Non-Stationary Search Equilibrium*

Giuseppe Moscarini†
Yale University
and
NBER

Fabien Postel-Vinay‡
University of Bristol
and
Université de Paris I
(Panthéon-Sorbonne)

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Abstract

We study the transitional dynamics of the Burdett and Mortensen (1998) equilibrium search model. Our exercise provides the first analysis of aggregate dynamics of a popular class of search wage-posting models. We assume that firms offer and commit to time-dependent wage contracts. We show that, when all firms have the same productivity, they post different time-dependent contracts, paying workers a higher value the larger is the initial size of the firm. That is, equilibrium exhibits a *Rank-Preserving property.* The same property holds in equilibrium when firms have heterogeneous productivity, where more productive firms offer a larger value and employ more workers at all points in time, if (but not only if) they have more employees to begin with.
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†Address: Department of Economics, Yale University, PO Box 208268, New Haven CT 06520-8268. Tel. +1-203-432-3596. E-mail `giuseppe.moscarini@yale.edu`. Web `http://www.econ.yale.edu/faculty1/moscarini.htm`

‡Address: Department of Economics, University of Bristol, 8 Woodland Road, Bristol BS8 1TN, UK. Tel: +44 117 928 8431. E-mail `Fabien.Postel-Vinay@bristol.ac.uk`. Web `www.efm.bris.ac.uk/economics/staff/vinay/`. Postel-Vinay is also affiliated with CEPR (London), IZA (Bonn) and PSE (Paris).

1 Introduction

We study the transitional dynamics of the Burdett and Mortensen (1998) equilibrium search model. Our exercise provides the first analysis of aggregate dynamics of a popular class of search wage-posting models.

The Burdett and Mortensen (1998, henceforth BM) search and wage posting model provided the first successful formalization of the hypothesis that cross-sectional wage dispersion is largely a consequence of labor market frictions. In so doing the BM model has started what has now established itself as the most promising line of research in the analysis of wage inequality, as the vibrant and empirically very successful literature organized around that hypothesis continues to show.

That literature, however, is invariably cast in steady state. Ever since the inception of the BM model, job search scholars have regarded the characterization of its out-of-steady-state behavior as a daunting problem, essentially because one of the model's state variables, which is also the main object of interest, is the endogenous *distribution* of wage (or job value) offers. This is an infinite-dimensional object, endogenously determined in equilibrium as the distribution across a continuum of firms of strategies that are all best responses to one another.

We find a way around this problem by considering a class of equilibria satisfying what we call the *Rank-Preserving property*, i.e. equilibria in which the workers' ranking of firms is time-invariant. We show that this class of equilibria is generic if all firms are equally productive. We further show that the same property holds in equilibrium when firms have heterogeneous productivity, where more productive firms offer a larger value and employ more workers at all points in time, if (but not only if) they have more employees to begin with. We view the fact that the workers' ranking of firms also reflects the hierarchy of productivity in a Rank-Preserving Equilibrium in the presence of productive heterogeneity across firms as a very appealing property of the model. It parallels a similar property of BM's static equilibrium, and in ensures constrained-efficient labor reallocation at all dates.

Besides being of intrinsic theoretical interest, our characterization of the dynamics of the BM model opens the analysis of aggregate labor market dynamics as a whole potential new field of application of search/wage-posting models. Unlike the typical representative-agent model, the BM model makes predictions about the business cycle behavior of wage distributions, firm size distributions, or patterns of labor reallocation across firms. In a companion paper (Moscarini and Postel-Vinay, 2008), we quantitatively gage those predictions against facts documented using various (often new) data sets. Moreover, by advocating the BM model as a potentially useful tool for the study of aggregate labor market dynamics, we hope to contribute to a synthesis between the BM approach and the "other", equally successful side of the search literature, organized around the matching framework (Pissarides, 1990; Mortensen and Pissarides, 1994), initially designed for the understanding of labor market flows and equilibrium unemployment.

The paper has four sections after this Introduction. In Sectin 2 we lat out the basic assumptions on the economic environment and state the typical firm's optimization problem. Dynamic equilibria are characterized in Section 3. Section 4 discusses the results and Section 5 sketches some extensions.

2 The Economy

2.1 The Environment

The model is a near-exact replica of the BM wage posting model with heterogeneous firm types. Time is continuous. The labor market is populated by a unit-mass of workers who can be either employed or unemployed. It is affected by search frictions in that unemployed workers can only sample job offers sequentially at some finite Poisson rate $\lambda_0 > 0$. Employed workers are allowed to search on the job, and face a sampling rate of job offers of $\lambda_1 > 0$. Firm-worker matches are dissolved at rate $\delta > 0$. Upon match dissolution, the worker becomes unemployed. All workers are ex-ante identical: they are infinitely lived, risk-neutral, equally capable at any job, and they attach a common lifetime value of U_t to being unemployed at date t.

Workers face a measure N of active firms operating constant-return technologies with heterogeneous productivity levels $p \sim \Gamma(\cdot)$ among firms, and density $\gamma = \Gamma'$. The sampling of firms by workers is not necessarily uniform, in that a type-p firm has a sampling weight of $q(p) > 0$.

Sampling weights are normalized to ensure that their cumulated sum $\Phi(p) := \int_{\underline{p}}^{p} q(x)\gamma(x)\,dx$ is a (sampling) cdf, i.e. $\Phi(\overline{p}) = 1$. The sampling density of a type-p firm is therefore $\varphi(p) := q(p)\gamma(p)$. This naturally encompasses the conventional case of uniform sampling which has $q(p) = 1$ for all p.[1]

At some initial date which we normalize at $t_0 = 0$, each firm of a given type p commits to a wage profile $\{w_t(p)\}_{t\in[0,+\infty)}$ over the infinite future. We generalize the BM restrictions placed on the set of feasible wage contracts to a non-steady-state environment by preventing firms from making wages contingent on anything else than calendar time.[2]

Any such profile $\{w_t(p)\}_{t\in[0,+\infty)}$ offered by any type-p firm yields a continuation value of $V_t(p)$ to any worker employed at that firm at any date t. The (time-varying) sampling distribution of job values is denoted as $F_t(\cdot)$, and its relationship to the sampling distribution of firm types $\Phi(\cdot)$ will be discussed momentarily. Because from the workers' viewpoint jobs are identical in all dimensions but the wage profile, employed jobseekers quit into higher-valued jobs only. This gradual self-selection of workers into better jobs implies that the distribution of job values in a cross-section of workers—which will be denoted as $G_t(\cdot)$—differs from the sampling distribution $F_t(\cdot)$.

[1]There are three possible interpretations of sampling weights, First, they reflect the different visibility of employers of different sizes, due to informational spill-overs across workers connected in social networks. Alternatively, they are a shortcut for directed search: if search has any element of directness, people will apply more to high paying firms (which higher-p firms will turn out to be in equilibrium). Finally, and perhaps most naturally, they may reflect the relative density of vacancies posted by a firm of productivity p, with random meetings between all vacancies and all job searchers mediated by a standard matching function. This last possibility endogenizes both sampling weights q and arrival rates λ_0, λ_1, as we discuss it in detail in Section 5.

[2]Or, less stringently, we allow firms to index wages to any aggregate variable that evolves monotonically over time (e.g. the unemployment rate). We thus rule out, among other things, wage-tenure contracts (Stevens, 2004; Burdett and Coles, 2003), offer-matching or individual bargaining (Postel-Vinay and Robin, 2002; Dey and Flinn, 2005; Cahuc, Postel-Vinay and Robin, 2006), or contracts conditioned on employment status (Carrillo-Tudela, 2007). Note, however, that the model can be generalized to allow for time-varying individual heterogeneity under the assumption that firms offer the type of piece-rate contracts described in Barlevy (2005). In that sense experience and/or tenure effects can be introduced into the model.

