Optimal retirement benefit systems in the presence of moral hazard

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October 2006

Abstract

Several countries that switched to funded private account retirement systems opted to complement such systems with explicit guarantees to retirees and agents saving for retirement. The motivation was that a social insurance system should provide a minimum standard of living in retirement. This paper studies the optimal design of a social insurance system that aims to provide retirees with a minimum standard of living in retirement. Particular attention is paid to moral hazard, i.e. the incentive to take more risk once the guarantees are in place. Surprisingly, the simple policy of giving agents a fixed annuity in retirement is shown to be an optimal policy in the baseline model.

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*Contact: panageas@wharton.upenn.edu. I would like to thank Andrew Abel, Patrick Kehoe, Naryana Kocherlakota, Nicholas Souleles and participants of seminars at Wharton, the Minneapolis FED, the Philadelphia FED, and the University of Lausanne for helpful comments and discussions. Any errors are mine.

1 Introduction

In the last few decades several countries around the world have experienced dramatic changes in the ways their citizens prepare for retirement. Many countries with substantially different economic structures and histories like Australia, Chile, Mexico and Sweden have replaced mostly unfunded with funded retirement systems[1]. Even in many countries where such changes have not occurred, there has been intense political debate on proposals to switch from unfunded to funded retirement systems. Under such systems individuals are able to invest their savings for retirement in a portfolio of stocks and bonds of their choosing. The switch from defined benefit to defined contribution plans in the private sector has had similar effects.

These trends imply an increased importance of financial markets and especially of the stock market for retirement savings. A common argument given by proponents of the above mentioned changes is that the rate of return for retirement funds in the stock market can be substantially higher than the rate of return in a pay as you go system -especially in the presence of global aging. Additionally, agents can use the added flexibility to better tailor their portfolio to their own preferences.

A risk that is recognized uniformly by proponents and opponents of the move towards retirement systems that are based on the principles of private choice and full funding is that they are too exposed to market risk: A downturn in the stock market could result in pressures to “bailout” the retirees and the workers close to retirement who have to rely on financial markets which are notoriously volatile.

A commonly proposed remedy for this problem is to offer explicit government guarantees to current and prospective retirees: Variants of such proposals call for the government to explicitly guarantee a minimum level of retirement income, or a minimum return on retirement assets or returns on a benchmark portfolio. Needless to say, this raises the concern of moral hazard. As Becker (2005) put it in the context of the US discussion on privatizing social security:

> As in Chile and other countries with private retirement accounts, the government would guarantee retirees a minimum income - similar to, but larger than, the present minimum Social Security guarantee. Unfortunately, such guarantees create a “moral

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[1] Mitchell and Lachance (2003) report that more than 20 countries have established individual accounts.

> hazard" - that is savers may want to make risky investments that give high payoffs if they succeed because the government partly bails them out. Or they may not save at all. [Becker (2005)]

Despite this concern with moral hazard, a very large number of countries that have made the switch to a fully funded system have accompanied this switch with explicit guarantees to retirees. For instance, in Chile retirees are guaranteed a minimum level of retirement income, irrespective of the level of their funds or their withdrawals[2]. The idea to include such guarantees was also present in the discussion to privatize the U.S. social security system[3]. Presumably, such guarantees are so popular for the same reason that led to the very appearance of social security systems in Europe 130 years ago: namely to provide a minimum standard of living to people who cannot rely on their labor income any longer. Such retirement benefit guarantees are intended to achieve precisely this goal, while obtaining the obvious benefits of a fully funded retirement system with individual choice.

Accepting the (political) necessity of such complements to fully funded systems leads to a host of questions: What form should they take? Should the government guarantee a minimum income, a minimum rate of return, or a minimum level of assets upon entering retirement? If individuals reduce their savings and increase the risk in their portfolios in response to such guarantees, how successful are such guarantees likely to be in achieving the stated goal? How large is the cost of such guarantees likely to be and who will finance them?

The existing literature has addressed mostly issues related to the cost and the financing of such guarantees[4]. This research has increased our understanding of the quantitative magnitude of such guarantees. However, two important issues have not been addressed yet:

a) First, the existing literature has not developed a normative theory, in order to guide the choice of one type of guarantee over another. Different countries have taken different approaches

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[2] See e.g. Pennacchi (1999) for a description of the Chilean and other guarantees in various Latin American countries.

[3] See e.g. Feldstein (2005b), Feldstein (2005a), Feldstein and Ranguelova (2001). See also the study by the Congressional Budget Office on the cost of such guarantees. [Sinclair, Lucas, Rehder-Harris, Simpson, and Topoleski (2006)]

[4] See Feldstein and Ranguelova (2001), Feldstein (2005a), Mitchell and Lachance (2003), Constantinides, Donaldson, and Mehra (2002), Smetters (2001), Pennacchi (1999), Sinclair, Lucas, Rehder-Harris, Simpson, and Topoleski (2006).

by guaranteeing retirement income, returns on assets, etc. This of course raises the question of finding an optimal way of achieving a minimum standard of living in retirement. Which magnitude (retirement income, assets, returns) should be guaranteed and to what extent?

b) Second and more importantly, existing papers do not take into account the distortions that would be introduced by government guarantees. Even though the importance of this issue is recognized throughout the literature[5], typically the guarantees are priced assuming that individual behavior is not going to be significantly affected by the presence of the guarantee. However, as a matter of theory, it is not clear why the presence of guarantees would have a negligible effect on individual behavior. Actually one might expect the opposite: The seminal paper by Bodie, Merton, and Samuelson (1992) illustrates that if an agent can expect to receive some income in the future, the agent's portfolio will contain a component that will perfectly offset variations in the net present value of that income. Viewing guarantees as an anticipated source of income and taking the Bodie, Merton, and Samuelson (1992) argument to its logical conclusion, should lead one to conjecture that such guarantees will be perfectly offset by the agent's portfolio choice.

In this paper the goal is to address these issues and develop an optimality theory of guarantees in the presence of moral hazard.

To motivate the need for guarantees, I assume the presence of an externality arising from bailout pressures of retired agents, whenever their standard of living drops below a given level. Working agents anticipate such pressures and divert productive resources to "protecting" their assets and their income from the government. From the perspective of the government this diversion of productive resources towards a privately rational, but socially useless activity is viewed as an inefficiency. In order to correct it, it can provide retirees with optimally designed pre-funded guarantees, that will ensure that the standard of living of retirees will stay above a minimum.

These considerations form the departure point for the rest of the analysis: In a next step, the paper considers a fairly general set of potential government guarantees and searches for an optimal one. Importantly, the analysis recognizes explicitly that agents' behavior will be affected by the presence of government intervention, and this is taken into account when designing optimal guarantees and determining their cost.

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[5] As Feldstein (2005b) points out in his presidential address "Social Insurance programs generally involve a tradeoff of preotection and distortion".

In particular, the paper considers a standard consumption and portfolio choice problem in the presence of government transfers to the agent. To avoid the perfect "offsetting" of the government transfers by increased risk taking, the model imposes a borrowing constraint, by requiring that the agent's financial wealth always stay non-negative[6]. Since the model has a fully funded system as a "backdrop" and a normative character, the guarantees are fully funded by raising appropriate taxes on the agent while she is working[7]. The analysis establishes an upper bound on the welfare that any set of transfers can attain, and illustrates that there exist multiple government policies that are optimal. Interestingly, the simplest conceivable policy of just transferring a constant income stream to the agent in retirement is optimal. However, an appropriate type of portfolio insurance policy that guarantees a minimum return on the agent's retirement portfolio -after some cumulated losses- is also optimal. A noteworthy implication of the analysis is that these two policies are equivalent from a welfare perspective, even though they imply completely different costs. This surprising finding implies that just focusing on the cost of guarantees may be misleading for welfare comparisons.

An additional outcome of the analysis, which is of practical importance, is that it derives *explicitly* a minimum level of funds that need to be available when entering retirement, if there is to exist *any* post-retirement set of transfers that will "keep" the agent's post-retirement consumption above the required minimum level. This helps one compute the income taxes that would have to be levied on the agent while she is working in order to ensure that such guarantees can be prefunded. It is also shown that the presence of moral hazard will tend to raise the magnitude of these minimum assets.

Simple closed form solutions are given for all quantities. An interesting result is that the presence of moral hazard will tend to substantially magnify the amount of transfers that are required to ensure a minimum standard of living.

In summary, this paper lends support to the view expressed by Feldstein (2005b) that by combining elements of a fully funded defined contribution system with some explicit guarantees can achieve the goal of making the retirement system robust to market downturns, even when one *takes account of the "moral hazard" effects.* The model presented here suggests that there may

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[6] It is also shown that this constraint arises endogenously as long as the government outlaws securitization of future government transfers.

[7] This is in contrast to e.g. Smetters (2001) where guarantees are unfunded, i.e. pay when needed.

be many equivalent ways to achieve the goal of a minimum standard of living in retirement and just looking at their costs can be misleading. Under certain assumptions, it also suggests that a particularly simple and optimal way of achieving the stated goal is to introduce a minimum constant "defined benefit" feature next to a purely privatized "defined contribution" system.

The paper is structured as follows. Section 2 sets up the model and the lays out the reasons for government intervention. Section 3 introduces a government with the task of keeping the agent's consumption above a minimum level by usage of appropriate taxes and transfers. Section 4 considers the agent's reaction to the presence of such intervention. Section 5 derives an upper bound to welfare (which coincides with the government objective function) no matter which set of admissible taxes/transfers is utilized. Section 6 illustrates two distinct ways of attaining that upper bound, which are hence optimal. Section 7 discusses the cost involved in these transfer schemes and identifies the lowest amount of funds that need to be available in order to achieve the stated goal of guaranteeing a minimum level of retirement consumption. Section 8 discusses extensions to heterogenous incomes and arbitrary stochastic discount factors. Section 9 concludes.

2 The model

2.1 Agents, preferences, and endowments

The model is very similar to the small open economy version of Blanchard (1985). At each point in time there is a continuum of agents of mass 1 who are alive. Agents face a constant probability of death q per unit of time dt. There is also constant arrival of new agents through birth at the rate of q per unit of time dt.

All agents are identical. They have constant relative risk aversion γ, and a constant discount rate ρ. Accordingly, each agent who is born at time t aims to maximize

$$F = E_t \int_t^{\infty} e^{-(\rho+q)(s-t)} \frac{(c_s)^{1-\gamma}}{1-\gamma} ds \qquad (1)$$

Once born, agents are endowed with a non-tradeable tree that produces a constant income stream of Y per unit of time dt. That tree stops delivering any fruit (and hence its output becomes $Y = 0$) after a duration of time equal to T. After this point of time the agent can only rely on her assets to sustain herself. Agents will be referred to as “workers” while their trees are producing fruit and will be referred to as “retirees” once their tree stops producing any further income.

2.2 Investment opportunity set

Agents can invest in a riskless and a risky asset. (The extension to multiple risky assets is straightforward). As is quite standard in the literature that studies incentive problems between a central planner and agents, I will fix the rates of return that agents can earn when accessing financial markets. Alternatively put, I will consider a small open economy and accordingly fix the stochastic discount factor. Section 8.2 shows how the results can be extended to setups where the stochastic discount factor is arbitrary.

