* 57, 555-573.

Fudenberg, D., and D.K. Levine (1983): "Subgame-Perfect Equilibria of Finite and Infinite Horizon Games," *Journal of Economic Theory* 31, 251-268

Fudenberg, D., and D.K. Levine (1994): "Efficiency and Observability in Games with Long-Run and Short-Run Players," *Journal of Economic Theory* 62, 103-135.

Fudenberg, D., D.K. Levine, and E. Maskin (1994): "The Folk Theorem with imperfect public information," *Econometrica* 62, 997-1040.

Fudenberg, D., D.K. Levine, and S. Takahashi (2007): "Perfect Public Equilibrium when Players are Patient," *Games and Economic Behavior* 61, 27-49.

Fudenberg, D., and E. Maskin (1986): "The Folk Theorem in Repeated Games with Discounting and with Incomplete Information," *Econometrica* 54, 533-554.

Fudenberg, D., and Y. Yamamoto (2009): "Perfect Public Ex-Post Equilibria of Repeated Games with Uncertain Outcomes," mimeo.

Gossner, O., and N. Vieille (2003): "Strategic Learning in Games with Symmetric Information," *Games and Economic Behavior* 42, 25-47.

Hart, S. (1985): "Nonzero-Sum Two-Person Repeated Games with Incomplete Information," *Mathematics of Operations Research* 10, 117-153.

Hörner, J., and S. Lovo (2009): “Belief-Free Equilibria in Games with Incomplete Information,” *Econometrica* 77, 453-487.

Hörner, J., S. Lovo, and T. Tomala (2009): “Belief-Free Equilibria in Games with Incomplete Information: the N-Player Case,” mimeo.

Kandori, M. (1992a): “Social Norms and Community Enforcement,” *Review of Economic Studies* 59, 63-80.

Kandori, M. (1992b): “Repeated Games Played by Overlapping Generations of Players,” *Review of Economic Studies* 59, 81-92.

Kandori, M. (2003): “Randomization, Communication, and Efficiency in Repeated Games with Imperfect Public Monitoring,” *Econometrica* 71, 345-353.

Kandori, M. (2008): “Weakly Belief-Free Equilibria in Repeated Games with Private Monitoring,” mimeo.

Kandori, M. and H. Matsushima (1998): “Private Observation, Communication, and Collusion,” *Econometrica* 66, 627-652.

Miller, D. (2007): “The Dynamic Cost of Ex Post Incentive Compatibility in Repeated Games of Private Information,” mimeo.

Piccione, M. (2002): “The Repeated Prisoner’s Dilemma with Imperfect Private Monitoring,” *Journal of Economic Theory* 102, 70-83.

Radner, R., R. Myerson, and E. Maskin (1986): “An Example of a Repeated Partnership Game with Discounting and with Uniformly Inefficient Equilibria,” *Review of Economic Studies* 53, 59-70.

Sorin, S. (1984): “Big Match with Lack of Information on One Side (Part I),” *International Journal of Game Theory* 13, 201-255.

Sorin, S. (1985): “Big Match with Lack of Information on One Side (Part II),” *International Journal of Game Theory* 14, 173-204.

Takahashi, S. (2008): “Community Enforcement when Players Observe Partners’ Past Play,” mimeo.

Wiseman, T. (2005): “A Partial Folk Theorem for Games with Unknown Payoff Distributions,” *Econometrica* 73, 629-645.

Wiseman, T. (2008): “A Partial Folk Theorem for Games with Private Learning,” mimeo.

Yamamoto, Y. (2007): "Efficiency Results in N Player Games with Imperfect Private Monitoring," *Journal of Economic Theory* 135, 382-413.

Yamamoto, Y. (2009): "A Limit Characterization of Belief-Free Equilibrium Payoffs in Repeated Games," *Journal of Economic Theory* 144, 802-824.