2.2 The Contract Posting Problem

Firms post wage profiles over an infinite horizon that solve the following problem:

$$\Pi_0\left(L_0\left(p\right);p\right) = \max_{\{w_t\}} \int_0^{+\infty} \left(p - w_t\right) L_t\left(p\right) e^{-rt} dt \tag{1}$$

subject to:

$$\rho V_t\left(p\right) = \dot{V}_t\left(p\right) + w_t - \delta\left[V_t\left(p\right) - U_t\right] + \lambda_1 \int_{V_t(p)}^{+\infty} \left[x - V_t\left(p\right)\right] dF_t\left(x\right) \tag{2}$$

$$\dot{L}_t\left(p\right) = -\left[\delta + \lambda_1 \overline{F}_t\left(V_t\left(p\right)\right)\right] L_t\left(p\right) + \frac{q\left(p\right)}{N}\left[\lambda_0 u_t + \lambda_1\left(1 - u_t\right) G_t\left(V_t\left(p\right)\right)\right] \tag{3}$$

$$w_t \geq \underline{w}, \tag{4}$$

where $L_t\left(p\right)$ denotes a type-p firm's workforce at date t,[3] $\underline{w}$ is the exogenous institutional minimum wage, U_t is the workers' lifetime value of unemployment, r (ρ) is the firms' (workers') discount rate,[4] and $\overline{F}_t\left(\cdot\right) = 1 - F_t\left(\cdot\right)$ designates the survivor function associated with $F_t\left(\cdot\right)$. When solving (1), the typical firm of productivity p also is also constrained by its given initial size $L_0\left(p\right)$.

The firm's problem has two state variables that the firm controls through the wage. First, the chosen path of wages translates through the Hamilton-Jacobi-Bellman equation (2) into a value $V_t\left(p\right)$ for the worker of employment at that type-p firm. The worker's opportunity cost $\rho V_t\left(p\right)$ equals the capital gain plus the flow wage minus the capital loss when the match is destroyed exogenously at rate δ, plus the capital gain that occurs at rate $\lambda_1 \overline{F}_t\left(V_t\left(p\right)\right)$ when the worker receives and offer which also turns out to provide him with a higher value. This offer is drawn from the endogenous offer distribution $F_t\left(\cdot\right)$, which is the cross-section distribution at time t of all such values offered by other firms.

The value $V_t\left(p\right)$ offered by a type-p firm translates into inflows and outflows of workers. The only friction in the model is search, so the boundaries of the firm are defined by attrition, retention and hiring. Equation (3), describes the evolution of the firm's employment. Following standard practice, we impose a law of large numbers at the individual firm's level and we treat the evolution of firm size as deterministic, although it is the result of various random events. These include

[3] Incidentally, this implies that the *density* of firm types among workers at date t is given by $NL_t\left(p\right)\gamma\left(p\right)/\left(1 - u_t\right)$.

[4] Although in some of what follows we will occasionally comply with standard practice and impose a common discount rate on firms and workers (i.e. assume $r = \rho$), this restriction is by no means essential. Indeed other cases, such as the case of myopic workers for example, are of potential interest (see Moscarini and Postel-Vinay, 2008).

separations—both exogenous at rate δ and endogenous at rate $\lambda_1 \overline{F}_t (V_t (p))$ when a worker receives a better offer—which reduce employment, and accessions from both unemployment (at rate λ_0) and from other firms that are paying their workers less than $V_t (p)$.

At the individual firm's level, the sampling and cross-sectional distributions of job values $F_t (\cdot)$ and $G_t (\cdot)$ are given macroeconomic quantities that no individual firm can affect with its choice. Given all firm's choices of wages, and the implied worker values $V_t (p)$ and firm sizes $L_t (p)$, they are defined by

$$F_t (W) = \int_{\underline{p}}^{\overline{p}} \mathbb{I} \{V_t (x) \leq W\} q (x) d\Gamma (x) \tag{5}$$

$$G_t (W) = \frac{\int_{\underline{p}}^{\overline{p}} L_t (x) \mathbb{I} \{V_t (x) \leq W\} d\Gamma (x)}{\int_{\underline{p}}^{\overline{p}} L_t (x) d\Gamma (x)} \tag{6}$$

where $\mathbb{I} \{\cdot\}$ is an indicator function. Notice that both are normalized to be proper c.d.f.'s. Also notice an important restriction that was kept implicit so far: the definitions in (5) and (6) are only valid in symmetric equilibria where there is no dispersion in firm size conditional on p (i.e. $p \mapsto V_t (p)$ and $p \mapsto L_t (p)$ are well-defined mappings for all t). Although this restriction will receive some further discussion below, we will essentially limit our attention to such equilibria in the rest of the paper.

Similarly, a single firm cannot affect the value of unemployment, which solves the HJB equation:[5]

$$\rho U_t = \dot{U}_t + b + \lambda_0 \int_{U_t}^{+\infty} (x - U_t) dF_t (x) \tag{7}$$

with b denoting the income flow in unemployment, or the unemployment rate u_t, which solves

$$\dot{u}_t = \delta (1 - u_t) - \lambda_0 u_t, \qquad \text{with} \quad u_0 = 1 - N \int_{\underline{p}}^{\overline{p}} L_0 (x) d\Gamma (x) \text{ given.} \tag{8}$$

2.3 An Equivalent Value-Posting Problem

The formulation of the contract-posting problem spelled out in equations (1) - (8) above can be simplified somewhat. First, to simplify notation, we redefine the firm's employment by normalizing

[5] In formulating (1), we assume for simplicity that any job offer posted in equilibrium is preferred to unemployment, i.e. $\inf_p V_t (p) \geq U_t$ at all t. This is achieved by assuming that the minimum wage $\underline{w}$ is sufficiently higher than b for unemployed workers to find even the least valuable job offer worth accepting.

by its sampling weight

$$\ell_t(p) := \frac{N}{q(p)} L_t(p)$$

so that initial unemployment is derived from the initial distribution of employment $u_0 = 1 - \int_{\underline{p}}^{\overline{p}} \ell_0(x)\, d\Phi(x)$ and

$$\dot{\ell}_t(p) = -\left[\delta + \lambda_1 \overline{F}_t(V_t(p))\right] \ell_t(p) + \lambda_0 u_t + \lambda_1 (1-u_t) G_t(V_t(p)). \tag{9}$$

The problem of the firm is then (1) subject to (2), (4), (7), (8), (9).

Next, the firm's objective (1) can be recast as follows by substitution of the workers' value function (2) and integration by parts using (9):

$$\begin{aligned}\int_0^{+\infty} (p - w_t)\, \ell_t(p)\, e^{-rt} dt = -V_0(p)\, \ell_0(p) \\ + \int_0^{+\infty} \Bigg\{ \left[p + \lambda_1 \int_{V_t}^{+\infty} x dF_t(x) + \delta U_t + (r-\rho) V_t\right] \ell_t(p) \\ -V_t \left[\lambda_0 u_t + \lambda_1 (1-u_t) G_t(V_t)\right] \Bigg\} e^{-rt} dt.\end{aligned}$$

This formulation decomposes the discounted sum of future profits accruing to the firm into the sum of two terms: the total present value of the firm measured by the second (integral) term, less the first term $V_0(p)\,\ell_0(p)$ which equals the total value transferred by the firm to its initial workforce at date $t = 0$.

For any given initial value $V_0(p)$ (and temporarily ignoring the minimum wage constraint (4) for simplicity — we will reintroduce it later on), the initial contract posting problem (1)-(2) can be restated as the following mathematically equivalent problem:

$$\begin{aligned}\Pi_0(\ell_0(p); p) = \max_{\left\{V_t \in \left[\frac{b}{r}, \frac{\overline{p}}{r+\delta}\right]\right\}} \int_0^{+\infty} \Bigg\{ \left[p + \lambda_1 \int_{V_t}^{+\infty} x dF_t(x) + \delta U_t + (r-\rho) V_t\right] \ell_t(p) \\ - V_t \left[\lambda_0 u_t + \lambda_1 (1-u_t) G_t(V_t)\right] \Bigg\} e^{-rt} dt\end{aligned} \tag{10}$$

subject to (9) and $\ell_0(p)$ given.

Notice that values can be chosen WLOG in the compact set $\left[\frac{b}{r}, \frac{\overline{p}}{r+\delta}\right]$. Any value strictly below b/r will be declined by the workers, who will quit to unemployment as $U_t \geq b/r$, hence all such values yield and equivalent profit and can be ignored. Values above $\overline{p}/(r+\delta)$ exceed what any firm can physically deliver. Therefore, this is a well-defined optimal control problem even ignoring the minimum wage constraint.

While control problems generally admit piece-wise continuous solutions, in this particular case we must further restrict the optimal path of worker values, say $\{V_t^\star(p)\}_{t>0}$ to be continuously differentiable at all dates and right-continuous at $t=0$ as it has to solve the original HJB equation (2). Moreover, any such optimal path will turn out to be independent of the initial value $V_0(p)$ (see below). As a consequence, the the initial contract posting problem (1) is literally equivalent to the reformulated problem (10) with the initial workers' value being defined as $V_0(p)=\lim_{t\searrow 0}V_t^\star(p)$.