In particular, I will assume that agents can invest in the money market, where they receives a constant strictly positive interest rate $r > 0$. In addition they can invest in a risky security with a price per share that evolves as

$$\frac{dP_t}{P_t} = \mu dt + \sigma dB_t$$

where $\mu > r$ and $\sigma > 0$ are given constants and B_t is a one-dimensional Brownian motion on a complete probability space (Ω, F, P).[8] The realization of this Brownian motion will be the only source of uncertainty in this economy.

As is well understood, dynamic trading in the stock and the bond leads to a dynamically complete market. (See e.g. Duffie (2001) or Karatzas and Shreve (1998)). As Karatzas and Shreve (1998) show, the assumptions of a constant interest rate and risk premium imply the existence of a unique stochastic discount factor (or state price density) which is given by:

$$H(t) = \exp\left\{-\int_0^t \kappa dB_s - rt - \frac{1}{2}\kappa^2 t\right\}, \quad H(0) = 1 \tag{2}$$

where κ is the Sharpe ratio, defined as

$$\kappa = \frac{\mu - r}{\sigma}$$

Using this stochastic discount factor, the no-arbitrage price of any claim that delivers dividends

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[8] I shall denote by $F = \{F_t\}$ the P-augmentation of the filtration generated by B.

equal to D_s is given by[9]:

$$E_t \int_t^{\infty} \frac{H_s}{H_t} D_s ds$$

The agent can also enter into contracts with a competitive life insurance company as in Blanchard (1985). As Blanchard (1985) I shall assume that the agent's hazard rate of death is a constant q, so that the insurance company can offer the agent an income stream of q per unit of time dt, in exchange for receiving one dollar when the agent dies.

2.3 Portfolio and wealth processes

An agent chooses a portfolio process π_t and a consumption process c_t. The portfolio process π_t is the *dollar amount* invested in the risky asset (the "stock market") at time t. The rest, $W_t - \pi_t$, is invested in the money market. Short selling and borrowing are both allowed. The agent has no bequest motives. As Blanchard (1985) shows, it is optimal in this case for the agent to enter an annuity contract: The agent receives from the insurance company an income stream of qW_t per unit of time dt while she is alive. In exchange, the entire remaining wealth of the agent gets transferred to the insurance company when the agent dies. Accordingly, the wealth process of a retired agent evolves as

$$dW_t = qW_t dt + \pi_t \{\mu dt + \sigma dB_t\} + \{W_t - \pi_t\} r dt + dt - c_t dt \tag{3}$$

and the wealth process of a working agent is given by:

$$dW_t = qW_t dt + \pi_t \{\mu dt + \sigma dB_t\} + \{W_t - \pi_t\} r dt + dt - c_t dt + Y dt$$

An additional requirement is that wealth must remain non-negative throughout:

$$W_t \geq 0 \text{ for all } t \geq 0 \tag{4}$$

This constraint excludes uncollateralized borrowing. The motivation for this constraint is given in section 3.

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[9] From a macroeconomic perspective one can also think of H_t as the marginal utility of consumption of the world-representative agent.

2.4 Redistribution and political pressures

This section introduces the reason for the existence of government guarantees. Since the aim of the paper is to take the need for such guarantees as given and discuss their optimal design, this section proposes an intentionally simple and concise framework for why such guarantees are needed.

In particular, let c_t^R denote the average consumption of surviving retirees who retired at times prior to t. I will assume that if c_t^R drops below a given level ξ, then there will be pressures to redistribute the resources of working agents (assets and remaining income from the tree) towards retirees[10].

Retirees can apply such pressures at no cost. However, working agents have the ability to protect their resources against such demands. Once born, they can choose between two types of trees. Trees of type "A" produce an income stream of Y. Trees of type "B" produce an income stream of $(1-x)Y$ where x is a number between zero and one. The advantage of type "B" trees is that they are surrounded by a "fence" that allows agents to fully protect their assets and the remaining output of the tree against redistribution demands. By contrast trees of type "A" provide no protection whatsoever: Whenever $c_t^R < \xi$ retirees can tax away the full assets and remaining income of agents who posses trees of type "A". One should think of the opportunity cost of a type "B" (namely the lost output xY) as the resources that are lost to tax evasion, distortions in labor choice, lobbying with politicians to prevent redistribution, etc. In extreme cases one should also think of increased security measures to prevent the peace in a society as part of this cost.

The simple assumptions of the previous paragraph will help in making an unambiguous prediction about the choice of a new born agent to plant a tree of type "B". Assuming a coefficient of relative risk aversion $\gamma \geq 1$, any positive probability that $c_t^R < \xi$ at some point t in the future, will be sufficient to make a working agent choose to protect her resources, by sacrificing a fraction x of the income Y. (Else she risks losing all of her resources -assets and remaining income- to redistribution, which is an outcome that yields negative infinity.) It is important to remark here, that

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[10] It seems reasonable to make such pressures contingent on consumption falling below a given level: If one set up a more elaborate model, where expressing political redistribution demands is associated with a fixed non-monetary cost (say the time spent protesting), then demands will be expressed whenever the marginal utility of consumption is high enough. This will in turn happen when consumption is sufficiently low. To keep the presentation concise, I take such an outcome as a primitive building block directly.

weaker assumptions on the potential for expropriation would also lead to the same conclusion, as long as a sufficiently large fraction of resources can be expropriated[11].

To conclude that all new-born agents would choose to plant trees of type "B", it remains to check whether there is a positive probability that average consumption in the retiree cohort can fall below the minimum level ξ. The answer is affirmative: In a Merton (1971) type setup, as the one underlying the present model, an individual retiree's consumption is given by[12]:

$$c_t^* = \left(\lambda_s e^{\rho(t-s)} \frac{H_t}{H_s} \right)^{-\frac{1}{\gamma}} \quad \text{for all } t > s \tag{5}$$

for a constant λ_s that depends on the agent's assets at the time $s < t$ that she enters retirement. Since $\left(\frac{H_t}{H_s}\right)^{-\frac{1}{\gamma}}$ follows a geometric Brownian motion, and the consumption processes of retirees are perfectly correlated, the probability that $c_t^R < \xi$ is non-zero.

To conclude, the fear of redistribution will make new-born agents choose trees of type "B". This raises then the question of whether the government could somehow intervene and safeguard that the consumption of retirees never falls below the level ξ in the first place. This is the main question of the paper: Namely how to design optimal transfer schemes that will keep the standard of living of retirees above ξ and hence will eliminate pressures for redistribution and their associated distortions.

3 Introducing a role for the government

Motivated by the observation that the savings and consumption choices of the representative retiree can present an externality to new-born agents, it is interesting to examine how a government can

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[11] The advantage of "full expropriation" is to just guarantee that it is always optimal to protect one's resources, whereas weaker assumptions would just require some additional parametric assumptions.

[12] For a derivation of this equation see Cox and Huang (1989), Karatzas, Lehoczky, and Shreve (1987), Karatzas and Shreve (1998) Chapter 3. The intuition behind this equation is that in a complete market an agent's marginal utility of consumption is proportional to the stochastic disount factor, adjusted by the probability of survival to time t $(e^{-q(t-s)})$:

$$e^{-(\rho+q)(s-t)} c_t^{-\gamma} = \lambda_s e^{-q(t-s)} \frac{H_t}{H_s}$$

fix this problem by providing the representative retiree with optimally chosen transfers that will keep her consumption above the minimum level ξ at all times.

The government's objective is to maximize the lifetime expected utility (1) of a new born agent (which clearly includes her utility once retired).[13] Since all agents are identical, the government offers each agent the same path of taxes and transfers over their lifetime.

The goal of the paper is to determine the optimal structure of post retirement transfers that will safeguard that

$$c_t \geq \xi \tag{6}$$

Since the backdrop of the paper is a fully funded system, these transfers will be funded by a lump sum tax on the agent once in retirement. Clearly, raising such a tax will only be feasible if the agent has accumulated a minimum amount of assets by the time she retires. Sections 3.1-7 addresses the optimal design of transfers and derives the minimum level of assets that are required for the existence of transfer processes that will safeguard (6).

Having determined the structure of optimal post retirement transfers, it is then possible to apply backwards induction and show that the only purpose of any government action prior to retirement is to make sure that an agent arrives at retirement with the endogenously derived minimum level of assets. Assuming that the government can raise a tax on an agent's income and place the resulting proceeds in an interesting bearing account, section 7 derives the tax rate that will ensure that the agent reaches retirement with enough assets that will allow funding of her post retirement guarantee.

Given that the time of retirement is central to the analysis, the paper will adopt the following timing convention. Since the setup is time-invariant, time 0 will be the time at which the representative agent retires. Hence the representative agent will be assumed to be born at time $-T$. Moreover, since all quantities depend on ratios of the stochastic discount factor between two points in time, I will take the value of the stochastic discount factor at time 0 to be equal to 1 without loss of generality.

3.1 Admissible government transfers

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[13] Therefore the paper will not discuss the transitional problems associated with switching from no government intervetion to some form of pre-funded government intervention

Post retirement the government can make transfers to the retiree. These transfers can be made contingent on quantities that are "exogenous" to the agent (i.e. the returns in the stock market). The government cannot directly observe (or at least verify) an individual agent's consumption, portfolio, or wealth process. Therefore, transfers cannot depend on these quantities.[14] This will form the source of the moral hazard problem, since the agent will need to be induced to choose a consumption process that satisfies (6) given the government's transfers. One exception is that the government will be assumed to know an agent's wealth upon entering retirement, namely W_0. This assumption will be relaxed later in the text.

The following definition formalizes the informational requirements:

Definition 1 *Let* $\widetilde{\mathcal{F}}_t$ *be the filtration generated by the brownian motion* B_t *and knowledge of the retirees assets at the time that she enters retirement* (W_0). *An admissible cumulative transfer process* G_t *is a non-decreasing, progressively measurable (with respect to* $\widetilde{\mathcal{F}}_t$*) process starting at* $G_0 = 0$ *and satisfying:*

$$E\int_0^{\infty} e^{-qs} H_s dG_s < \infty$$

With some abuse of mathematical precision, the non-negative increments of the process G_t, namely $dG_s \geq 0$, will be referred to as the "transfers" to the agent.

It is useful to discuss the requirements of Definition 1. The requirement that the process be non-decreasing and start at 0 captures the fact that G_t progressively adds all the positive transfers to the agent.

Progressive measurability with respect to $\widetilde{\mathcal{F}}_t$ is the requirement that captures "exogenous" information and knowledge of W_0. The government is assumed to observe the brownian path, and hence all the quantities that can depend on that brownian path (for instance the stock market). However, it cannot observe directly an agent's consumption choice, portfolio choice or wealth (except at time 0). It can at most *infer* these choices from its knowledge of the brownian path, along with its knowledge of an agent's optimizing behavior.

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[14] There is an analogy to standard principal agent models here. Just as in the standard principal agent model the principal cannot write contracts that are contingent on the agent's effort choice, here it is impossible to write contracts that depend directly on agent's consumption, portfolio choices or assets.

Given the assumption that G_t is progressively measurable with respect to $\mathcal{F}_t$, the fair value of a claim delivering the cumulative transfer process G_t is given by[15]:

$$L_t = E_t \int_t^{\infty} e^{-q(s-t)} \frac{H_s}{H_t} dG_s \tag{7}$$

Before formalizing the government's problem, it remains to discuss how these transfers will be financed. Given the "backdrop" of the analysis, which is a fully funded system, I will assume that these transfers are funded by levying a lump sum tax D_0 on the agent at the time of retirement (time 0) , where[16]:

$$D_0 = E_0 \int_0^{\infty} e^{-qs} H_t dG_t.$$

An alternative interpretation of this setup is that the government passes a law that requires the agent to buy a contract that offers the payoffs dG_t from competitive financial firms. The competitive financiers will charge D_0 for such a contract.