Supplementary Materials

S.1 Proof of Lemma A5

Lemma A5. *Suppose that a profile $\vec{\alpha}$ is ex-post enforceable and has statewise full rank for (i,ω) and (j,ω') satisfying $\omega \neq \omega'$. Then, $k^*(\vec{\alpha},\lambda) = \infty$ for direction λ such that $\lambda_i^\omega \neq 0$ and $\lambda_j^{\omega'} \neq 0$.*

Let (i,ω) and (j,ω') be such that $\lambda_i^\omega \neq 0$, $\lambda_j^{\omega'} \neq 0$, and $\omega' \neq \omega$. Let $\vec{\alpha}$ be a profile that has statewise full rank for all (i,ω) and (j,ω').

First, we claim that for every $K > 0$, there exist $z_i^\omega = (z_i^\omega(y))_{y\in Y}$ and $z_j^{\omega'} = (z_j^{\omega'}(y))_{y\in Y}$ such that

$$\pi^\omega(a_i, \alpha_{-i}^{\theta_{-i}(\omega)}) \cdot z_i^\omega = \frac{K}{\delta\lambda_i^\omega} \tag{8}$$

for all $a_i \in A_i$,

$$\pi^{\omega'}(a_j, \alpha_{-j}^{\theta_{-j}(\omega')}) \cdot z_j^{\omega'} = 0 \tag{9}$$

for all $a_j \in A_j$, and

$$\lambda_i^\omega z_i^\omega(y) + \lambda_j^{\omega'} z_j^{\omega'}(y) = 0 \tag{10}$$

for all $y \in Y$. To prove that this system of equations indeed has a solution, eliminate (10) by solving for $z_j^{\omega'}(y)$. Then, there remain $|A_i| + |A_j|$ linear equations, and its coefficient matrix is $\Pi_{(i,\omega)(j,\omega')}(\vec{\alpha})$. Since statewise full rank implies that this coefficient matrix has rank $|A_i| + |A_j|$, we can solve the system.

By assumption, there is $\tilde{w} : Y \to \boldsymbol{R}^{I\times|\Omega|}$ that enforces $\vec{\alpha}$. Let $\tilde{v} = (\tilde{v}_i^\omega)_{(i,\omega)}$ be the average payoff when players play $\vec{\alpha}$ today and receive the continuation payoff $\tilde{w}$. Let $K > \max_{y\in Y} \lambda \cdot \tilde{w}(y)$, and choose $(z_i^\omega(y))_{y\in Y}$ and $(z_j^{\omega'}(y))_{y\in Y}$ to satisfy (8) through (10). Then, let

$$w_l^{\omega''}(y) = \begin{cases} \tilde{w}_i^\omega(y) + z_i^\omega(y) & \text{if} \quad (l,\omega'') = (i,\omega) \\ \tilde{w}_j^{\omega'}(y) + z_j^{\omega'}(y) & \text{if} \quad (l,\omega'') = (j,\omega') \\ \tilde{w}_l^{\omega''}(y) & \text{otherwise} \end{cases}$$

for each $y \in Y$. Also, let

$$v_l^{\omega''} = \begin{cases} \tilde{v}_i^\omega + \dfrac{K}{\lambda_i^\omega} & \text{if} \quad (l,\omega'') = (i,\omega) \\ \\ \tilde{v}_l^{\omega''} & \text{otherwise} \end{cases}.$$

We claim that this (v,w) satisfies constraints (i) through (iii) in LP-Average. Since $\tilde{w}$ ex-post enforces $\vec{\alpha}$, constraints (i) and (ii) are satisfied for all $(l,\omega'') \in (\boldsymbol{I}\times\Omega)\setminus\{(i,\omega),(j,\omega')\}$. Also, we obtain