Couching the contract posting problem as the choice of a path of values as in (10) rather than the choice of a wage path as in (1) brings about an important simplification in that (10) is a problem featuring only one state variable, $\ell_t(p)$, with a fixed initial value.

Before we move on to solving (10), we should clarify that our formulation of the contract-posting game and the firm's best-response problem contains the assumption that firms are bound by an *equal treatment constraint*: a firm must pay all of its workers the same wage, irrespective of when they were hired, from where, and of the outside offers that some of them may have received. In particular, the firm does renege on its promised wage, cannot condition the wage on tenure or received outside offers, and more generally does not respond to outside offers to its employees, but lets them go if they are offered more.[6]

[6] As argued in Moscarini (2005), not responding to outside offers is a sequential equilibrium of an ascending (English) auction between the incumbent and the poacher, and the unique equilibrium which survives natural refinements. The more productive of the two firms wins without offering more than it does to its other workers, because it can always respond to any attempt by the competitor to outbid it, even if the competitor trembles. In this case, our assumption of no ex-post competition is not particularly restrictive. If the auction is instead simultaneous with either one bid or a sealed bid, as in Bertrand (Postel-Vinay and Robin, 2002), then firms would bid their maximum valuation and our assumption has bite.

2.4 Optimality Conditions

The current value Hamiltonian of problem (10) is defined by:

$$\mathcal{H}_t(p) = \left[p + \lambda_1 \int_{V_t}^{+\infty} x dF_t(x) + \delta U_t + (r-\rho) V_t\right] \ell_t(p) - V_t \left[\lambda_0 u_t + \lambda_1 (1-u_t) G_t(V_t)\right] + \mu_t(p) \left\{- \left[\delta + \lambda_1 \overline{F}_t(V_t)\right] \ell_t(p) + \lambda_0 u_t + \lambda_1 (1-u_t) G_t(V_t)\right\},$$

where $\mu_t(p)$ is the costate variable. Denoting the optimal value offered by a type-p firm by $V_t(p)$, the optimality conditions are:

$$\lambda_0 u_t + \lambda_1 (1-u_t) G_t(V_t(p)) + (\rho - r) \ell_t(p) = \lambda_1 \left[\mu_t(p) - V_t(p)\right] \left[f_t(V_t(p)) \ell_t(p) + (1-u_t) g_t(V_t(p))\right] \tag{11}$$

$$\dot{\mu}_t(p) = \left[r + \delta + \lambda_1 \overline{F}_t(V_t(p))\right] \mu_t(p) - \left[p + \lambda_1 \int_{V_t(p)}^{+\infty} x dF_t(x) + \delta U_t + (r-\rho) V_t(p)\right] \tag{12}$$

$$\lim_{t \to +\infty} e^{-rt} \mu_t(p) \ell_t(p) = 0. \tag{13}$$

Supplementing this latter set of conditions with the state equations (7), (8) and (9), we obtain a system of partial differential equations characterizing the solution to an individual firm's maximization problem *for a given path of sampling distributions* $\{F_t(\cdot)\}_{t \in [0,+\infty)}$. Given a solution to that system, the optimal wage path can be retrieved using (2). The main difficulty, however, lies in characterizing the *equilibrium* $\{F_t(\cdot)\}_{t \in [0,+\infty)}$, i.e. the path of sampling distributions which is consistent with the above dynamic system simultaneously for the whole population of firms. This task will be carried out in the following section. Before we turn to that, however, it is worth spelling out some economic interpretation of the above optimality conditions.

As usual in economic applications of optimal control, the costate variable $\mu_t(p)$ is interpreted as the imputed (or shadow) unit value of the state variable $\ell_t(p)$ at date t. Because (1) is formally a maximization of the total value of the firm, $\mu_t(p)$ is indeed the shadow value *to the firm-worker match* (rather than to the firm) of the marginal unit of labor. The *firm's* shadow value of the marginal unit of labor $\pi_t(p)$ is obtained by subtracting the worker's value: $\pi_t(p) := \mu_t(p) - V_t(p)$ and solves the following Euler equation:

$$\dot{\pi}_t(p) = \left(r + \delta + \lambda_1 \overline{F}_t(V_t(p))\right) \pi_t(p) - (p - w_t(p)) \tag{14}$$

which in turn was obtained by subtracting (2) from (12).

The first-order condition (11) reflects a balance between the firm's present-value cost and benefit of marginally changing its posted value at date t. The RHS of (11) equals $\pi_t(p) \cdot \frac{\partial \dot{\ell}_t(p)}{\partial V_t}$ and clearly reflects the benefit of offering a marginally higher value stemming from the larger workforce achieved through the implied higher retention and hiring rates. To see how the LHS of (11) reflects the cost of a marginal increase in the value transferred to workers, it may help to view an employer's commitment to transferring a certain value to its workers as that employer running up a debt to its employees. The (net) interest paid by the employer on a stock of debt of $V_t(p)$ to each of its workers equals the workers' overall discount rate, $\rho + \delta + \lambda_1 \overline{F}_t(V_t(p))$ (which results from the combination of sheer time discounting at rate ρ plus a "depreciation rate" of $\delta + \lambda_1 \overline{F}_t(V_t(p))$ reflecting future match dissolution, either through job destruction or the worker quitting), less the firm's discount (or interest) rate r. A unit increase in the value offered to all of the firm's employees then adds $\ell_t(p)$ to the firm's stock of debt. The marginal cost of such an addition to the stock of debt is an increase in the debt burden which in turn results from the net interest paid on that debt being raised by $\left[\rho - r + \delta + \lambda_1 \overline{F}_t(V_t(p))\right] \ell_t(p)$ plus an extrinsic expansion/contraction term $\dot{\ell}_t(p)$ reflecting the fact that the stock of debt is by nature indexed to workforce size. The sum of these latter two terms is equal to Equation (11)'s LHS.

Equation (12) describes the dynamics of the shadow value of the marginal unit of labor. It has a straightforward asset-pricing-type interpretation, whereby the firm's marginal employee is viewed as an asset priced at $\mu_t(p)$. The annuity value of the marginal employee, $\left(r + \delta + \lambda_1 \overline{F}_t(V_t(p))\right) \mu_t(p)$, must then equal the return on the corresponding asset which is the sum of a dividend term (in square brackets) plus a capital gain term $\dot{\mu}_t(p)$. That dividend term is the sum of a profit flow of $p - w_t(p)$ accruing to the employer (see Equation (14)) and an (expected) flow income of $w_t(p) + \delta U_t + \lambda_1 \int_{V_t(p)}^{+\infty} x dF_t(x)$ accruing to the worker (see Equation (2)).

3 Equilibrium Characterization

Definition 1 (Equilibrium) *An equilibrium of the dynamic contract-posting game is a vector of differentiable functions* $[V_t(p), \mu_t(p), \ell_t(p), U_t, u_t]$ *which solve the optimality conditions* (11), (12) *and* (13), *the state equations* (7), (8) *and* (9), *and the consistency conditions* (5) *and* (6) *given* $\ell_0(p)$ *and* $u_0 = 1 - \int \ell_0(p)\, dp$.

Having defined the object of interest, we now return to the main hurdle that we face in describing it and solving the contract-posting dynamic game. We need to find strategies whose distributions across firms evolve according to F_t, induce a distribution G_t of values across employed workers, and are all best-responses to each other.

We begin by arguing that F_t and G_t cannot have atoms in equilibrium at almost all points in time, so a density f_t and g_t indeed exists almost everywhere in p and t. The argument is the same as in BM: if there was an atom of firms offering the same value for an interval of time of nonzero length, one of them could gain by deviating and offering an ε more, winning the competition against the atom every time it arises, at an infinitesimal cost. Atoms in G_t exist if and only if atoms in F_t do. This argument, however, does not rule out atoms at countable points in time, where the paths of values offered by a set of firms of positive productivity measure happen to cross simultaneously, an issue that did not arise in BM's steady state analysis.

Next, we focus on a particularly tractable and natural class of equilibria, which satisfy what we call a *Rank-Preserving* (RP) property. We show conditions under which all equilibria must satisfy this property, implying uniqueness of equilibrium as an added bonus. The conditions have a natural economic interpretation. Finally, we fully characterize the dynamics of firm size, wage and value offers in a RP equilibrium.