Finally, I will require that:

$$W_0 \geq \frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} K\xi \tag{8}$$

where

$$\phi = \frac{-\left(\rho - r - \frac{\kappa^2}{2}\right) + \sqrt{\left(\rho - r - \frac{\kappa^2}{2}\right)^2 + 2\left(\rho + q\right)\kappa^2}}{\kappa^2} \tag{9}$$

and

$$K = \frac{\gamma}{\frac{\gamma-1}{\gamma}\frac{\kappa^2}{2} + \gamma\left(r + q\right) + \left(\rho - r\right)} \tag{10}$$

A full discussion of condition (8) will wait until section 7. For now I just remark, that without this condition there would be no combination of D_0, G_t that will safeguard $c_t \geq \xi$. It will also turn out to be the case, that this level of assets will help in determining the minimum amount of savings that the government needs to enforce prior to retirement so as to ensure that the subsequent transfers to the retiree are pre-funded.

It is now possible formulate the government's objective.

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[15] This is a consequence of the martingale representation theorem.

[16] An alternative way of writing this equation is as :$L_0 = D_0$, since $H_0 = 1$.

Problem 1 *Assuming (8), the government's objective is to determine an admissible cumulative transfer process G_t and an initial tax D_0 so as to maximize:*

$$V = \max_{G_t, D_0} E_0 \int_0^\infty e^{-(\rho+q)s} \frac{c_s^{1-\gamma}}{1-\gamma} ds \tag{11}$$

subject to

$$\begin{aligned} c_t &\geq \xi \text{ for all } t > 0 & (12) \\ D_0 &= E_0 \int_0^\infty e^{-qt} H_t dG_t & (13) \end{aligned}$$

and subject to the constraint that c_t solves the agent's optimization problem given G_t

$$\begin{aligned} c_t &= \arg \max_{<c_t, \pi_t>} E_t \int_t^\infty e^{-(\rho+q)(s-t)} \frac{c_s^{1-\gamma}}{1-\gamma} ds & (14) \\ \textit{subject to} &: \\ dW_t &= qW_t + \pi_t \{\mu dt + \sigma dB_t\} + \{W_t - \pi_t\} r dt - c_t dt + dG_t & (15) \\ W_{0+} &= W_0 - D_0 & (16) \\ W_t &\geq 0 \text{ for all } t > 0 & (17) \end{aligned}$$

There are several remarks on the above setup: As can be seen from equation (11), the government's objective is to maximize a representative agent's welfare in retirement subject to certain requirements. This section focuses on the post retirement value function exclusively. Section 7.2 employs backwards induction to relate the solution to problem 1 to the government's ultimate goal, which is to maximize the lifetime expected utility of a new born agent.

Equation (12) captures the requirement that transfers should induce a consumption process that keeps consumption above the level ξ in order to avoid the assumed negative externalities associated with redistribution demands. Equation (13) requires that the guarantee given to the agent should be financed by the tax raised upon entering retirement.

Equations (14)-(17) capture the requirement, that the consumption process be optimal from the consumer's perspective taking *the governmental taxes and transfers as given.* Equation (14) is the consumer's objective, while equation (15) presents the wealth evolution equation, taking into account the presence of transfers. Equation (16) states that the consumer's financial assets W_0 will be reduced by the tax D_0, so that the agent's post tax assets are given by W_{0+}.

Equation (17) requires that assets be non-negative at all times. I shall refer to this constraint as the borrowing constraint and it will play a key role in this paper. In practical terms, this constraint implies that the agent has no ability to borrow against future transfers, by -say- securitizing them[17].

For the purposes of this paper, equation (17) is the key constraint of the analysis. Without this constraint, it would be impossible for the government to find a set of taxes and transfers that would induce the consumer to choose a consumption path that satisfies (12). The reason is that the magnitude of the tax D_0 raised at time $t = 0$ is exactly equal to the expected net present value of the government's transfers to the agent. If the agent was unconstrained in her ability to transfer resources between dates and states, she could completely "undo" the effects of these taxes and transfers by appropriate trading strategies.

The easiest way to see this is to ignore the constraint (17) momentarily. In this case the agent is facing dynamically complete markets, so that her consumption choices are constrained by a single intertemporal budget constraint[18]:

$$E_0\left(\int_0^\infty e^{-qs}H_s c_s ds\right) \leq W_0 - D_0 + E_0\left(\int_0^\infty e^{-qs}H_s dG_s\right) \tag{18}$$

Using (13), it is immediate that the above equation reduces to

$$E_0\left(\int_0^\infty e^{-qs}H_s c_s ds\right) \leq W_0$$

which is the intertemporal budget constraint in the complete absence of taxes and transfers. Alternatively put, in the absence of a borrowing constraint, the agent's feasible consumption choices would be unchanged by the presence of taxes and transfers, and so would her optimal consumption choices. Therefore, the government's tax and transfers would have no effect on the agent's consumption choices. This result is a manifestation of the well understood principle of Ricardian Equivalence[19]. As long as the agent who is taxed is the same agent that receives the future transfers *and markets are dynamically complete*, Ricardian Equivalence asserts that government intervention will have no effects.

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[17] This seems plausible, as long as the government can outlaw such securitization. But if the government outlaws such securitization, then the only way the agent could borrow against this future income would be by reputation (unsecured lending). However, as is well known from the seminal Bulow and Rogoff (1989) paper, unsecured lending based on reputation cannot be supported. To conclude, as long as the government can outlaw securitization of these transfers, the constraint (17) results naturally from the results in Bulow and Rogoff (1989).

[18] See e.g. Cox and Huang (1989), or Karatzas, Lehoczky, and Shreve (1987).

[19] See e.g. Abel (2003)

The presence of a borrowing constraint such as (17), however, makes taxes and transfers non-neutral. The reason is that a borrowing constraint implies stronger restrictions than (18) on the agent's feasible consumption choices. Hence, by a judicious choice of an initial tax and subsequent transfers, the government can affect the agent's consumption.

Sections 4-7 are devoted to the study of problem 1.

4 The agent's consumption choices in the presence of government intervention and borrowing constraints

To solve problem 1 it is instructive to take an intermediate step: This section examines how different forms of transfers will affect the agent's optimal consumption choices.

Specifically, suppose that at the time that the agent enters retirement (time 0) the government taxes her by an amount D_0 and then promises an admissible cumulative transfer process G_t. It is now natural to ask how the agent's consumption choices will be affected by this intervention in the presence of the constraint (17). The following result shows how to obtain the optimal consumption process in this case and is due to He and Pages (1993):

Proposition 1 *Let $\mathcal{D}$ be the set of non-increasing, non-negative and progressively measurable processes that start at $X(0) = 1$. Then, the consumer's value function $V(W_0)$ can be expressed as:*

$$V(W_0) = \min_{\lambda>0,\ X_s\in\mathcal{D}} \left[\begin{array}{c} E\left(\int_0^\infty e^{-(\rho+q)s}\max_{c_s}\left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s}H_sX_sc_s\right)ds + \lambda\int_0^\infty e^{-qs}H_sX_s dG_s\right) \\ +\lambda\left(W_0 - D_0\right) \end{array} \right] \tag{19}$$

Let X_t^*, λ^* denote the process X_t and the constant λ that minimize the above expression. Then the optimal consumption process c_t^* for a consumer faced with the borrowing constraint (17) is:

$$c_t^* = \left(\lambda^* e^{\rho t} H_t X_t^*\right)^{-\frac{1}{\gamma}} \text{ for all } t > 0 \tag{20}$$

Moreover, the process X_t^* decreases only when the associated wealth process (W_t) falls to zero and is otherwise constant, i.e.:

$$\int_0^\infty W_t dX_t^* = 0 \tag{21}$$

Finally, the resulting wealth process for any $t > 0$ satisfies:

$$W_t = \frac{E_t\left(\int_t^\infty e^{-q(s-t)} X_s^* H_s c_s^* ds\right)}{X_t^* H_t} - \frac{E_t\left(\int_t^\infty e^{-q(s-t)} X_s^* H_s dG_s\right)}{X_t^* H_t} \tag{22}$$

Interestingly, comparing equations (20) and (5) reveals a striking similarity between the versions of the consumption /portfolio problem with and without government intervention. The optimal consumption process has the same structure except that H_t is replaced by $H_t X_t^*$. Given that the sole difference between the two problems is the presence of a transfer process in the presence of borrowing constraints, one can interpret X_t^* as the Lagrange multiplier process associated with the borrowing constraint: Recall that in simple one-dimensional maximization problems, a Lagrange multiplier helps convert a constrained problem into an unconstrained by appropriately altering the "shadow" prices. By analogy, here the process X_t^* allows one to transform the problem of an agent faced with borrowing constraints in an economy where the stochastic discount factor is H_t into the problem of an agent who is unconstrained, but instead faces the stochastic discount factor $H_t X_t^*$.

5 Government transfers and their welfare effects: an upper bound

Turning to problem 1, Proposition 1 gives an intuitive way to summarize the effects of the incentive compatibility requirement (equations [14]-[17]).

It asserts that every government transfer process G_t will be associated with a constant λ^G and a Lagrange multiplier process X^G resulting from the minimization problem in equation (19). Given this duality between a choice of G_t and the resulting pair $\left(\lambda^G, X_t^G\right)$, there is a straightforward way to obtain an upper bound to problem 1. In particular consider the following problem:

Problem 2 *Maximize:*

$$J(W_0) = \max_{c_t, X_t \in \mathcal{D}, \lambda > 0} E_0 \int_0^\infty e^{-(\rho+q)s} \frac{c_s^{1-\gamma}}{1-\gamma} ds \tag{23}$$

subject to:

$$\begin{aligned} E_0\left(\int_0^\infty e^{-qs} H_s c_s ds\right) &\leq W_0 && (24)\\ c_t &\geq \xi && (25)\\ c_t &= \left(\lambda e^{\rho t} H_t X_t\right)^{-\frac{1}{\gamma}} && (26) \end{aligned}$$

Problem 2 is the problem of a central planner who can choose directly the consumption of the agent, subject to an intertemporal budget constraint, a constraint on the minimum consumption level (equation [25]) and the additional requirement that any chosen consumption process should have a representation in the form of equation (26).

In effect, problem 2 allows the central planner to choose freely the pair (λ, X_t) without being concerned whether there exist any optimal transfer process G_t that will make (λ, X_t) the optimal solution of the minimization problem in (19).