$$\begin{aligned}
&(1-\delta)g_i^\omega(a_i,\alpha_{-i}^{\theta_{-i}(\omega)})+\delta\pi^\omega(a_i,\alpha_{-i}^{\theta_{-i}(\omega)})\cdot w_i^\omega \\
=&(1-\delta)g_i^\omega(a_i,\alpha_{-i}^{\theta_{-i}(\omega)})+\delta\pi^\omega(a_i,\alpha_{-i}^{\theta_{-i}(\omega)})\cdot(\tilde{w}_i^\omega+z_i^\omega) \\
=&(1-\delta)g_i^\omega(a_i,\alpha_{-i}^{\theta_{-i}(\omega)})+\delta\pi^\omega(a_i,\alpha_{-i}^{\theta_{-i}(\omega)})\cdot\tilde{w}_i^\omega+\frac{K}{\lambda_i^\omega} \\
\leq&\tilde{v}_i^\omega+\frac{K}{\lambda_i^\omega}=v_i
\end{aligned}$$

for all $a_i \in A_i$ with equality if $a_i \in \mathrm{supp}\alpha_i^{\theta_i(\omega)}$. Here, the second equality follows from (8) and the inequality comes from the fact that $\tilde{w}$ enforces $\vec{\alpha}$. This shows that (v,w) satisfies constraints (i) and (ii) for (i,ω). Likewise, using (9), we can show that (v,w) satisfies constraints (i) and (ii) for (j,ω'). Furthermore, from (10) and $K>\max_{y\in Y}\lambda\cdot\tilde{w}(y)$,

$$\begin{aligned}
\lambda\cdot w(y)&=\lambda\cdot\tilde{w}(y)+\lambda_i^\omega z_i^\omega(y)+\lambda_j^{\omega'}z_j^{\omega'}(y) \\
&=\lambda\cdot\tilde{w}(y)<K=\lambda\cdot v
\end{aligned}$$

for all $y\in Y$, and hence constraint (iii) holds.

Therefore, $k^*(\vec{\alpha},\lambda)\geq\lambda\cdot v=\lambda\cdot\tilde{v}+K$. Since K can be arbitrarily large, we conclude $k^*(\vec{\alpha},\lambda)=\infty$.

S.2 Proof of Claim 13

Claim 13.

- **(a)** *Let $\vec{\alpha}$ denote the type-independent profile (a_1,D_2) for some $a_1\in A_1$. Then,* $k^*(\vec{\alpha},\lambda')\leq\lambda'\cdot g(a_1,D_2)$.

- **(b)** *Let $\vec{\alpha}$ denote the type-independent profile (a_1,C_2) for some $a_1\in A_1$. Then* $k^*(\vec{\alpha},\lambda')\leq\lambda'\cdot g(a_1,C_2)-\frac{1-\beta}{\beta}(g_2^{\omega_2}(a_1,D_2)-g_2^{\omega_2}(a_1,C_2))$.

Part (a) follows as in FL, because the signal distribution does not depend on the state if player 2 chooses D_2.

For part (b), suppose that (v,w) satisfy constraints (i) through (iii) of the LP problem corresponding to $\vec{\alpha}$ and λ'. From player 2's IC constraint for state ω_2, we have

$$\begin{aligned}\beta(q_H(w_2^{\omega_2}(H)-w_2^{\omega_2}(L))+q_M(w_2^{\omega_2}(M)-w_2^{\omega_2}(L)))\\ \geq \frac{1-\delta}{\delta}(g_2^{\omega_2}(a_1,D_2)-g_2^{\omega_2}(a_1,C_2)).\end{aligned} \quad (11)$$

Then,

$$\begin{aligned}v_2^{\omega_2}-v_2^{\omega_1} =&(1-\delta)(g_2^{\omega_2}(a_1,C_2)-g_2^{\omega_1}(a_1,C_2))\\ &+\delta(\pi^{\omega_2}(a_1,C_2)\cdot w_2^{\omega_2}-\pi^{\omega_1}(a_1,C_2)\cdot w_2^{\omega_1})\\ =&(1-\delta)(g_2^{\omega_2}(a_1,C_2)-g_2^{\omega_1}(a_1,C_2))+\delta\pi^{\omega_1}(a_1,C_2)\cdot(w_2^{\omega_2}-w_2^{\omega_1})\\ &-\delta(1-\beta)(q_H(w_2^{\omega_2}(H)-w_2^{\omega_2}(L))+q_M(w_2^{\omega_2}(M)-w_2^{\omega_2}(L)))\\ \leq&(1-\delta)(g_2^{\omega_2}(a_1,C_2)-g_2^{\omega_1}(a_1,C_2))+\delta(v_2^{\omega_2}-v_2^{\omega_1})\\ &-\frac{(1-\delta)(1-\beta)}{\beta}(g_2^{\omega_2}(a_1,D_2)-g_2^{\omega_2}(a_1,C_2))\end{aligned}$$