3.1 The Rank-Preserving Property

A tractable class of equilibria, and in many cases the only type of equilibrium, has the following property:

Definition 2 (Rank-Preserving Property) *An equilibrium is Rank-Preserving if firms post val-*

ues that are strictly increasing in p for all t.

A direct consequence of the above definition is that in a RPE workers rank firms according to productivity at all dates. The following two properties hold true at all dates under the RP assumption:

$$F_t\left(V_t\left(p\right)\right) \equiv \Phi\left(p\right),$$
$$\left(1-u_t\right) G_t\left(V_t\left(p\right)\right) = \int_{\underline{p}}^{p} \ell_t\left(x\right) d\Phi\left(x\right).$$

In addition to considerably simplifying equilibrium determination (see below), the RP assumption is theoretically appealing for at least two reasons. First, it parallels a well-known property of the static equilibrium characterized by BM, which is to have a unique equilibrium where workers rank firms according to productivity. Second, RPE feature constrained-efficient labor reallocation at all dates: if workers consistently rank more productive firms higher than less productive ones, then job-to-job moves will always be up the productivity ladder.[7] The following natural question is therefore to ask about the generality of these rank-preserving equilibria. Given the definition of a RPE spelled out above the following two propositions can be established:

Proposition 1 (Ranked Initial Firm Size Implies Rank-Preserving Equilibrium) *If the initial state of the economy is such that $\ell_0\left(p\right)$ is non decreasing in p (i.e. higher-p firms are no smaller in sampling-weight-adjusted terms), then any equilibrium of the dynamic value-posting game is necessarily rank-preserving.*

The proof of this first proposition is in Appendix A. It builds on Caputo's (2003) comparative dynamic characterization of optimal controls in infinite horizon problems, which itself is based on the second-order condition of the primal-dual problem corresponding to (10).

This Proposition has a simple economic intuition, thus it appears to be a robust conclusion. In BM's steady state model, more productive firms offer higher wages due to a single-crossing property of their steady state profits, which in turn reflects two very basic economic forces. First, a higher

[7]We thank Pat Kline for pointing this out to us.

wage implies a larger firm size, as a more generous offer makes it easier to poach workers and to fend off competition. Second, a larger firm size is more valuable to a more productive firm, because each worker produces more. Therefore, by a simple monotone comparative statics argument, it must be the case that more productive firms offer more, employ more workers, and earn higher profits. Simply put, a productive firm can afford paying more, and is willing to do so to attract workers, because its opportunity cost of not producing is higher. Key to this argument is the fact that firm size is an endogenous object, and BM look for an appropriate firm size distribution which guarantees a stationary allocation.

In our dynamic model, firm size is a state variable, and its *initial* value is a parameter of the model, arbitrarily fixed , not an endogenous object. Therefore, in order to get a start on monotone comparative statics, it is sufficient (but not necessary) that the initial size distribution shares the key property of BM's steady state distribution; namely, it is increasing in productivity. In the proof, we begin by invoking Theorem 2 of Caputo (2003), which in this case is equivalent to a single-crossing property of the Hamiltonian of the value-posting problem. Given a ranked initial size, a more productive firm still wants and can afford to pay more, now in terms of values accruing to workers. The initial ranking of sizes by productivities is preserved throughout, so values offered to workers remain ranked by firm productivity at all points in the future, even if the firm were to stop and re-optimize. This condition is only sufficient. We conjecture that it is not necessary, and we are exploring this issue.

Proposition 2 (Rank-Preserving Stationary Allocations) *For r in a neighborhood of zero, the set of necessary optimality conditions for problem* (1) *has a unique steady-state symmetric solution which is "rank-preserving" in the sense that:*

- *steady-state worker value $V_\infty(p)$ is non decreasing in p;*
- *steady-state firm size $\ell_\infty(p)$ is non decreasing in p.*

The proof is in Appendix B. This latter result should not come as a surprise to those familiar with the BM model: if $r \to 0$, then firms only care about steady-state profits and our initially dynamic optimization problem becomes confounded with the static BM problem, which has a unique solution that is RP in the sense indicated in Proposition 2.

Taken together, Propositions 1 and 2 are statements about the generality of RPE. Specifically these propositions establish that RPE are generic within the set of dynamic equilibria such that there is no dispersion in firm size among firms of a common type p. Note that steady-state symmetric equilibria are necessarily in that class—as can be seen from the steady-state version of (3) which shows that steady-state firm size only depends on the value offered in steady-state, $V_\infty(p)$. The flip side of those arguments is that the RP property can transitionally break down because of the entry of new firms. For example an entrant firm with a productivity level somewhere in the interior of Γ's support and an initial size of zero might be tempted to break ranks in one direction or the other, depending on the shape of $f_t(\cdot)$ and $g_t(\cdot)$. With this caveat in mind, we now proceed to a characterization of RPE.

3.2 Evolution of the Firm Size Distribution in RPE

Let us consider the stock of workers employed at a firm of type-p or less, $\int_{\underline{p}}^{p} \ell_t(x)\, d\Phi(x)$. In a RPE (assuming one exists), those firms hire workers from unemployment and lose workers to their more productive competitors (firms of type higher than p). The stock of workers under consideration thus evolves according to:[8]

$$\int_{\underline{p}}^{p} \dot{\ell}_t(x)\, d\Phi(x) = \lambda_0 u_t \Phi(p) - \left[\delta + \lambda_1 \overline{\Phi}(p)\right] \int_{\underline{p}}^{p} \ell_t(x)\, d\Phi(x).$$

The latter equation now solves as:

$$\int_{\underline{p}}^{p} \ell_t(x)\, d\Phi(x) = e^{-\left[\delta+\lambda_1\overline{\Phi}(p)\right]t} \left(\int_{\underline{p}}^{p} \ell_0(x)\, d\Phi(x) + \lambda_0 \Phi(p) \int_0^t u_s e^{\left[\delta+\lambda_1\overline{\Phi}(p)\right]s} ds \right) \tag{15}$$

Now differentiating with respect to p, on obtains a closed-form expression for the workforce of any type-p firm:

$$\ell_t(p) = e^{-\left[\delta+\lambda_1\overline{\Phi}(p)\right]t} \left[\ell_0(p) + \lambda_1 t \int_{\underline{p}}^{p} \ell_0(x)\, d\Phi(x) + \lambda_0 \int_0^t \left[1 + \lambda_1 (t-s) \Phi(p)\right] u_s e^{\left[\delta+\lambda_1\overline{\Phi}(p)\right]s} ds \right] \tag{16}$$

[8]Note that the following law of motion can also by obtained by integration of (3) w.r.t. p. Details available on request.

The steady-state versions of (15) and (16) are:

$$\ell_\infty(p) = \frac{\delta\lambda_0(\delta+\lambda_1)}{(\delta+\lambda_0)\left[\delta+\lambda_1\overline{\Phi}(p)\right]^2} \qquad \text{and} \qquad \int_{\underline{p}}^{p} \ell_\infty(x)\, d\Phi(x) = \frac{\delta\lambda_0\Phi(p)}{(\delta+\lambda_0)\left[\delta+\lambda_1\overline{\Phi}(p)\right]}. \tag{17}$$

This is the point at which the necessity for sampling weights appears. Note from equation (17) that the steady-state size ratio of the largest to the smallest firm in the market in units of (non-normalized) employment is

$$\frac{L_\infty(\overline{p})}{L_\infty(\underline{p})} = \frac{\ell_\infty(\overline{p})\, q(\overline{p})}{\ell_\infty(\underline{p})\, q(\underline{p})} = \left(1+\frac{\lambda_1}{\delta}\right)^2 \frac{q(\overline{p})}{q(\underline{p})}.$$

With uniform sampling ($q(p) \equiv 1$ throughout), this ratio would equal $\left(1+\frac{\lambda_1}{\delta}\right)^2$, which is in the order of 25-30 given standard estimates of λ_1 and δ. Now of course the data counterpart of that size ratio is virtually infinite. More generally, it appears that the BM model requires a sampling distribution that is very heavily skewed toward high-productivity firms in order to replicate the observed distribution of firm sizes.

Before going any further into characterizing Rank-Preserving Equilibria, we should notice that the analysis of firm size and employment dynamics carried out in this paragraph would apply to any job ladder model in which a similar concept of RPE can be defined. Indeed nothing in the dynamics of L_t or u_t depends on the particulars of the wage setting mechanism, so long as this is such that employed jobseekers move from lower-ranking into higher-ranking jobs in the sense of a time-invariant ranking. Therefore, this model's predictions about everything relating to firm sizes are in fact much more general than the wage- (or value-) posting assumption retained in the BM model.