It is reasonable to conjecture as a result, that problem 2 is a "relaxed" version of the optimization problem 1 and hence the value function of 2 dominates the value function of problem 1. The next proposition proves this assertion and determines the solution of problem 2:

Proposition 2 *For any* $\lambda > 0$, *let the process* $X_t^*(\lambda)$ *be defined as*

$$X_t^* = \min\left[1, \frac{\xi^{-\gamma}/\lambda}{\max_{0\leq s\leq t}\left(e^{\rho s}H_s\right)}\right] \tag{27}$$

[*To simplify notation,,* X_t^* *will be used as a shorthand for* $X_t^*(\lambda)$]. *Assuming (8), the value function of problem (2) is given by:*

$$J(W_0) =$$

$$= \min_{\lambda\geq 0}\left[E\left(\int_0^\infty e^{-(\rho+q)s}\frac{\left(\lambda e^{\rho s}H_sX_s^*\right)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda\int_0^\infty e^{-qs}H_s\left(\lambda e^{\rho s}H_sX_s^*\right)^{-\frac{1}{\gamma}} + \lambda W_0\right)\right] \tag{28}$$

$$= \min_{\lambda\geq 0}\left[-\frac{K\xi^{1-\gamma}}{\gamma\phi(\phi-1)}\left(\frac{\lambda}{\xi^{-\gamma}}\right)^{\phi} + K\frac{\gamma}{1-\gamma}\lambda^{1-\frac{1}{\gamma}} + \lambda W_0\right] \tag{29}$$

where K *is given in (10) and* ϕ *is given in (9). Letting* λ^* *be the scalar that minimizes (29), the optimal triplet that solves problem (2) is given by* λ^*, $X_t^*(\lambda^*)$ *,and the* c_t *that is implied by equation (26) for* $\lambda^*, X_t^*(\lambda^*)$ *.*

Finally, let $\mathcal{G}$ *be the class of all admissible transfer processes that lead to a consumption process such that (12) is satisfied and let* $V(W_0)$ *be given as in equation (19). Then the following upper bound characterizes the value function of problem 1:*

$$\max_{G_t\in\mathcal{G}} V(W_0) \leq J(W_0) \tag{30}$$

Proposition 2 illustrates that there exists an upper bound to the value function of the original problem 1 against which one can measure any government policy. At a practical level, the usefulness of Proposition 2 is to provide a very simple test for the optimality of an admissible transfer process G_t: As long as an admissible process G_t attains the upper bound given by proposition 2, such a process must be optimal.

To gain some intuition on Proposition 2, it is useful to ask why the process X_t^* is optimal for problem 2. Taking logs in equation (26) and subtracting $\log\xi$ on both sides gives:

$$\log c_t - \log\xi = -\frac{1}{\gamma}\left[\log\lambda + \rho t + \log H_t + \log X_t\right] - \log\xi \tag{31}$$

The above equation implies that $c_t \geq \xi$ if and only if $\log c_t - \log\xi \geq 0$. Hence the constraint (12) will be satisfied if and only if there exists a process $X_t \in \mathcal{D}$ that will safeguard that the right hand side of (31) will always be non-negative. The process of equation (27) does have this property[20]. Moreover, this process has an additional property. It is the *largest* X_t^* that will satisfy (31).[21]

In practical terms this means that among all decreasing processes that will enforce the requirement (25), X_t^* is the process that will minimize the difference between the consumer's optimal consumption in the absence of government intervention (equation [5]) and in its presence (equation [20]) . Hence it achieves the goal of having $c_t \geq \xi$ while distorting the consumer's consumption choices as little as possible.

6 Optimal Transfer Processes

This section illustrates two distinct policies that can attain the upper bound of (29).

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[20] To see this, consider two cases. The first case is $\lambda \max_{0\leq s\leq t} e^{\rho s} H_s \leq \xi^{-\gamma}$. In that case $X_t^* = 1$ and accordingly

$$\lambda e^{\rho t} H_t X_t^* = \lambda e^{\rho t} H_t \leq \lambda \max_{0\leq s\leq t} e^{\rho s} H_s \leq \xi^{-\gamma}$$

The second case is $\lambda \max_{0\leq s\leq t} e^{\rho s} H_s \geq \xi^{-\gamma}$. Then $X_t^* = \frac{\xi^{-\gamma}/\lambda}{\max_{0\leq s\leq t}(e^{\rho s} H_s)}$ and accordingly

$$\lambda e^{\rho t} H_t X_t^* = \lambda e^{\rho t} H_t \frac{\xi^{-\gamma}/\lambda}{\max_{0\leq s\leq t}\left(e^{\rho s} H_s\right)} = \xi^{-\gamma} \frac{e^{\rho t} H_t}{\max_{0\leq s\leq t}\left(e^{\rho s} H_s\right)} \leq \xi^{-\gamma}$$

Hence $\lambda e^{\rho t} H_t X_t^* \leq \xi^{-\gamma}$ as asserted.

[21] This is a consequence of the Skorohod Equation. For a reference on the Skorohod equation see e.g. Karatzas and Shreve (1991).

6.1 A constant income stream

The simplest form of government transfer process is a constant income stream: The government collects an initial tax of $D_0 = \frac{y_0}{r+q}$ and in exchange it delivers a constant stream of y_0 in annuity. Such a simple policy turns out to be optimal as long as y_0 is chosen appropriately.

The following Proposition illustrates this fact:

Proposition 3 *Let y_0 be given as:*

$$y_0 = (r+q)\,K\xi\left(\frac{\frac{1}{\gamma}+\phi-1}{\phi-1}\right) \tag{32}$$

where K is given in (10) and ϕ is given in (9). Assume that the government raises an initial tax of $D_0 = \frac{y_0}{r+q}$ and promises a constant stream of payments equal to y_0 in annuity. Then, the agent will choose a consumption path that satisfies $c_t \geq \xi$ and her value function will attain the upper bound given in Proposition 2.

This is a somewhat surprising result. It asserts that by simply promising the agent a constant benefit forever, one can attain the upper bound of Proposition 2. Moreover, there is a simple closed form solution for y_0 depending solely on the parameters.

One can decompose $\frac{y_0}{\xi}$ into two components:

$$\frac{y_0}{\xi} = \underbrace{(r+q)\,K}_{\text{cost with exclusion}}\ \underbrace{\left(\frac{\frac{1}{\gamma}+\phi-1}{\phi-1}\right)}_{\text{Cost of Moral Hazard}} \tag{33}$$

To understand the first term, it is useful to examine the constant K first. This constant is (up to an adjustment for the probability of death q) the wealth to consumption ratio in the Merton (1971) model. A thought experiment is useful in order to further explore this term: Suppose that the government promises an income stream of $\overline{y}$ to an agent who is not subject to the borrowing constraint (17). Assuming that this agent has 0 financial wealth, her consumption will be:

$$c_t = \frac{1}{K}\frac{\overline{y}}{r+q}$$

The term $\frac{\overline{y}}{r+q}$ is the net present value of the promised income stream and $\frac{1}{K}$ is the consumption to wealth ratio in a Merton model. Moreover, if the government wants to make sure that $c_t = \xi$, then $\overline{y}$ needs to be set as $(r+q)\,K\xi$. If the government could *completely and unexpectedly* exclude agents from financial markets from this point on, without making them experience a consumption drop then $\overline{y} = (r+q)\, K\xi$ for all future times. This motivates the term “cost with exclusion”.

The second term in (33) is due to the fact that the government cannot undertake such sudden exclusions from financial markets, and instead has to cope with the fact that agents will take more risks due to the guarantee. There are several interesting remarks about the second term. First:

$$\frac{\frac{1}{\gamma}+\phi-1}{\phi-1} > 1$$

since $\phi > 1$. Second, this ratio has intuitive properties: For instance, increased risk aversion (γ) will reduce the amount of y_0 that needs to be promised to the agent in order to make sure that $c_t \geq \xi$. This is intuitive: The larger is the risk aversion of the agent, the less risks she will take in the stock market, and the cheaper it is to insure her against adverse consumption variation. In the limit, as $\gamma \rightarrow \infty$, the second term converges to 1, and hence the effect of moral hazard vanishes. It can also be shown that the second term increases as the Sharpe ratio in the market increases. This is equally intuitive: A higher Sharpe ratio will incentivize the agent to invest in the stock market and it will become more expensive to insure a minimum consumption level.

Figure 1 gives a quantitative assessment of the components that enter the ratio of guaranteed income to minimum consumption (y_0/ξ). The figure shows that if the government wants to ensure a minimum consumption of one dollar, it needs to deliver more than one dollar in guaranteed income. What drives this ratio above one is mostly the cost of moral hazard, as can be seen by the decomposition of y_0/ξ into the two components given in (33).

The fact that a constant income policy can attain the upper bound of Proposition 2 is both reassuring and surprising: The constant income policy has very low informational requirements. The government can implement this policy without even knowing the realization of the brownian paths, or the exact magnitude of the agents’ assets at time 0. Hence, even though the model setup gives the government the ability to observe the stochastic discount factor and the agent’s assets at time 0, this simple “constant income” policy is optimal despite the fact that it doesn’t exploit this information. From a practical perspective, the policy has the additional advantage that it is very simple.

The constant income policy is not the unique optimal policy however. The next section presents an alternative approach to achieving the upper bound in (29).

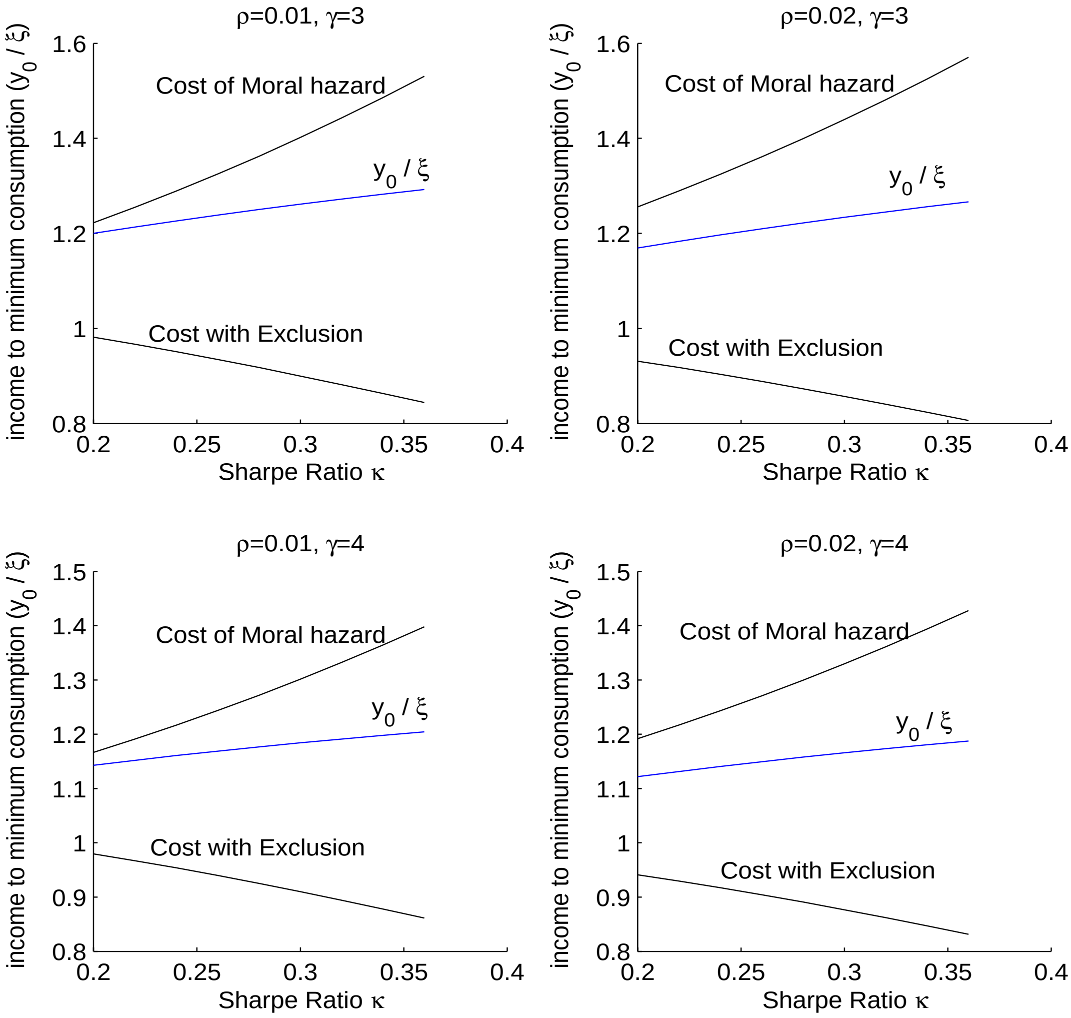

FIGURE 1: The ratio of guaranteed income to the minimum level of guaranteed consumption $\left(\frac{y_0}{\xi}\right)$ and its components. The product of the lines "Cost of Moral hazard" and "Cost with Exclusion" are equal to $\frac{y_0}{\xi}$. The Sharpe ratio used in the calibrations is reported on the x-axis. The preference parameters are given at the top of each figure. The probability of death q is fixed at 0.04 in all figures and the interest rate is equal to 0.02.