Arranging,

$$v_2^{\omega_2}-v_2^{\omega_1}\leq g_2^{\omega_2}(a_1,C_2)-g_2^{\omega_1}(a_1,C_2)-\frac{1-\beta}{\beta}(g_2^{\omega_2}(a_1,D_2)-g_2^{\omega_2}(a_1,C_2)).$$

So we have

$$\lambda'\cdot v\leq\lambda'\cdot g(a_1,C_2)-\frac{1-\beta}{\beta}(g_2^{\omega_2}(a_1,D_2)-g_2^{\omega_2}(a_1,C_2)).$$

This proves the desired result.

S.3 Proof of Claim 14

Claim 14. *Suppose that* $\Theta_2=\{(\omega_1),(\omega_2)\}$.

(a) *Let $\vec{\alpha}$ denote the type-contingent profile such that player 1 chooses $a_1\in A_1$ and player 2 chooses C_2 for state ω_1 while D_2 for state ω_2. Then $k^*(\vec{\alpha},\lambda')\leq\lambda'\cdot g(\vec{\alpha})-(g_2^{\omega_1}(a_1,D_2)-g_2^{\omega_1}(a_1,C_2))$.*

(b) *Let $\vec{\alpha}$ denote the type-contingent profile such that player 1 chooses $a_1\in A_1$ and player 2 chooses D_2 for state ω_1 while C_2 for state ω_2. Then $k^*(\vec{\alpha},\lambda')\leq\lambda'\cdot g(a_1,C_2)-\frac{1-\beta}{\beta}(g_2^{\omega_2}(a_1,D_2)-g_2^{\omega_2}(a_1,C_2))$.*

Part (a). Suppose that (v,w) satisfy constraints (i) through (iii) of the LP problem corresponding to $\vec{\alpha}$ and λ.From player 2's IC constraint for state ω_1, we have

$$\begin{aligned}
&q_H(w_2^{\omega_1}(H)-w_2^{\omega_1}(L))+q_M(w_2^{\omega_1}(M)-w_2^{\omega_1}(L)) \\
&\qquad \geq \frac{1-\delta}{\delta}(g_2^{\omega_1}(a_1,D_2)-g_2^{\omega_1}(a_1,C_2)).
\end{aligned}$$

Then,

$$\begin{aligned}
v_2^{\omega_2}-v_2^{\omega_1} =&(1-\delta)(g_2^{\omega_2}(a_1,D_2)-g_2^{\omega_1}(a_1,C_2)) \\
&+\delta(\pi^{\omega_2}(a_1,D_2)\cdot w_2^{\omega_2}-\pi^{\omega_1}(a_1,C_2)\cdot w_2^{\omega_1}) \\
=&(1-\delta)(g_2^{\omega_2}(a_1,D_2)-g_2^{\omega_1}(a_1,C_2))+\delta\pi^{\omega_2}(a_1,D_2)\cdot(w_2^{\omega_2}-w_2^{\omega_1}) \\
&-\delta(q_H(w_2^{\omega_1}(H)-w_2^{\omega_1}(L))+q_M(w_2^{\omega_1}(M)-w_2^{\omega_1}(L))) \\
\leq&(1-\delta)(g_2^{\omega_2}(a_1,D_2)-g_2^{\omega_1}(a_1,C_2))+\delta(v_2^{\omega_2}-v_2^{\omega_1}) \\
&-(1-\delta)(g_2^{\omega_1}(a_1,D_2)-g_2^{\omega_1}(a_1,C_2)).
\end{aligned}$$

Then the result follows as in the proof of Claim 13(b).