3.3 Wage Contracts in RPE

We now go back to the dynamical system characterizing the behavior of the typical individual firm, and analyze it in a RPE. The system in question is comprised of the set of optimality conditions (11) - (13) plus the set of state equations (9), (2) and (8). For simplicity, we now assume equal discount rates for workers and employers from now on (i.e. $r = \rho$).

The RP assumption changes the system (11) - 13) into:

$$\left(\lambda_0 u_t + \lambda_1 \int_{\underline{p}}^{p} \ell_t(x)\, d\Phi(x)\right) V_t'(p) = 2\lambda_1 \varphi(p)\, \ell_t(p)\left(\mu_t(p) - V_t(p)\right) \tag{18}$$

$$\dot{\mu}_t(p) = \left(r + \delta + \lambda_1 \overline{\Phi}(p)\right) \mu_t(p) - \lambda_1 \int_{p}^{+\infty} V_t(x)\, d\Phi(x) - \delta U_t - p \tag{19}$$

$$\lim_{t \to +\infty} e^{-rt} \mu_t(p) = 0. \tag{20}$$

Differentiation of (19) w.r.t. p yields (primes denote differentiation w.r.t. p while dots denote time differentiation):

$$\dot{\mu}_t'(p) = \left(r + \delta + \lambda_1 \overline{\Phi}(p)\right) \mu_t'(p) - \lambda_1 \varphi(p)\left(\mu_t(p) - V_t(p)\right) - 1. \tag{21}$$

This, together with (18) and the definition of the firm's shadow value of the marginal woreker $\pi_t(p) := \mu_t(p) - V_t(p)$, gives the following system of two PDEs in $(\mu_t'(p), \pi_t(p))$:

$$\begin{aligned} \dot{\mu}_t'(p) &= R(p)\, \mu_t'(p) + R'(p)\, \pi_t(p) - 1 \\ \mu_t'(p) &= \pi_t'(p) + B_t(p)\, \pi_t(p). \end{aligned} \tag{22}$$

where $R(p) := r + \delta + \lambda_1 \overline{\Phi}(p)$ is the effective discount factor of the firm, and

$$B_t(p) := \frac{2\lambda_1 \varphi(p)\, \ell_t(p)}{\lambda_0 u_t + \lambda_1 \int_{\underline{p}}^{p} \ell_t(x)\, d\Phi(x)}.$$

The system (22) can be solved numerically subject to some initial and boundary conditions. 'Initial' conditions are given by the steady-state solution to (22), which is characterized as:

$$\begin{aligned} \mu_\infty'(p) &= \frac{1 + \lambda_1 \varphi(p)\, \pi_\infty(p)}{r + \delta + \lambda_1 \overline{\Phi}(p)} \\ \pi_\infty(p) &= \frac{\left[\delta + \lambda_1 \overline{\Phi}(p)\right]^2}{r + \delta + \lambda_1 \overline{\Phi}(p)} \left(\int_{\underline{p}}^{p} \frac{dx}{\left(\delta + \lambda_1 \overline{\Phi}(x)\right)^2} + \frac{\pi_\infty(\underline{p})\,(r + \delta + \lambda_1)}{(\delta + \lambda_1)^2}\right). \end{aligned} \tag{23}$$

Now turning to boundary conditions, standard arguments prove that the lowest-type firms have no reason to pay more than the minimum wage: type $\underline{p}$ firms can only hire from unemployment and lose workers to poachers anyway, so trying to prevent poaching by raising wages is pointless for those firms in a RPE. While this implies that the minimum wage constraint (4) will bind at all dates for the lowest-type firm, it also implies that the following (time-invariant) boundary conditions are satisfied:

$$\pi_t\left(\underline{p}\right) \equiv \frac{\underline{p}-\underline{w}}{r+\delta+\lambda_1}$$
$$\mu_t'\left(\underline{p}\right) \equiv \frac{1+\lambda_1\varphi\left(\underline{p}\right)\pi_t\left(\underline{p}\right)}{r+\delta+\lambda_1}, \tag{24}$$

where the second condition is obtained by combining the first one with the $\dot{\mu}_t'(p)$ equation in (22). These boundary conditions can be further simplified by imposing $\underline{p}=\underline{w}$, a kind of free-entry condition holding throughout the adjustment toward the new steady state, which implies $\pi_t\left(\underline{p}\right) \equiv 0$. The minimum productivity $\underline{p}$ that can survive in the market is $\underline{w}$, as any firm with $p > \underline{w}$ can make positive profits by offering $\underline{w}$, and possibly even more by offering a higher wage while no firm with $p < \underline{w}$ can ever make any profits.

We note that (22) can also be written more compactly as one PDE in the firm's shadow value of the marginal worker:

$$\frac{\partial^2\pi_t(p)}{\partial t\partial p}+\frac{\partial}{\partial t}\left[B_t(p)\pi_t(p)\right]=\frac{\partial}{\partial p}\left[R(p)\pi_t(p)\right]+R(p)B_t(p)\pi_t(p)-1. \tag{25}$$

Once either the PDE in (25) is solved for $\pi_t(p)$ or (22) is solved for $(\mu_t'(p),\pi_t(p))$, wages can be retrieved from (14) (written under the RP assumption):

$$w_t(p)=p-\left(r+\delta+\lambda_1\overline{\Phi}(p)\right)\pi_t(p)+\dot{\pi}_t(p),$$

which has the following familiar steady-state solution:

$$w_\infty(p)=p-\left(\delta+\lambda_1\overline{\Phi}(p)\right)^2\left(\int_{\underline{p}}^{p}\frac{dx}{\left(\delta+\lambda_1\overline{\Phi}(x)\right)^2}+\frac{\underline{p}-\underline{w}}{(\delta+\lambda_1)^2}\right). \tag{26}$$

This is exactly the BM solution for the heterogeneous firm case (see equation (47) in Burdett and Mortensen, 1998). This confirms that our contracts are consistent with the BM steady-state wage-posting equilibrium *if the labor market is at a steady state.* It is no longer the case off steady-state, however: posting a time-invariant wage is not, in general,[9] a firm's best response to all other firms posting time-invariant wages.[10]

We now look back to the minimum wage constraint. The only firm for which the minimum wage constraint (4) is binding at the steady state characterized above is the lowest-type firm, $\underline{p}$. It may be the case, however, that the constraint temporarily binds for some higher-type firms over the transition to that steady state, in which case the economy no longer behaves according to (22) as this system was derived ignoring the minimum wage constraint (4). In our companion paper, Moscarini and Postel-Vinay (2008), we describe an algorithm that constructs an equilibrium in which $\underline{w}$ is allowed to temporarily bind for some firms (at the lower end of the p-distribution) with the restriction that it only bind over some initial period. In other words, any firm can choose to post the minimum wage for a while right after the occurrence of the productivity shock, but once it ceases to do so it is not allowed to return to the minimum wage.

The aforementioned numerical algorithm can be used to numerically solve (22) and simulate our model's dynamic equilibrium to study its quantitative properties. This is the objective pursued in Moscarini and Postel-Vinay (2008). As for the present theoretical analysis, we thus conclude our equilibrium characterization with the following two remarks. First, as we already mentioned, our constructive characterization of a RPE implies uniqueness within that class of equilibria. Second, our analysis still leaves two open questions: (conditions for) existence of an equilibrium, which in the RP case reduces to the relatively tractable problem of existence of a solution to a system of PDEs; and the much harder issue of equilibrium play when our sufficient conditions for RP fails.

[9] A pedagogically interesting exception is the case of myopic workers ($\rho = +\infty$), fully characterized and discussed in our companion paper Moscarini and Postel-Vinay (2008).

[10] To see this, notice that (14) and (18) yield two different growth rates for $\pi_t(p)$ if all wages are constant and the economy is off its steady state (so that firm sizes change over time). Under constant wages, Equation (14) gives a $\pi_t(p)$ which evolves as an exponential of time. But then with a constant wage and constant wages offered elsewhere, $V_t'(p)$ is constant over time, so dividing (18) by $\ell_t(p)$ tells us that $\pi_t(p)$ is proportional to the gross hiring rate, and so $\pi_t(p)$ cannot be exponential in time (because the hiring rate is not an exponential function of time in a RPE). All this implies that posting a constant wage in the face of competitors themselves posting constant wages violates the firm's set of necessary optimality conditions.