6.2 Portfolio Insurance

The above approach to attaining the upper bound of Proposition 2 is not unique. The approach presented in this section also succeeds in attaining the same upper bound. To describe this approach, let λ^* be the scalar that minimizes (29). Then define the government's transfer process as:

$$dG_t = -\left(\frac{1}{\gamma} + \phi - 1\right) K\xi \frac{dX_t^*}{X_t^*} \tag{34}$$

where $X_t^* (\lambda^*)$ is the process defined in (27).[22]

This section shows the following results:

a) The process (34) attains the upper bound of Proposition 2.

b) The process (34) has a very intuitive economic interpretation. In particular, the process (34) represents a type of minimum return guarantee (portfolio insurance) on the agent's optimal portfolio of stocks and bonds.

The following proposition formalizes the first claim:

Proposition 4 *Let λ^* be the scalar that minimizes (29) and X_t^* be the process that is given in (27). Consider an agent who anticipates transfers given by (34) and faced with an initial tax of D_0, where D_0 satisfies (13). Then*

a) her value function will coincide with the upper bound given in (29)

b) That agent will invest

$$\pi_t = \frac{\kappa}{\sigma} K\xi \left[(\phi - 1) \left(\frac{Z_t}{\xi^{-\gamma}} \right)^{\phi - 1} + \frac{1}{\gamma} \left(\frac{Z_t}{\xi^{-\gamma}} \right)^{-\frac{1}{\gamma}} \right] \tag{35}$$

dollars *in the stock market and consume*

$$c_t = Z_t^{-\frac{1}{\gamma}} \tag{36}$$

where:

$$Z_t = \lambda^* e^{\rho s} H_s X_s^* \tag{37}$$

---

[22] Note that $\eta < 0$ since $dX_t^* < 0, X_t^* > 0$ and hence $dG_t > 0$.

The agent's optimal wealth process W_t will be given by:

$$W_t = -K\left(\xi^{-\gamma}\right)^{-\frac{1}{\gamma}}\left(\frac{Z_t}{\xi^{-\gamma}}\right)^{\phi-1} + KZ_t^{-\frac{1}{\gamma}} \tag{38}$$

c) The initial tax D_0 associated with (34) is given by:

$$D_0 = K\xi\frac{\frac{1}{\gamma}+\phi-1}{\phi-1}\left(\frac{\lambda^*}{\xi^{-\gamma}}\right)^{\phi-1}$$

The portfolio policy (35) will aid in the interpretation of (34) as a form of portfolio insurance. To obtain some intuition on the nature of (34), consider first the following puzzling feature of (35): As $Z_t \to \xi^{-\gamma}$, equation (38) implies that $W_t \to 0$ whereas the portfolio of the agent becomes:

$$\lim_{Z_t \to \xi^{-\gamma}} \pi_t = \left(\frac{1}{\gamma}+\phi-1\right)K\xi\frac{\kappa}{\sigma} > 0 \tag{39}$$

This result may seem surprising at first. How is it that the agent's dollar holdings of stock do not go to zero as $W_t \to 0$? To understand the nature of the puzzle, note first that if the agent holds a positive amount of stocks $\pi_t > 0$ when her financial wealth is $W_t = 0$ that means that she must also be holding $W_t - \pi_t = -\pi_t < 0$ in bonds. But then if the stock market experiences a realization of a return that is less than $-r$ over the next interval dt, wouldn't that make the financial wealth of the investor negative?

The resolution of the puzzle is in the nature of the transfers that the agent receives. First, from the definition of Z_t and X_t^* in equations (37) and (27) it follows that X_t^* decreases when and only when $Z_t = \xi$. Accordingly G_t increases when and only when $Z_t = \xi^{-\gamma}$, that is when[23] $c_t = \xi$ and $W_t = 0$. Simply put, the agent starts receiving transfers from the government when her wealth becomes 0 and the stock market experiences further negative returns[24]. That way the agent becomes hedged against negative returns when her wealth is equal to zero and hence can afford to hold stock. This motivates the name "portfolio insurance" for the transfer process (34).

It is important to caution against a caveat: The process G_t given in (34) does not condition the payments to the agent on $W_t = 0$, because the government doesn't posses this information directly. However, the government rationally anticipates the agent's portfolio choice and consumption decisions, as a function of the realization of the returns in the stock market. By setting the process G_t

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[23] This follows from (36) and (37).

[24] Note that the state price density H_t and the stock market P_t are perfectly negatively correlated.

equal to (34), the government rationally anticipates that its payments will reach the agent when her wealth is equal to $W_t = 0$ and will be just enough to safeguard a non-negative wealth.

An alternative way of thinking about the transfer process G_t in (34) is that the government makes a recommendation to the agent on how she should invest and consume. That recommendation is given by (36) and (35). Based on this recommendation and its observation of stock market realizations, the government can compute the agent's wealth and make "just enough" transfers to the agent when needed, so as to keep her inferred wealth above 0. Proposition 4 asserts that given this transfer structure, the consumer will indeed find it optimal to follow the government's "recommendation".

6.3 Comparing the two policies

Given that both policies attain the upper bound of equation (29), this means that they imply the same value function for the agent, and hence are equivalent from a welfare perspective[25].

However, the two policies do differ. They make transfers in different magnitudes at different states of the world. The initial taxes that they imply are also different. Indeed, the cost of the constant income policy is:

$$D_0^{const.} = \frac{y_0}{r+q} = K\xi \left(\frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} \right) \tag{40}$$

whereas by proposition 4, the cost of the portfolio insurance policy is:

$$D_0^{p.i.} = K\xi \frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} \left(\frac{\lambda^*}{\xi^{-\gamma}} \right)^{\phi - 1} \tag{41}$$

All the terms in $D_0^{p.i.}$ are explicit, except for λ^* which is determined implicitly by solving the minimization problem of (29). Equation (20) however implies that:

$$c_0^{-\gamma} = \lambda^* \leq \xi^{-\gamma}$$

since $H_0 = X_0^* = 1$. It can also be shown by applying the implicit function theorem to (29) that λ^* is a declining function of W_0. Dividing (41) by (40) gives:

$$\frac{D_0^{p.i.}}{D_0^{const.}} = \left(\frac{\lambda^*}{\xi^{-\gamma}} \right)^{\phi - 1} = \left(\frac{c_0}{\xi} \right)^{-\gamma(\phi - 1)} \leq 1 \tag{42}$$

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[25] The derivations in the appendix also show that they imply exactly the same consumption process "path by path".

since $c_0 \geq \xi$ and $\phi > 1$. Hence the "portfolio insurance" policy has a cost that cannot be larger than the cost of the "constant income" policy.

This may seem puzzling, since the two policies imply the same value function, while keeping $c_t \geq \xi$. To resolve the puzzle, note first that the cost of the two policies coincides when $c_0 = \xi$. It can also be shown that under both policies $c_0 = \xi$ if and only if $W_0 = D_0^{const.}$, so that post tax wealth is equal to $W_{0+} = 0$. Hence the cost of the two policies differs only when the borrowing constraint is *not* binding, but is identical when the borrowing constraint *does* bind. Alternatively phrased, the constant income policy has a higher cost, because it promises transfers in all states of the world, as opposed to the portfolio insurance policy that only delivers payments when the borrowing constraint is binding. However, when the borrowing constraint binds both policies promise the same transfers in net present value.

This observation is helpful, because it hints to the reason why the two policies have different costs but are equivalent from a welfare perspective. The constant income policy delivers transfers also in states of the world where the borrowing constraint doesn't bind, so that the agent can "undo" them by appropriate savings and portfolio choices. Alternatively phrased, Ricardian Equivalence applies in these states. By contrast the portfolio insurance policy delivers transfers only in the states of the world where the borrowing constraint binds, and hence is cheaper. Nevertheless, since the two policies only differ in states where transfers are subject to the Ricardian Equivalence theorem, they are equivalent from a welfare perspective.

The above discussion illustrates a more fundamental point about the evaluation of government guarantees. Determining the cost of government guarantees, as is routinely done in the literature, can be misleading from a welfare perspective. To perform such comparisons one should examine how different guarantees will affect the behavior of agents, and then evaluate the resulting welfare.

7 Minimum level of assets and implications for pre-retirement savings

7.1 Minimum assets

The previous sections illustrated several equivalent ways of giving transfers to retirees that will safeguard a consumption process above the level ξ. A maintained assumption was (8). I will now illustrate that this assumption is not only sufficient, but it is also necessary for the existence of transfer processes that will induce a consumption process that satisfies $c_t \geq \xi$.

Proposition 5 *An admissible transfer process G_t that will induce $c_t \geq \xi$ will exist if and only if condition (8) holds.*

The practical implication of this proposition is that it gives an exact lower bound on the assets that need to be available upon retirement in order to ensure the feasibility of attaining the goal $c_t \geq \xi$. Hence, in contrast to existing literature that typically takes this lower bound as exogenously given, the present analysis goes a step further and provides a link between the level of minimal consumption that can be guaranteed and the amount of assets that need to have been accumulated.

Letting $W^{\min}$ be the minimum amount of assets implied by (8), one can decompose $W^{\min}$ in a manner similar to section 6.1 in two parts:

$$\frac{W^{\min}}{\xi} = \underbrace{K}_{\text{Merton's wealth to consumption ratio}} \underbrace{\frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1}}_{\text{Cost due to moral hazard}}$$

Figure 2 plots the left hand side of the above equation and the associated value of K for various combinations of the parameters. For reasonable parameter choices the resulting numbers are close to 20. For each dollar of minimum guaranteed consumption, the government needs to be able to raise an initial tax close to 20 dollars.