Part (b). Suppose that (v,w) satisfy constraints (i) through (iii) of the LP problem corresponding to $\vec{\alpha}$ and λ. From player 2's IC constraint for state ω_2, we obtain (11). Then we have

$$\begin{aligned}
v_2^{\omega_2}-v_2^{\omega_1} =&(1-\delta)(g_2^{\omega_2}(a_1,C_2)-g_2^{\omega_1}(a_1,D_2)) \\
&+\delta(\pi^{\omega_2}(a_1,C_2)\cdot w_2^{\omega_2}-\pi^{\omega_1}(a_1,D_2)\cdot w_2^{\omega_1}) \\
\leq&(1-\delta)(g_2^{\omega_2}(a_1,C_2)-g_2^{\omega_1}(a_1,C_2)) \\
&+\delta(\pi^{\omega_2}(a_1,C_2)\cdot w_2^{\omega_2}-\pi^{\omega_1}(a_1,C_2)\cdot w_2^{\omega_1}) \\
=&(1-\delta)(g_2^{\omega_2}(a_1,C_2)-g_2^{\omega_1}(a_1,C_2))+\delta\pi^{\omega_1}(a_1,C_2)\cdot(w_2^{\omega_2}-w_2^{\omega_1}) \\
&-\delta(1-\beta)(q_H(w_2^{\omega_2}(H)-w_2^{\omega_2}(L))+q_M(w_2^{\omega_2}(M)-w_2^{\omega_2}(L))) \\
\leq&(1-\delta)(g_2^{\omega_2}(a_1,C_2)-g_2^{\omega_1}(a_1,C_2))+\delta(v_2^{\omega_2}-v_2^{\omega_1}) \\
&-\frac{(1-\delta)(1-\beta)}{\beta}(g_2^{\omega_2}(a_1,D_2)-g_2^{\omega_2}(a_1,C_2)).
\end{aligned}$$

Here, the first inequality follows from player 2's IC constraint for state ω_1, and the second inequality follows from (11). Then the result follows as in the proof of Claim 13(b).

S.4 Inefficiency when Player 2 Knows the State

Consider the game studied in Section 6.4. For $\varepsilon > 0$, let $\lambda(\varepsilon) = ((0,-\varepsilon),(1,0))$. We will show that for some parameters, there is ε such that $k^*(\lambda(\varepsilon)) < \lambda(\varepsilon)\cdot g(C_1,C_2)$, so that the efficient outcome $g(C_1,C_2)$ is not in Q.

Claim 15.

(a) *Let $\vec{\alpha}$ denote the type-independent profile (a_1,D_2) for some $a_1 \in A_1$. Then for any $\varepsilon > 0$, $k^*(\vec{\alpha},\lambda(\varepsilon)) \leq \lambda(\varepsilon)\cdot g(a_1,D_2)$.*

(b) *Let $\vec{\alpha}$ denote the type-independent profile (a_1,C_2) for some $a_1 \in A_1$. Then for any $\varepsilon > 0$, $k^*(\vec{\alpha},\lambda(\varepsilon)) \leq \lambda(\varepsilon)\cdot g(a_1,C_2) - \frac{(1-\beta)\varepsilon}{\beta}(g_2^{\omega_2}(a_1,D_2) - g_2^{\omega_2}(a_1,C_2))$.*

Claim 16. *Suppose that player 2 knows the state.*

(a) *Let $\vec{\alpha}$ denote the type-contingent profile such that player 1 chooses $a_1 \in A_1$ and player 2 chooses C_2 for state ω_1 while D_2 for state ω_2. Then for any $\varepsilon > 0$, $k^*(\vec{\alpha},\lambda(\varepsilon)) \leq \lambda(\varepsilon)\cdot g(\vec{\alpha}) - \varepsilon(g_2^{\omega_1}(a_1,D_2) - g_2^{\omega_1}(a_1,C_2))$.*

(b) *Let $\vec{\alpha}$ denote the type-contingent profile such that player 1 chooses $a_1 \in A_1$ and player 2 chooses D_2 for state ω_1 while C_2 for state ω_2. Then for any $\varepsilon > 0$, $k^*(\vec{\alpha},\lambda(\varepsilon)) \leq \lambda(\varepsilon)\cdot g(a_1,C_2) - \frac{(1-\beta)\varepsilon}{\beta}(g_2^{\omega_2}(a_1,D_2) - g_2^{\omega_2}(a_1,C_2))$.*

The proofs of the above claims are analogous to those of Claims 13 and 14, so that we omit them.