4 Discussion

4.1 Homogeneous Firms

The original motivation of the search literature exemplified by BM is to explain the large cross-sectional variation in worker wages that remains even after controlling for observable and unobservable worker and firm characteristics. The hallmark of the BM research program is to produce a robust failure of the law of one price, whereas identical agents make identical trades a different prices. In the light, the most striking and meaningful version of the BM model is the simplest setting where all firms and workers are identical. BM prove that the unique equilibrium must be in asymmetric strategies and entail wage dispersion among identical workers.

The proof of Proposition 1 can be adapted to show that, in this case, whenever we start with a continuous size distribution $\ell_0(p)$ with no atoms, equilibrium is always RP. The logic is simple. Now p plays the role of the rank in the initial $\ell_0(p)$. An initially larger firm wants to offer more to its workers, although they produce just as much as anywhere else, because the retention effect of a more generous offer is more powerful the larger firm size. Since there exists always a ranking of firms when $\ell_0(p)$ has no atoms, then the RP always holds in equilibrium.

4.2 Time Consistency of Equilibrium Contracts

As well known, wage- (or, in this case, contract-) posting models of frictional markets require a credible commitment by firms to fulfill the terms of the promise. Taking advantage of job search frictions and imperfect recall of past offers, a firm is tempted to exploit its bargaining power and to renege on the contract right after the worker accepts the offer, to drive down the wage to its reservation value. Coles (2001) provides an equilibrium reputational foundation for commitment to the wage offer. In our dynamic context, a time-consistent contract should set to zero the firm's shadow marginal value of the workforce $\pi_t(p) = \mu_t(p) - V_t(p) = 0$, in order to align the marginal value of the match $\mu_t(p)$ to that of the worker $V_t(p)$. As pointed out by Stevens (2004), the firm should effectively sell itself to the worker, extracting all rents, and then let the worker appropriate all the flow output. Besides the obvious issue of credibility, liquidity constraints are a powerful counter argument. Stevens proposes wage-tenure contracts, that Burdett and Coles (2003) develop further. We return later to this extension of the contract space.

Similarly, a firm would like to fight outside offers to its own employees, and counter-offer when its offer is not sufficient to poach a worker. Postel-Vinay and Robin (2002, 2004) investigate this idea. The same tension exists in our setting. Moscarini (2004) illustrates an alternative reputational mechanism to enforce this kind of commitment not to respond to outside offers to own employees.

In our model, the optimal contract is in open-loop form, a pre-determined function of time. By the principle of optimality, however, for any given time path of F_t and G_t, no firm wants to deviate from the initially chosen contract if this deviation has no impact on those aggregate paths. If firms could coordinate and re-optimize collectively, they would want to deviate, but this is not an issue under a Nash equilibrium concept. For sequential equilibrium one needs to specify the continuation strategies of all firms after such a deviation. If firms' actions are observed only by the parties involved, the worker and at best a competing firm, then any deviation will trigger a cascade of reactions that will involve at best a countable number of firms and workers down the road, a zero measure set. Therefore, a firm should not expect to change the continuous distribution of values offered and earned at any time in the future. Hence, our equilibrium is also sequential, thus contracts are time-consistent, given the commitment power vis-a-vis workers. This argument breaks down if all actions are publicly observed.

5 Extensions [in progress]

5.1 Vacancy creation

So far, we have taken the arrival rates of offers λ_0 and λ_1, the match destruction rate δ, the interest rate r and the sampling weight q as given constants. We are extending the analysis to allow for these variables to be arbitrary, smooth functions of time. The aggregate, $p-$independent variables like λ_0, λ_1, δ and r pose no particular problems.

Suppose that a firm p needs to post vacancies in order to hire. Suppose that posting v_t vacancies has a flow cost $c(v_t)$, where c is convex and smooth. Let $v_t(p)$ be the density of vacancies posted by $p-$ firms at time t. Let $v_t = \int v_t(p)\, d\Gamma(p)$ denote the aggregate stock of vacancies. Suppose that a random CRS matching function m mediates meetings between unemployed and employed job searchers with open vacancies. Specifically, suppose that each employed worker finds open vacancies at a rate σ times that of unemployed workers. Then contact rates are endogenous and time-dependent.

$$\lambda_{0t} = \frac{m\left(u_t + \sigma \int L_t(p)\, d\Gamma(p), \int v_t(p)\, d\Gamma(p)\right)}{u_t + \sigma \int L_t(p)\, d\Gamma(p)}$$

$$\lambda_{1t} = \sigma \lambda_{0t}$$

Sampling weights have the same property:

$$q_t(p) = \frac{v_t(p)}{\int v_t(p')\, d\Gamma(p')}$$

so that indeed $\Phi_t(p) = \int_{\underline{p}}^{p} q_t(p')\, d\Gamma(p')$ is a proper cdf at all points in time.

The choice of vacancies must be added to the control problem. While it cannot affect the arrival rate of offers to workers, as each firm is too small to make itself easily visible, it does affect the chance that the job application will land on a particular firm type, namely, the sampling distribution. The solution of the model is now significantly more involved, as contract offers and thus their distribution depend on sampling weights, and vice versa. We are currently working on this extension. The model with constant λ_0 and λ_1 can be justified by assuming that each $p-$firm can post only a fixed number of vacancies $q(p)$, that is the vacancy creation cost function is flat at zero until $q(p)$ and infinite thereafter, and that the matching function is $m(u, v) = \min \langle u, v \rangle$ and selects always the second argument.

5.2 Stochastic aggregate productivity

So far, we have analyzed only deterministic transitional dynamics of a contract-posting model. More general and interesting, but also considerably harder, is characterizing a rational expectations equilibrium when such exogenous parameters as the productivity distribution, the arrival rates of offers, the match destruction rate, follow known stochastic process. Increases and decreases in productivity may have asymmetric effects. Conceptually, the thorny issue is the specification of the contract space. On the one hand, some commitment to offers is necessary for the market to work. On the other hand, commitment despite aggregate shocks appears strong and implausible. We are currently exploring wage contracts that depend on two states, aggregate productivity as well as the unemployment rate as we have allowed for so far.

The RP property remains a natural property of equilibrium in a stochastic model. If more productive firms have a tendency to offer more and grow larger, after every realization of aggregate shocks they will start with a larger employment, and from then on, until the next shock, the logic of our exercise will apply, preserving the rank. Therefore, the mutual feedback between ranking of firm sizes and ranking of employment contracts preserves both rankings indefinitely.

5.3 Wage-Tenure Contracts

In our companion paper Moscarini and Postel-Vinay (2008) we compute the dynamics of a calibrated version of our baseline model, and we find that, if we start from firm sizes that are uniformly below steady state, firms tend to smoothly backload wages in time. This result obtains even if firms cannot offer wage-tenure contracts and workers are risk-neutral, and is driven by the smooth transitional dynamics of unemployment. We illustrate several strategic forces at play behind this backloading in that particular configuration of parameters. This example shows that time effects in wages do not require explicit wage-tenure contracts a la Stevens (2004) and Burdett and Coles (2003).

Introducing wage-tenure contracts alone in our setting appears uninteresting, as the underlying aggregate dynamics will create a strong tension and push firms towards modifying the terms of new contracts, giving rise to complicated within-firm differentiation. An alternative is to allow for wage contracts that are contingent both on tenure and the state of unemployment. It is not a priori clear, however, what this hybrid model would accomplish from a theoretical viewpoint, beyond greater realism.

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Appendix

A Proof of Proposition 1

Consider the following generic dynamic optimization problem:

$$\Pi_t(\ell_t;p)=\max_{\{V_s\}}\int_t^{+\infty}\left\{\left[p+\lambda_1\int_{V_s}^{+\infty}xdF_s(x)+\delta U_s+(r-\rho)V_s\right]\ell_s - V_s\left[\lambda_0 u_s+\lambda_1(1-u_s)G_s(V_s)\right]\right\}e^{-r(s-t)}ds \quad (\mathcal{P})$$

subject to: $\dot{\ell}_s=-\left(\delta+\lambda_1\overline{F}_s(V_s)\right)\ell_s+\lambda_0 u_s+\lambda_1(1-u_s)G_s(V_s)$

ℓ_t given.