The above discussion has some clear implications for pre-retirement savings. Namely, the government needs to make sure that the agent arrives at retirement with an amount of savings equal to $W^{\min}$. Assuming that the only policy instrument that the government has at its disposal prior to retirement is a proportional tax χ on Y, then the simplest imaginable policy is to set χ equal to

$$\chi = \frac{(r+q)\, e^{-(r+q)T}}{1 - e^{-(r+q)T}} \frac{K\xi}{Y} \left(\frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} \right) \tag{43}$$

and then place the proceeds into a riskless account. I will assume that the parameters are such that $\chi < 1$. A simple computation yields then:

$$\int_0^T \chi Y e^{-(r+q)t} dt = e^{-(r+q)T} W^{\min}$$

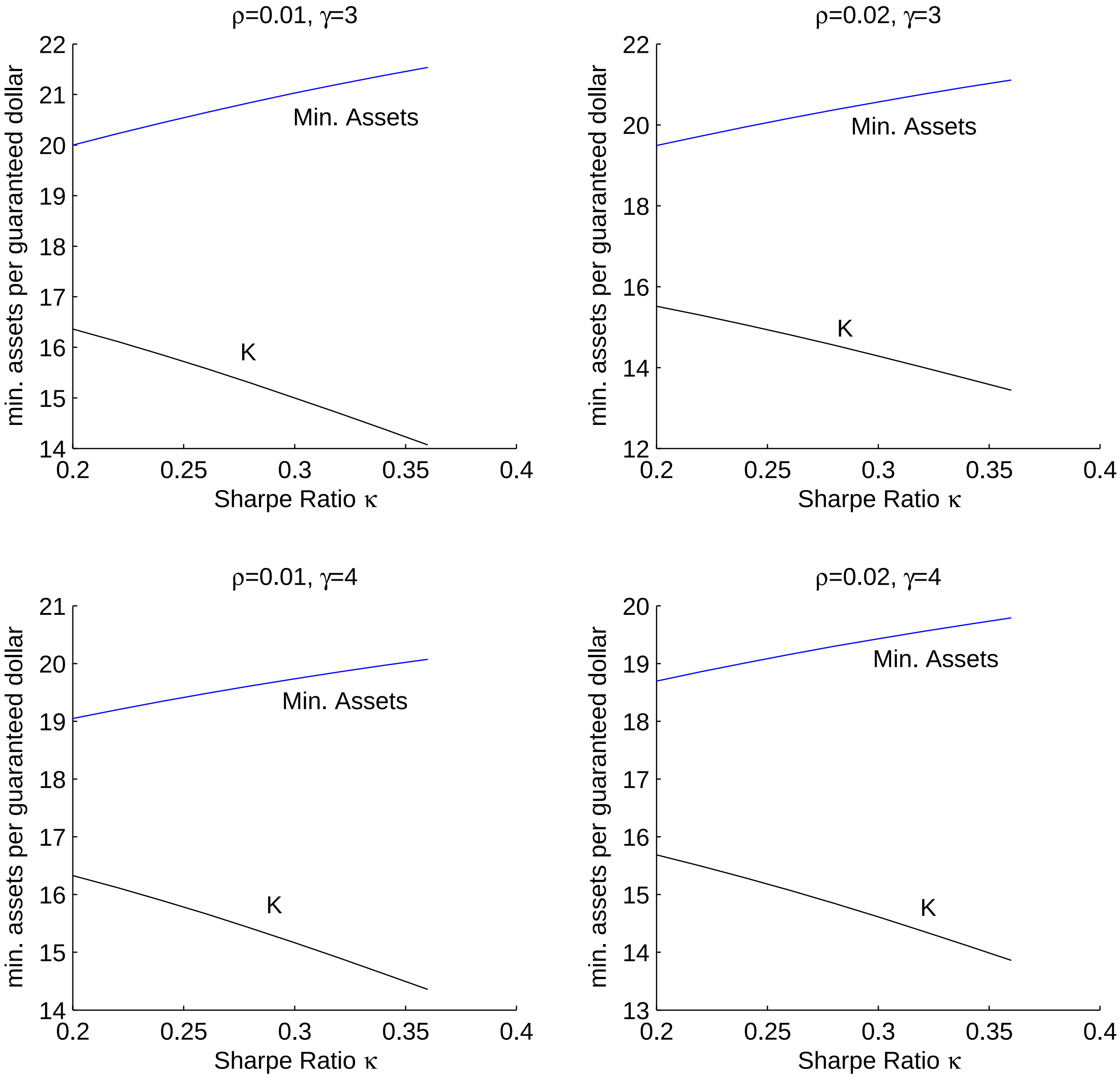

FIGURE 2: The top line in each figure depicts the minimum assets per guaranteed dollar of consumption $\left(\frac{W_0^{\min}}{\xi}\right)$. The bottom line depicts the wealth to consumption ratio in the Merton model (K). The preference parameters are given at the top of each figure. The probability of death q is fixed at 0.04 in all figures and the interest rate is equal to 0.02.

In words, the net present value of the tax proceeds are exactly equal to the net present value of $W^{\min}$. Given the non-negativity constraint on the agent's personal financial wealth, such a policy will imply that the agent's total assets once she enters retirement can be no less than the amount that is accumulated in the riskless account, namely $W^{\min}$. Hence condition (8) will be automatically verified. To gain a quantitative sense, Figure 3 illustrates the resulting tax rates for various levels of $\frac{\xi}{Y}$, Sharpe ratios and preference parameters.

7.2 The pre-retirement problem of the government

Sections 3-6 considered only the post-retirement value function of the agent. The analysis established the upper bound (29) to the value function of the agent and showed that it can be attained, as long as condition (8) is satisfied. The previous subsection also established that a feasible policy will only exist as long as (8) is satisfied. Assuming that the government's only policy instrument prior to retirement is a constant tax rate on Y, then the unique way of achieving condition (8) is by raising a tax rate that is at least equal to χ in equation (43).

An intuitive argument also shows that it is not optimal to raise the pre-retirement tax rate above that level. Given that the agent is faced with borrowing constraints prior to retirement, any government intervention that taxes income today and returns it in the form of a lump sum payment upon entering retirement will reduce the agents ability to smooth consumption. The following proposition states this formally:

Proposition 6 *The optimal pre-retirement tax rate that will ensure (8) is given by (43).*

This proposition shows that the optimal policy of the government (pre-retirement) is to set the tax rate on income as low as possible, but subject to the constraint that the agent has accumulated assets equal to $W^{\min}$ at retirement.

To summarize, the optimal government policy over the agent's life cycle is to set taxes equal to (43) prior to the agent's retirement, place the proceeds in riskless assets, and then use the compounded amount to finance either of the two types of guarantees advocated in section 6.

This describes the optimal government policy, assuming that it is optimal for the government to intervene in the first place. To determine if government intervention is optimal one needs to compare the lifetime expected utility of a new-born agent in the presence of government interven-

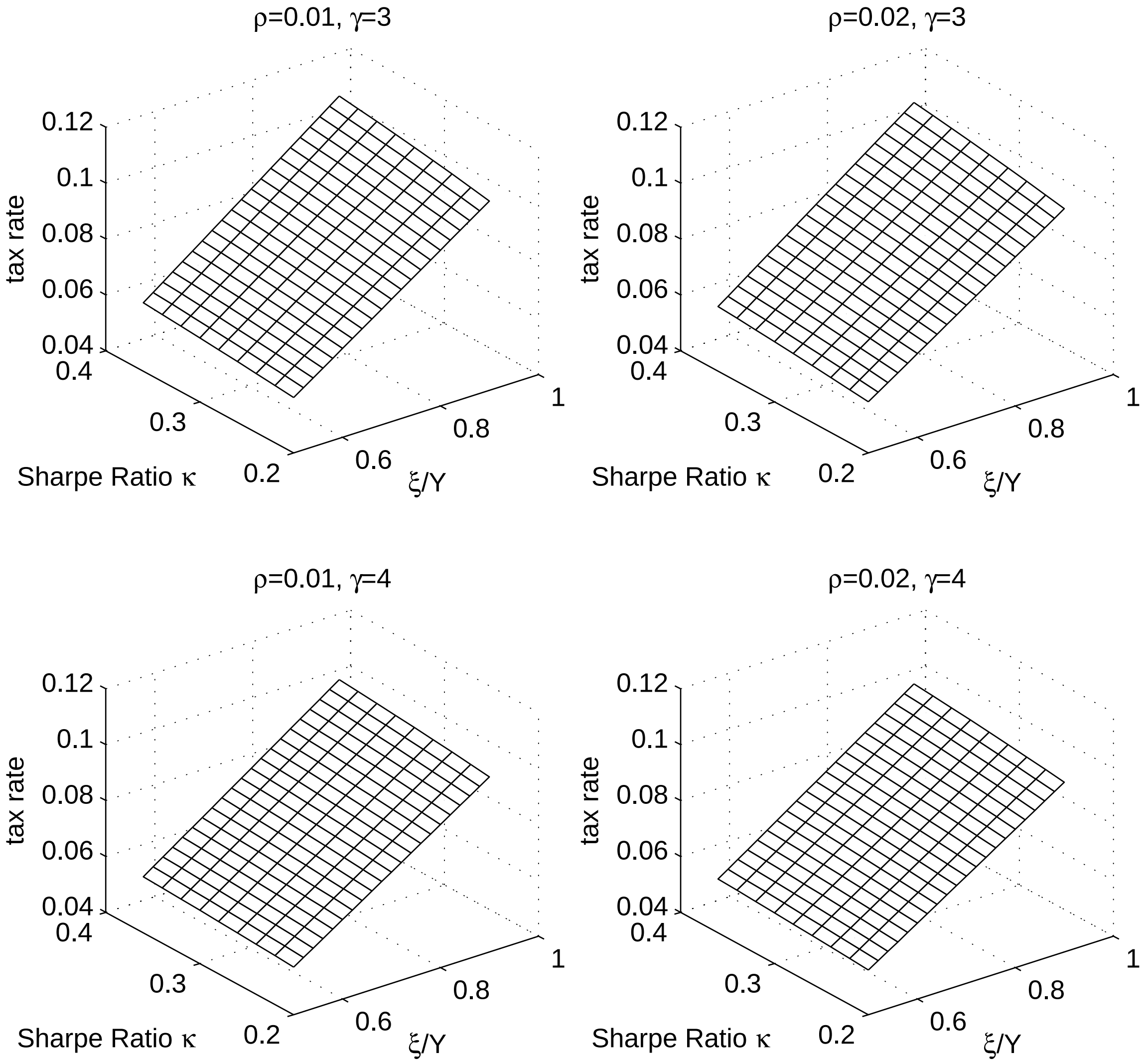

FIGURE 3: The top line in each figure depicts the minimum assets per guaranteed dollar of consumption $\left(\frac{W_0^{\min}}{\xi}\right)$. The bottom line depicts the wealth to consumption ratio in the Merton model (K). The preference parameters are given at the top of each figure. The probability of death q is 0.04. In all figures the interest rate is equal to 0.02.

tion and in its absence. To revisit the setup of section 2.4, the trade-off is that the presence of government intervention distorts agents' optimal consumption choices. However, by safeguarding that no agent's consumption in retirement can fall below ξ, the presence of government intervention safeguards that agents have no incentive to engage in counterproductive activities to precaution against redistribution.