Claim 17. *Let $\vec{\alpha}$ be the type-independent profile (a_1,D_2) for some $a_1 \in A_1$. If*

$$0 < \varepsilon < \frac{g_1^{\omega_2}(C_1,C_2) - g_1^{\omega_2}(D_1,D_2)}{g_2^{\omega_1}(C_1,C_2) - g_2^{\omega_1}(D_1,D_2)}$$

then $\lambda(\varepsilon)\cdot g(C_1,C_2) > k^*(\vec{\alpha})$.

Proof. Arranging $\varepsilon < \frac{g_1^{\omega_2}(C_1,C_2) - g_1^{\omega_2}(D_1,D_2)}{g_2^{\omega_1}(C_1,C_2) - g_2^{\omega_1}(D_1,D_2)}$, we obtain $\lambda(\varepsilon)\cdot g(C_1,C_2) > \lambda(\varepsilon)\cdot g(D_1,D_2)$. Since the stage game is a prisoner's dilemma, we know that $\lambda(\varepsilon)\cdot g(D_1,D_2) > \lambda(\varepsilon)\cdot g(C_1,D_2)$. Plugging this into the above inequality, we have $\lambda(\varepsilon)\cdot g(C_1,C_2) > \lambda(\varepsilon)\cdot g(a_1,D_2)$ for each $a_1 \in A_1$. This implies that $\lambda(\varepsilon)\cdot g(C_1,C_2) > k^*(\vec{\alpha})$, as the above claim shows that $\lambda(\varepsilon)\cdot g(\vec{\alpha}) > k^*(a_1,D_2)$. *Q.E.D.*

Claim 18. *Suppose that player* 2 *knows the state, and let* $\vec{\alpha}$ *be the type-independent profile* (a_1, C_2) *for some* $a_1 \in A_1$*, or the type-contingent profile such that player* 1 *chooses* $a_1 \in A_1$ *and player* 2 *chooses* D_2 *for state* ω_1 *while* C_2 *for state* ω_2*. If*

$$\varepsilon > \frac{\beta(g_1^{\omega_2}(D_1,C_2) - g_1^{\omega_2}(C_1,C_2))}{\beta(g_2^{\omega_1}(C_1,C_2) - g_2^{\omega_1}(D_1,C_2)) + (1-\beta)(g_2^{\omega_2}(D_1,D_2) - g_2^{\omega_2}(D_1,C_2))},$$

then $\lambda(\varepsilon) \cdot g(C_1,C_2) > k^*(\vec{\alpha})$*.*

Proof. Arranging

$$\varepsilon > \frac{\beta(g_1^{\omega_2}(D_1,C_2) - g_1^{\omega_2}(C_1,C_2))}{\beta(g_2^{\omega_1}(C_1,C_2) - g_2^{\omega_1}(D_1,C_2)) + (1-\beta)(g_2^{\omega_2}(D_1,D_2) - g_2^{\omega_2}(D_1,C_2))},$$

we get

$$\lambda(\varepsilon) \cdot g(C_1,C_2) > \lambda(\varepsilon) \cdot g(D_1,C_2) - \frac{(1-\beta)\varepsilon}{\beta}(g_2^{\omega_2}(D_1,D_2) - g_2^{\omega_2}(D_1,C_2)).$$