By the optimality principle, a solution to $(\mathcal{P})$ coincides with a solution to the contract posting problem (10) over $[t,+\infty)$ provided that the initial condition for ℓ_t in $(\mathcal{P})$ is set at the value taken on by ℓ_t along the optimal path in the sense of the solution to problem (10).

The proof of Proposition 1 involves an application of Caputo's (2003) comparative dynamic results.[11] In order to apply those results we let $V_t^\star(\ell_t;p)$ denote the closed-loop optimal control to problem $(\mathcal{P})$, and we also take up Caputo's (2003) notation $D_{xy}\left[V_t^\star(\ell_t;p),(\ell_t,t;p)\right]$ for the (x,y) element of the Hessian matrix of the primal-dual problem associated with $(\mathcal{P})$ (see Caputo, 2003, equations 14-23 and Theorem 2).

We begin by establishing two intermediate results, from which the proposition will follow.

Lemma 1 *Along the optimal path, for all $t\geq 0$:*

$$\frac{\partial V_t^\star}{\partial p}(\ell_t;p)\geq 0 \qquad \text{and} \qquad \frac{\partial^2\Pi_t}{\partial p\partial\ell_t}(\ell_t;p)>0.$$

Proof. We apply Theorem 2 in Caputo (2003), which is a statement of the second-order necessary condition for the primal-dual problem corresponding to $(\mathcal{P}_a)$. That second-order condition implies, inter alia, the following:

$$-D_{pp}\left[V_t^\star(\ell_t;p),(\ell_t,t;p)\right]\equiv\frac{\partial V_t^\star}{\partial p}\cdot\frac{\partial^2\Pi_t}{\partial p\partial\ell_t}\cdot\left[\lambda_1 f_t(V_t^\star)\ell_t+\lambda_1(1-u_t)g_t(V_t^\star)\right]\geq 0. \qquad (27)$$

Thus $\partial V_t^\star/\partial p$ has the same sign as $\partial^2\Pi_t/\partial p\partial\ell_t$.

Application of the Dynamic Envelope Theorem (e.g. Caputo, 1990) next establishes that:

$$\frac{\partial \Pi_t}{\partial p}(\ell_t;p)=\int_t^{+\infty}\ell_s e^{-r(s-t)}ds,$$

implying:

$$\frac{\partial^2\Pi_t}{\partial p\partial\ell_t}(\ell_t;p)=\int_t^{+\infty}\frac{\partial\ell_s}{\partial\ell_t}e^{-r(s-t)}ds. \qquad (28)$$

[11] Although problem $(\mathcal{P})$ is nonautonomous, it can be reexpressed as an autonomous problem by treating time as an additional predetermined state variable τ_s such that $\dot{\tau}_s=1$ and $\tau_t=t$. This is the generic type of problem analyzed by Caputo (2003).

The proof of the proposition is then completed by establishing the following:

Claim: $\partial \ell_s / \partial \ell_t > 0$ *for* $s \geq t$ *along the optimal path.*

To prove this claim, we go back to the law of motion of ℓ_s along the optimal path which is given by:

$$\dot{\ell}_s = -\left(\delta + \lambda_1 \overline{F}_s \left(V_s^\star \left(\ell_s; p\right)\right)\right) \ell_s + \lambda_0 u_s + \lambda_1 \left(1 - u_s\right) G_s \left(V_s^\star \left(\ell_s; p\right)\right).$$

Differentiation w.r.t. ℓ_t yields:

$$\begin{aligned} \frac{\partial \dot{\ell}_s}{\partial \ell_t} &= \left\{ -\left(\delta + \lambda_1 \overline{F}_s \left(V_s^\star\right)\right) + \left[\lambda_1 f_s \left(V_s^\star\right) \ell_s + \lambda_1 \left(1 - u_s\right) g_s \left(V_s^\star\right)\right] \cdot \frac{\partial V_s^\star}{\partial \ell_s} \right\} \cdot \frac{\partial \ell_s}{\partial \ell_t} \\ &\equiv \Psi_s \left(V_s^\star, \ell_s\right) \cdot \frac{\partial \ell_s}{\partial \ell_t}. \end{aligned}$$

Given the initial condition $\partial \ell_s / \partial \ell_t = 1$ at $s = t$, the differential equation above can be rewritten as:

$$\frac{\partial \ell_s}{\partial \ell_t} = \exp \int_t^s \Psi_x \left(V_x^\star \left(\ell_x; p\right), \ell_x\right) dx > 0. \tag{29}$$

□

Lemma 1 is not sufficient to establish that the RP property must hold in equilibrium: for that we need to determine how the optimal $V_t^\star$ responds to differences in the state variable, i.e. firm size (ℓ_t). This is what the next proposition is about.

Lemma 2 *Along the optimal path, for all* $t \geq 0$*:*

$$\frac{\partial V_t^\star}{\partial \ell_t} \left(\ell_t; p\right) \geq 0 \qquad \text{and} \qquad \frac{\partial^2 \Pi_t}{\partial \ell_t^2} \left(\ell_t; p\right) > 0.$$

Proof. We apply Theorem 2 in Caputo (2003) again, which also implies:

$$\begin{aligned} -D_{\ell\ell} \left[V_t^\star \left(\ell_t; p\right), \left(\ell_t, t; p\right)\right] \equiv \frac{\partial V_t^\star}{\partial \ell_t} \cdot \Bigg\{ & \lambda_1 f_t \left(V_t^\star\right) \cdot \left[\frac{\partial \Pi_t}{\partial \ell_t} - V_t^\star\right] \\ & + \frac{\partial^2 \Pi_t}{\partial \ell_t^2} \cdot \left[\lambda_1 f_t \left(V_t^\star\right) \ell_t + \lambda_1 \left(1 - u_t\right) g_t \left(V_t^\star\right)\right] + \left(r - \rho\right) \Bigg\} \geq 0. \end{aligned} \tag{30}$$

Next, the Euler equation for problem ($\mathcal{P}$) (which can also be viewed as one of the envelope conditions for the HJB equation associated with ($\mathcal{P}$); see e.g. Caputo, 2003) writes down as:[12]

$$\frac{\partial^2 \Pi_t}{\partial \ell_t \partial t} + \frac{\partial^2 \Pi_t}{\partial \ell_t^2} \cdot \dot{\ell}_t = \left(r + \delta + \lambda_1 \overline{F}_t \left(V_t^\star\right)\right) \frac{\partial \Pi_t}{\partial \ell_t} - p - \lambda_1 \int_{V_t^\star}^{+\infty} x dF_t \left(x\right) - \delta U_t - \left(r - \rho\right) V_t^\star. \tag{31}$$

Differentiating w.r.t. ℓ_t along the optimal path yieds:

$$\begin{aligned} \frac{\partial}{\partial t} \frac{\partial^2 \Pi_t}{\partial \ell_t^2} + \frac{\partial}{\partial \ell_t} \frac{\partial^2 \Pi_t}{\partial \ell_t^2} \cdot \dot{\ell}_t - \left(\delta + \lambda_1 \overline{F}_t \left(V_t^\star\right)\right) \frac{\partial^2 \Pi_t}{\partial \ell_t^2} & \\ + \left[\lambda_1 f_t \left(V_t^\star\right) \ell_t + \lambda_1 \left(1 - u_t\right) g_t \left(V_t^\star\right)\right] \frac{\partial V_t^\star}{\partial \ell_t} \cdot \frac{\partial^2 \Pi_t}{\partial \ell_t^2} & \\ = \left(r + \delta + \lambda_1 \overline{F}_t \left(V_t^\star\right)\right) \frac{\partial^2 \Pi_t}{\partial \ell_t^2} - \lambda_1 f_t \left(V_t^\star\right) \frac{\partial V_t^\star}{\partial \ell_t} \cdot \left[\frac{\partial \Pi_t}{\partial \ell_t} - V_t^\star\right] - \left(r - \rho\right) \frac{\partial V_t^\star}{\partial \ell_t}. & \end{aligned}$$

[12]Note that $\partial \Pi_t / \partial \ell_t$ is the shadow joint value to the firm and the worker of the marginal match in the specific firm considered, which coincides with μ_t in the main text.