One would expect that government intervention will raise the overall expected utility of an agent at birth, as long as the fraction (x) of forgone output is large enough. Deriving an exact condition on the magnitude of x that will justify intervention is hard in general. It is possible though to give a sufficient condition on x that will justify intervention in a special case. That special case is when ξ is given by:

$$\xi = \frac{1}{K} \frac{\phi - 1}{\frac{1}{\gamma} + \phi - 1} \frac{1 - e^{-(r+q)T}}{(r+q)} Y \tag{44}$$

What makes this value of ξ special, is most easily seen by assuming that the government decides to adopt a "constant income" policy post retirement. In that case equations (44), (43) and (32) imply that the agent receives a constant income throughout his life equal to (32). Moreover, she starts her life with 0 wealth. In that case the results of sections 6.1 and Proposition 2 can be used to show that the agent's value function is given by

$$F = \left(\frac{\gamma}{1-\gamma} - \frac{1}{\gamma\phi\left(\phi - 1\right)} \right) K \xi^{1-\gamma} \tag{45}$$

It remains now to compute the value function of the agent in the complete absence of government intervention. Computing the exact value function of the agent for that scenario is fairly hard, because the presence of a deterministic drop in income after T periods will make the Lagrange multiplier process associated with the borrowing constraint both time and state dependent. However, it is straightforward to obtain an upper bound to the value function of the agent at birth, by just ignoring the borrowing constraints. To see how, note that the net present value of the agent's resources at birth is equal to

$$\left(1 - x\right) Y \int_0^T e^{-(r+q)s} ds = \frac{\left(1 - x\right)\left(1 - e^{-(r+q)T}\right)}{r + q}$$

since an agent sacrifices x percent of her resources in order to guard against redistribution. As Bodie, Merton, and Samuelson (1992) shows, in the absence of borrowing constraints and any government intervention, an agent's value function is given by the Merton (1971) solution treating the capitalized value of the income as wealth:

$$F_1 = K^{\gamma} \frac{\left(\frac{(1-x)\left(1-e^{-(r+q)T}\right)}{r+q} Y \right)^{1-\gamma}}{1-\gamma} \tag{46}$$

Comparing (46) and (45) shows that a sufficient condition for $F > F_1$ is:

$$\frac{(1-x)^{1-\gamma}}{1-\gamma} \leq \left(\frac{\gamma}{1-\gamma} - \frac{1}{\gamma\phi\left(\phi-1\right)} \right) \left(\frac{\phi-1}{\frac{1}{\gamma}+\phi-1} \right)^{1-\gamma}$$

8 Discussion

8.1 Heterogeneity in Income

Clearly assuming that everyone receives the same income Y pre-retirement is a far cry from reality. It is noteworthy however, that allowing income heterogeneity presents no conceptual problem for the analysis. For instance assume that each agent i who is a member of the cohort born at time $-T$ earns an income stream Y_s^i that follows an independent process from the income process of other agents in the same cohort, so that:

$$\int_i Y_s^i = Y$$

Assuming that the government taxes each agent's income at the rate χ and invests the proceeds in riskless bonds, it follows that agent $i's$ assets at retirement are at least:

$$W^i \geq \chi \int_{-T}^{0} e^{-(r+q)s} Y_s^i ds$$

By using these assets, equation (8) implies that the government can provide a floor on this agent's consumption at retirement of at least:

$$\xi^i = \frac{\phi-1}{\frac{1}{\gamma}+\phi-1} \frac{1}{K} \chi \int_{-T}^{0} e^{-(r+q)s} Y_s^i ds$$

Since

$$c_t^i \geq \xi^i \text{ for all } t \geq 0$$

it follows that:

$$\int_i c_t^i \geq \int_i \xi^i = \frac{\phi - 1}{\frac{1}{\gamma} + \phi - 1} \frac{1}{K} \chi \int_{-T}^{0} e^{-(r+q)s} \left(\int_i Y_s^i \right) ds = \frac{\phi - 1}{\frac{1}{\gamma} + \phi - 1} \frac{1}{K} \chi Y \frac{1 - e^{-(r+q)T}}{(r + \theta)\, e^{-(r+q)T}} \equiv \overline{\xi}$$

Hence, the presence of idiosyncratic variation in income profiles does not impede the goal of securing a minimum average consumption level in the cohort of retirees that retire at time 0. Repeating the same argument cohort by cohort, implies then that it is possible to guarantee an average consumption across all retirees above $\overline{\xi}$.

8.2 Arbitrary stochastic discount factors and multiple assets and sources of uncertainty

The exogeneity of the stochastic discount factor and the assumption of a single source of risk and a single asset are not as restrictive as they may seem at first. Even if the stochastic discount factor was driven by multiple sources of uncertainty and the risk premia and interest rates were time varying, most of the results of the paper would survive. The only indispensable assumption is that markets be dynamically complete in the absence of the borrowing constraint and the stochastic discount factor be a continuous function of time.

A close examination of the proof that (28) provides an upper bound to problem 1 reveals that none of the steps depend on the functional form of H_t. The proof goes through for any continuous stochastic discount factor H_t.

It is also possible to show that there always exist variants of the "portfolio insurance" policy that will attain the upper bound of proposition 2 for any stochastic discount factor. The result that seems however to not be true in general is that the "constant income" policy also attains the upper bound of proposition 2.

These observations imply that many of the results of the paper would survive, even if one closed the model in general equilibrium[26]. In that case, the prices of the guarantees and all the parametric formulas would be altered. However, qualitatively the characterization of the upper bound and the existence of at least a policy that attains it, would remain unchanged.

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[26] Of course in general equilibrium care should be taken to make sure that it is feasible to keep retiree consumption above any lower bound. If the aggregate endowment followed a lognormal process, this would only be possible if one reformulated the constraint $c_t \geq \xi$ so as to make ξ proportional to aggregate consumption.

9 Conclusion

This paper presented an optimality theory on how to design a retirement system with the ability to guarantee a minimum standard of living to retirees.

The key results of the paper can be summarized as follows:

First, there can be multiple optimal solutions to ensuring a minimum standard of living in retirement. Two such solutions are a constant income policy and a portfolio insurance policy.

Second, the cost of a policy can be a misleading indicator of its implications for welfare. The two solutions that were discussed in this paper have identical implications for welfare, yet their costs are different in general.

Third, the paper showed that the presence of moral hazard will tend to increase the cost of guarantees. The simple policy that delivers a constant income in retirement illustrates this best: In order to safeguard that an agent's consumption will not fall below a minimum amount (say a dollar), more than one dollar needs to be given in retirement income.

Fourth, the model derives explicitly a minimum amount of assets that need to be available in retirement so as to safeguard that consumption will not drop below a minimum level. Calibration exercises indicate that this minimum level of assets is about 20 times the guaranteed consumption level.

Several issues are unexplored by the present paper. A first question concerns unobserved preference heterogeneity. If agents have different risk aversions, or discount factors, then the government needs to offer menus of contracts in the spirit of discriminatory pricing. It appears straightforward to extend the analysis to allow for this possibility. A particularly interesting question that would emerge in such a setting is whether the need to enforce sorting into different types of contracts would affect the optimal security design or not.

A second question concerns the implications of such guarantees for asset prices. Even though the results of the paper go through for arbitrary stochastic discount factors, it is certain that extensive coverage of retirees by these guarantees would affect the stochastic discount factor in general equilibrium. Studying these two questions is left for future research.

A Appendix

A.1 Proof of proposition 1

Proof. Subject to minor modification the proof of this proposition is identical to the first theorem of He and Pages (1993) and is therefore omitted. However, it is possible to give a sketch of the main argument behind Proposition 1.

The consumer needs to choose her optimal consumption/portfolio path taking the process for transfers and the initial tax as given, while satisfying the constraint $W_t \geq 0$. The first step is to note that *for any* consumption / portfolio policy that satisfies (15)-(17), one can apply Ito's Lemma to $e^{-qt}H_tW_t$ and then integrate to obtain:

$$\int_0^t e^{-qs}H_sc_sds + e^{-qt}H_tW_t = W_0 - D_0 + \int_0^t e^{-qs}H_sdG_s + \int_0^t \psi_s dB_s$$

for an appropriate process[27] ψ_s that will depend on the agent's portfolio choice. Intuitively, this is just the dynamic budget constraint "integrated forward". Since $W_t \geq 0$, the above equation implies that:

$$\int_0^t e^{-qs}H_sc_sds - \left(W_0 - D_0 + \int_0^t e^{-qs}H_sdG_s + \int_0^t \psi_s dB_s\right) \leq 0 \text{ for all } t \geq 0 \tag{47}$$

For any non-increasing and positive process X_t (starting at $X_0 = 1$) and any positive constant λ, equation (47) implies:

$$\lambda \int_0^\infty \left[\int_0^t e^{-qs}H_sc_sds - \left(W_0 - D_0 + \int_0^t e^{-qs}H_sdG_s + \int_0^t \psi_s dB_s\right)\right] dX_t \geq 0$$

since dX_t is (weakly) decreasing.

Hence, for any consumption policy that satisfies the (15)-(17) it follows that:

$$\begin{aligned} E_0\int_0^\infty e^{-(\rho+q)s}\frac{c_s^{1-\gamma}}{1-\gamma}ds \quad \leq \quad & E_0\int_0^\infty e^{-(\rho+q)s}\frac{c_s^{1-\gamma}}{1-\gamma}ds \\ & +\lambda E_0 \int_0^\infty dX_t \int_0^t e^{-qs}H_sc_sds \\ & -\lambda E_0 \int_0^\infty dX_t \left(W_0 - D_0 + \int_0^t e^{-qs}H_sdG_s + \int_0^t \psi_s dB_s\right) \end{aligned} \tag{48}$$

Equation (48) suggests a natural interpretation for X_t as a process of (cumulative) Lagrange multipliers associated with the requirement (47), which follows from[28] $W_t \geq 0$. For short, from now I will refer to X_t as the process of Lagrange multipliers.

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[27] This is an implication of the martingale representation theorem. See e.g. He and Pages (1993) for details.

[28] As one might expect, the inequality in (48) can only become an equality if (21) holds, i.e. if X_t decreases only when $W_t = 0$ and remains otherwise constant.

Applying integration by parts to the second line of equation (48) and noting that $X(0) = 1$, and $\lim_{t\to\infty} X(t) \geq 0$ gives:

$$E_0 \int_0^\infty e^{-(\rho+q)s} \frac{c_s^{1-\gamma}}{1-\gamma} ds \leq$$

$$\begin{aligned}
&\leq E_0 \left(\int_0^\infty e^{-(\rho+q)s} \frac{c_s^{1-\gamma}}{1-\gamma} ds - \lambda \int_0^\infty e^{-qs} X_s H_s c_s ds + \lambda \int_0^\infty e^{-qs} H_s X_s dG_s \right) \\
&\quad +\lambda \left(W_0 - D_0\right) \\
&\leq E_0 \left(\int_0^\infty e^{-(\rho+q)s} \max_c \left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s} H_s X_s c_s \right) ds + \lambda \int_0^\infty e^{-qs} H_s X_s dG_s \right) \\
&\quad +\lambda \left(W_0 - D_0\right)
\end{aligned} \tag{49}$$

Since this inequality holds for any decreasing, positive progressively measurable process X_t and any positive constant λ, it must also hold for the process X_t that minimizes the right hand side of (49). By a similar argument, since the above inequality holds for any choice of c_t, π_t that satisfies (15)-(17) it must also hold for the optimal c_t, π_t. Hence:

$$V(W_0) = \max_{c_s, \pi_s} E_0 \int_0^\infty e^{-(\rho+q)s} \frac{c_s^{1-\gamma}}{1-\gamma} ds$$

$$\leq \min_{\lambda>0,\ X_s \in \mathcal{D}} \left[\begin{array}{c} E \int_0^\infty e^{-(\rho+q)s} \max_{c_s} \left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s} H_s X_s c_s \right) ds \\ +\lambda E \int_0^\infty e^{-qs} H_s X_s dG_s \\ +\lambda \left(W_0 - D_0\right) \end{array} \right]$$

These arguments establish that the right hand side of (19) provides an upper bound to the value function of the consumer. Establishing that it also provides a lower bound is achieved by showing that there exists a consumption / portfolio policy that satisfies (15)-(17) and attains this upper bound. This part of the proof is contained in He and Pages (1993) and the reader is referred to that paper for details. ■

A.2 Proof of Proposition 2

The proof of Proposition 2 is established in steps. The following Lemma will prove useful in establishing the last part of the proposition.