Also, since $\lambda(\varepsilon) \cdot g(D_1,C_2) > \lambda(\varepsilon) \cdot g(C_1,C_2)$ and $g_2^{\omega_2}(D_1,D_2) - g_2^{\omega_2}(D_1,C_2) = g_2^{\omega_2}(C_1,D_2) - g_2^{\omega_2}(C_1,C_2)$, we have

$$\lambda(\varepsilon) \cdot g(C_1,C_2) > \lambda(\varepsilon) \cdot g(C_1,C_2) - \frac{(1-\beta)\varepsilon}{\beta}(g_2^{\omega_2}(C_1,D_2) - g_2^{\omega_2}(C_1,C_2)).$$

Then from the above claims, we get the desired result. *Q.E.D.*

Claim 19. *Suppose that player* 2 *knows the state, and let* $\vec{\alpha}$ *be the type-contingent profile such that player* 1 *chooses* $a_1 \in A_1$ *and player* 2 *chooses* C_2 *for state* ω_1 *while* D_2 *for state* ω_2*. If*

$$\varepsilon > \frac{\beta(g_1^{\omega_2}(D_1,C_2) - g_1^{\omega_2}(C_1,C_2))}{\beta(g_2^{\omega_1}(C_1,C_2) - g_2^{\omega_1}(D_1,C_2)) + (1-\beta)(g_2^{\omega_2}(D_1,D_2) - g_2^{\omega_2}(D_1,C_2))}$$

and if $\beta \geq \frac{1}{2}$*, then* $\lambda(\varepsilon) \cdot g(C_1,C_2) > k^*(\vec{\alpha})$*.*

Proof. If $\beta \geq \frac{1}{2}$, then $\frac{1-\beta}{\beta} \leq 1$, so that

$$\begin{aligned}\lambda(\varepsilon) \cdot g(a_1,C_2) - \frac{(1-\beta)\varepsilon}{\beta}&(g_2^{\omega_2}(a_1,D_2) - g_2^{\omega_2}(a_1,C_2)) \\ &\geq \lambda(\varepsilon) \cdot g(a_1,C_2) - \varepsilon(g_2^{\omega_2}(a_1,D_2) - g_2^{\omega_2}(a_1,C_2)).\end{aligned}$$

Recall that in the proof of the previous claim, we have shown that

$$\lambda(\varepsilon)\cdot g(C_1,C_2) > \lambda(\varepsilon)\cdot g(a_1,C_2) - \frac{(1-\beta)\varepsilon}{\beta}(g_2^{\omega_2}(a_1,D_2) - g_2^{\omega_2}(a_1,C_2))$$

for each $a_1 \in A_1$. Thus it follows that

$$\lambda(\varepsilon)\cdot g(C_1,C_2) > \lambda(\varepsilon)\cdot g(a_1,C_2) - \varepsilon(g_2^{\omega_2}(a_1,D_2) - g_2^{\omega_2}(a_1,C_2)).$$

Then from the above claim, we obtain the desired result. *Q.E.D.*

From the last three claims, we can show that if $\beta \geq \frac{1}{2}$ and if

$$\frac{\beta(g_1^{\omega_2}(D_1,C_2) - g_1^{\omega_2}(C_1,C_2))}{\beta(g_2^{\omega_1}(C_1,C_2) - g_2^{\omega_1}(D_1,C_2)) + (1-\beta)(g_2^{\omega_2}(D_1,D_2) - g_2^{\omega_2}(D_1,C_2))} < \varepsilon < \frac{g_1^{\omega_2}(C_1,C_2) - g_1^{\omega_2}(D_1,D_2)}{g_2^{\omega_1}(C_1,C_2) - g_2^{\omega_1}(D_1,D_2)},$$

then $k^*(\lambda(\varepsilon)) < \lambda \cdot g(C_1,C_2)$. Such a ε indeed exists, if e_1 is sufficiently close to $p_H(u_1(H) - u_1(L)) + p_M(u_1(M) - u_1(L))$. (Note that if $e_1 \to p_H(u_1(H) - u_1(L)) + p_M(u_1(M) - u_1(L))$, then $g_1^{\omega_2}(D_1,C_2) - g_1^{\omega_2}(C_1,C_2) \to 0$.)