Rearranging:

$$\frac{d}{dt}\frac{\partial^2 \Pi_t}{\partial \ell_t^2} = \left(r + 2\delta + 2\lambda_1 \overline{F}_t\left(V_t^\star\right)\right)\frac{\partial^2 \Pi_t}{\partial \ell_t^2} - \frac{\partial V_t^\star}{\partial \ell_t}\cdot\left\{\lambda_1 f_t\left(V_t^\star\right)\cdot\left[\frac{\partial \Pi_t}{\partial \ell_t} - V_t^\star\right]\right.$$
$$\left. + \frac{\partial^2 \Pi_t}{\partial \ell_t^2}\cdot\left[\lambda_1 f_t\left(V_t^\star\right)\ell_t + \lambda_1\left(1-u_t\right)g_t\left(V_t^\star\right)\right] + (r-\rho)\right\}, \quad (32)$$

which can be integrated as:

$$\frac{\partial^2 \Pi_t}{\partial \ell_t^2} = \int_t^{+\infty}\frac{\partial V_s^\star}{\partial \ell_s}\cdot\left\{\lambda_1 f_s\left(V_s^\star\right)\cdot\left[\frac{\partial \Pi_s}{\partial \ell_s} - V_s^\star\right]\right.$$
$$\left. + \frac{\partial^2 \Pi_s}{\partial \ell_s^2}\cdot\left[\lambda_1 f_s\left(V_s^\star\right)\ell_s + \lambda_1\left(1-u_s\right)g_s\left(V_s^\star\right)\right] + (r-\rho)\right\}\cdot e^{-\int_t^s\left(r+2\delta+2\lambda_1\overline{F}_x\left(V_x^\star\right)\right)dx}ds \geq 0,$$

where the positive sign follows from (30). Lemma 2 then follows from (30) again, the positive sign of $\partial^2\Pi_t/\partial\ell_t^2$ just established and the fact that $\partial\Pi_t/\partial\ell_t - V_t^\star > 0$ (which is directly implied by the first order condition for problem $(\mathcal{P})$ and otherwise reflects the fact that any given firm makes a positive profit on its marginal worker). □

We are now in a position to complete the proof of Proposition 1. Going back to the law of motion of ℓ_t along the optimal path for a given firm type p:

$$\dot{\ell}_t = -\left(\delta + \lambda_1\overline{F}_t\left(V_t^\star\left(\ell_t;p\right)\right)\right)\ell_t + \lambda_0 u_t + \lambda_1\left(1-u_t\right)G_t\left(V_t^\star\left(\ell_t;p\right)\right)$$

and differentiating w.r.t. p, we obtain:

$$\frac{\partial}{\partial t}\frac{d\ell_t}{dp} = \left\{-\left(\delta + \lambda_1\overline{F}_t\left(V_t^\star\right)\right) + \left[\lambda_1 f_t\left(V_t^\star\right)\ell_t + \lambda_1\left(1-u_t\right)g_t\left(V_t^\star\right)\right]\cdot\frac{\partial V_t^\star}{\partial \ell_t}\right\}\cdot\frac{d\ell_t}{dp}$$
$$+ \left[\lambda_1 f_t\left(V_t^\star\right)\ell_t + \lambda_1\left(1-u_t\right)g_t\left(V_t^\star\right)\right]\cdot\frac{\partial V_t^\star}{\partial p}$$
$$\equiv \Psi_t\left(V_t^\star,\ell_t\right)\cdot\frac{d\ell_t}{dp} + \left[\lambda_1 f_t\left(V_t^\star\right)\ell_t + \lambda_1\left(1-u_t\right)g_t\left(V_t^\star\right)\right]\cdot\frac{\partial V_t^\star}{\partial p}, \quad (33)$$

which can be integrated as:

$$\frac{d\ell_t}{dp} = \left.\frac{d\ell_t}{dp}\right|_{t=0}\cdot e^{\int_0^t\Psi_x\left(V_x^\star,\ell_x\right)dx} + \int_0^t\left[\lambda_1 f_s\left(V_s^\star\right)\ell_s + \lambda_1\left(1-u_s\right)g_s\left(V_s^\star\right)\right]\cdot\frac{\partial V_s^\star}{\partial p}\cdot e^{\int_s^t\Psi_x\left(V_x^\star,\ell_x\right)dx}ds,$$

which is positive from Lemma 1 and the assumption about the initial condition. Firm size $\ell_t(p)$ is thus increasing in p throughout, which proves the proposition as:

$$\frac{dV_t^\star}{dp} = \frac{\partial V_t^\star}{\partial p} + \frac{\partial V_t^\star}{\partial \ell_t}\cdot\frac{d\ell_t}{dp}, \quad (34)$$

where all terms are positive from the result above and Lemmas 1 and 2. □

B Proof of Proposition 2

In this appendix we only prove that a steady-state solution for r not too large is necessarily RP. Uniqueness is established by construction later in the main text. Also in this proof we take up some of the notation introduced in Appendix A and drop all time subscripts when alluding to steady-state quantities.

The steady-state version of (33) writes as:

$$\frac{d\ell}{dp} = \frac{\lambda_1 f(V^\star)\,\ell + \lambda_1 (1-u)\, g(V^\star)}{\Psi(V^\star, \ell)} \cdot \frac{\partial V^\star}{\partial p}.$$

Substitution into (34) yields (using the definition of $\Psi(\cdot)$):

$$\frac{dV^\star}{dp} = -\frac{\delta + \lambda_1 \overline{F}(V^\star)}{\Psi(V^\star, \ell)} \cdot \frac{\partial V^\star}{\partial p}.$$

Showing that $\Psi(V^\star, \ell) \leq 0$ is therefore necessary and sufficient to prove the proposition (as $\partial V^\star / \partial p \geq 0$ from Lemma 1 in Appendix A). This is what we now do.

Writing (32) in steady-state, we obtain:

$$\frac{\partial^2 \Pi}{\partial \ell^2} \cdot \frac{\partial V^\star}{\partial p} = \frac{\partial^2 \Pi}{\partial p \partial \ell} \cdot \left(\frac{\partial V^\star}{\partial \ell}\right)^2 \cdot \frac{\lambda_1 f(V^\star)\,\ell + \lambda_1 (1-u)\, g(V^\star)}{r + 2\delta + 2\lambda_1 \overline{F}(V^\star)}.$$

Substitution into (30) (written in steady-state) yields:

$$\frac{\partial V^\star}{\partial p} \cdot \lambda_1 f(V^\star) \cdot \left[\frac{\partial \Pi}{\partial \ell} - V^\star\right] + \frac{\partial^2 \Pi}{\partial p \partial \ell} \cdot \left(\frac{\partial V^\star}{\partial \ell}\right)^2 \cdot \frac{\left[\lambda_1 f(V^\star)\,\ell + \lambda_1 (1-u)\, g(V^\star)\right]^2}{r + 2\delta + 2\lambda_1 \overline{F}(V^\star)} + (r - \rho) \frac{\partial V^\star}{\partial p}$$
$$= \frac{\partial^2 \Pi}{\partial \ell \partial p} \cdot \frac{\partial V^\star}{\partial \ell} \cdot \left[\lambda_1 f(V^\star)\,\ell + \lambda_1 (1-u)\, g(V^\star)\right]. \quad (35)$$

Now substituting (29) into (28) and time-differentiating leads to:

$$\frac{\partial}{\partial t} \frac{\partial^2 \Pi_t}{\partial p \partial \ell_t} = -1 + (r - \Psi_t) \frac{\partial^2 \Pi_t}{\partial p \partial \ell_t},$$

so that in steady state:

$$\frac{\partial^2 \Pi}{\partial p \partial \ell} = \frac{1}{r - \Psi}. \quad (36)$$

First note that from Lemma 1, which establishes that $\partial^2 \Pi_t / \partial p \partial \ell_t > 0$ throughout in a dynamic equilibrium, the only consistent steady-state solution thus has $\Psi < r$. Then once again substituting into (35) and rearranging yields the following quadratic equation in Ψ:

$$\Psi^2 - \left[r + A(r + 2\Delta)\right]\Psi - \Delta(r + \Delta) + rA(r + 2\Delta) = 0, \quad (37)$$

where $\Delta := \delta + \lambda_1 \overline{F}(V^\star)$ and $A := \frac{\partial V^\star}{\partial p} \cdot \left(r - \rho + \lambda_1 f(V^\star) \cdot \left[\frac{\partial \Pi}{\partial \ell} - V^\star\right]\right)$. As $r \to 0$, (37) has one strictly positive and one strictly negative root (the product of which equals $-\Delta^2$), the consistent one being the latter from the remark after Equation (36) above. Because all the coefficients of (37) are continuous functions of r, so are its roots, which proves the proposition. □