Lemma 1 *Take any $\lambda \in (0, \xi^{-\gamma}]$ and any process G_t and define*

$$\widehat{X}_t = \arg \min_{X_t \in \mathcal{D}} E \left(\int_0^\infty e^{-(\rho+q)s} \max_{c_s} \left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s} H_s X_s c_s \right) ds + \lambda \int_0^\infty e^{-qs} H_s \left(X_s - 1\right) dG_s \right) \tag{50}$$

Then:

$$\lambda E \left(\int_0^\infty H_s \left(\widehat{X}_s - 1 \right) dG_s \right) = E \int_0^\infty e^{-(\rho+q)s} \left(e^{\rho s} \lambda H_s \widehat{X}_s \right)^{1-\frac{1}{\gamma}} \left(1 - \frac{1}{\widehat{X}_s} \right) ds \tag{51}$$

Proof of Lemma 1. Let:

$$\Lambda_t \equiv 1 - \frac{1}{\widehat{X}_t} \tag{52}$$

Applying Ito's Lemma to Λ_t we obtain:

$$d\Lambda_t \equiv \frac{d\widehat{X}_t}{\left(\widehat{X}_t\right)^2} \tag{53}$$

Hence Λ_t changes when and only $\widehat{X}_t$ changes. By Theorem 1 of He and Pages (1993):

$$\int_0^\infty \left[E_t\left(\int_t^\infty \widehat{X}_s e^{-qs} H_s dG_s\right) - E_t\left(\int_t^\infty \widehat{X}_s e^{-qs} H_s c_s ds\right)\right] d\widehat{X}_t = 0 \tag{54}$$

where c_s is given explicitly by:

$$c_s = \left(e^{\rho s}\lambda H_s X_s\right)^{-\frac{1}{\gamma}} \tag{55}$$

Plugging (55) into (54) and then using (53) and observing that Λ_t changes when and only $\widehat{X}_t$ changes implies that:

$$\int_0^\infty \left(E_t \int_t^\infty \widehat{X}_s e^{-qs} H_s dG_s - E_t \int_t^\infty \widehat{X}_s e^{-qs} H_s \left(e^{\rho s}\lambda H_s \widehat{X}_s\right)^{-\frac{1}{\gamma}} ds\right) d\Lambda_t = 0$$

Then, for any admissible G_t and $\widehat{X}_t$ given by (50):

$$\begin{aligned}
&\lambda E\left(\int_0^\infty e^{-qs} H_s \left(\widehat{X}_s - 1\right) dG_s\right) = \\
&\quad \lambda E\left[\int_0^\infty e^{-qs} H_s \left(\widehat{X}_s - 1\right) dG_s - \int_0^\infty \left(E_t \int_t^\infty \widehat{X}_s e^{-qs} H_s dG_s\right) d\Lambda_t\right] \\
&\quad + \lambda E\left\{\int_0^\infty E_t\left[\int_t^\infty \widehat{X}_s e^{-qs} H_s \left(e^{\rho s}\lambda H_s \widehat{X}_s\right)^{-\frac{1}{\gamma}} ds\right] d\Lambda_t\right\}
\end{aligned} \tag{56}$$

Next consider the martingale:

$$\mathcal{M}_t = E_t \int_0^\infty \widehat{X}_s e^{-qs} H_s dG_s = \int_0^t \widehat{X}_s e^{-qs} H_s dG_s + E_t \int_t^\infty \widehat{X}_s e^{-qs} H_s dG_s \tag{57}$$

According to the martingale representation theorem, there exists a square integrable ψ_s such that:

$$\mathcal{M}_t = \mathcal{M}_0 + \int_0^t \psi_s dB_s \tag{58}$$

Combining (57) and (58) gives:

$$\begin{aligned}
d\left(E_t \int_t^\infty \widehat{X}_s e^{-qs} H_s dG_s\right) &= d\mathcal{M}_t - \widehat{X}_t e^{-qt} H_t dG_t \\
&= \psi_t dB_t - \widehat{X}_t e^{-qt} H_t dG_t
\end{aligned}$$

Now, fixing an arbitrary $\varepsilon > 0$, letting τ^{ε} be the first time t such that $|\Lambda_t| \geq \frac{1}{\varepsilon}$, applying integration by parts and using the fact that $\Lambda_0 = 0$, gives:

$$\begin{aligned}-E\int_0^{T\wedge\tau^{\varepsilon}} \left(E_t \int_t^{\infty} \widehat{X}_s e^{-qs} H_s dG_s\right) d\Lambda_t &= -E\int_0^{T\wedge\tau^{\varepsilon}} \Lambda_s \widehat{X}_s e^{-qs} H_s dG_s \\ &\quad +E\int_0^{T\wedge\tau^{\varepsilon}} \Lambda_s \psi_s dB_s \\ &\quad -E\left[\Lambda_{T\wedge\tau^{\varepsilon}} \left(E_{T\wedge\tau^{\varepsilon}} \int_{T\wedge\tau^{\varepsilon}}^{\infty} \widehat{X}_s e^{-qs} H_s dG_s\right)\right]\end{aligned}$$

Since ψ_s is square integrable and $|\Lambda_s|$ is bounded in $\left[0, \frac{1}{\varepsilon}\right]$ the second term on the right hand side of the above expression is 0. We also note that:

$$-E\left[\Lambda_{T\wedge\tau^{\varepsilon}} \left(E_{T\wedge\tau^{\varepsilon}} \int_{T\wedge\tau^{\varepsilon}}^{\infty} \widehat{X}_s e^{-qs} H_s dG_s\right)\right] = -E\left[X_{T\wedge\tau^{\varepsilon}} \Lambda_{T\wedge\tau^{\varepsilon}} J\right] \tag{59}$$

where

$$J = \left(E_{T\wedge\tau^{\varepsilon}} \int_{T\wedge\tau^{\varepsilon}}^{\infty} \frac{\widehat{X}_s}{\widehat{X}_{T\wedge\tau^{\varepsilon}}} e^{-qs} H_s dG_s\right) \leq E_{T\wedge\tau^{\varepsilon}} \int_{T\wedge\tau^{\varepsilon}}^{\infty} e^{-qs} H_s dG_s \tag{60}$$

since $\widehat{X}_t$ is non-increasing. Combining (60) with (59) and noting that $0 < \widehat{X}_t \leq 1$

$$-E\left[\widehat{X}_{T\wedge\tau^{\varepsilon}} \Lambda_{T\wedge\tau^{\varepsilon}} J\right] = E\left[\left(1 - \widehat{X}_{T\wedge\tau^{\varepsilon}}\right) J\right] \leq E_{T\wedge\tau^{\varepsilon}} \int_{T\wedge\tau^{\varepsilon}}^{\infty} e^{-qs} H_s dG_s \tag{61}$$

Given that:

$$E\int_0^{\infty} e^{-qs} H_s dG_s < \infty$$

it follows that:

$$E_{T\wedge\tau^{\varepsilon}} \int_{T\wedge\tau^{\varepsilon}}^{\infty} e^{-qs} H_s dG_s \to 0 \tag{62}$$

as $\varepsilon \to 0, T \to \infty$. This leads to the inequalities:

$$\begin{aligned}-E\int_0^{\infty} \left(E_t \int_t^{\infty} \widehat{X}_s e^{-qs} H_s dG_s\right) d\Lambda_t &\geq -E\int_0^{T\wedge\tau^{\varepsilon}} \left(E_t \int_t^{\infty} \widehat{X}_s e^{-qs} H_s dG_s\right) d\Lambda_t \\ &\geq -E\int_0^{T\wedge\tau^{\varepsilon}} \Lambda_s \widehat{X}_s e^{-qs} H_s dG_s\end{aligned}$$

Letting $\varepsilon \to 0, T \to \infty$, using the monotone convergence theorem, and using (61) and (62), gives

$$-\int_0^{\infty} \left(E_t \int_t^{\infty} \widehat{X}_s e^{-qs} H_s dG_s\right) d\Lambda_t = -E\int_0^{T\wedge\tau^{\varepsilon}} \Lambda_s \widehat{X}_s e^{-qs} H_s dG_s \tag{63}$$

Using (63) and the definition of Λ_t gives:

$$\lambda E\left[\int_0^{\infty} e^{-qs} H_s \left(\widehat{X}_s - 1\right) dG_s - \int_0^{\infty} \left(E_t \int_t^{\infty} \widehat{X}_s e^{-qs} H_s dG_s\right) d\Lambda_t\right] =$$

$$= E\left[\lambda \int_0^\infty e^{-qs} H_s \left(\widehat{X}_s - 1\right) dG_s - \lambda \int_0^\infty e^{-qs} H_s \widehat{X}_s \Lambda_s dG_s\right] = 0$$

Returning now to (56) and using the above equation yields:

$$\begin{aligned}\lambda E\left(\int_0^\infty e^{-qs} H_s \left(\widehat{X}_s - 1\right) dG_s\right) &= \lambda E\left\{\int_0^\infty E_t\left[\int_t^\infty \widehat{X}_s e^{-qs} H_s \left(e^{\rho s}\lambda H_s \widehat{X}_s\right)^{-\frac{1}{\gamma}} ds\right] d\Lambda_t\right\} && (64)\\ &= E\left[\int_0^\infty e^{-(\rho+q)t}\left(e^{\rho t}\lambda H_t \widehat{X}_s\right)^{1-\frac{1}{\gamma}} \Lambda_t dt\right] && (65)\end{aligned}$$

where (48) follows from a similar integration by parts argument as the one in equations (57)-(63). ■

The next Lemma uses Lemma 1 to prove (30).

Lemma 2 *For all admissible processes* $G_t \in \mathcal{G}$*:*

$$\max_{G_t \in \mathcal{G}} V(W_0) \leq \min_{\lambda \in (0,\xi^{-\gamma}]} \left[E\left(\int_0^\infty e^{-(\rho+q)s} \frac{\left(\lambda e^{\rho s} H_s X_s^*\right)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda \int_0^\infty e^{-qs} H_s \left(\lambda e^{\rho s} H_s X_s^*\right)^{-\frac{1}{\gamma}} + \lambda W_0\right)\right] \quad (66)$$

Proof. Proposition 1 along with Lemma 1 implies that for any admissible process G_t there exists a $\lambda^G > 0$ and a decreasing process $X_t^G \in \mathcal{D}$ that minimize (19) such that:

$$\begin{aligned}V(W_0) &= E\left(\int_0^\infty e^{-(\rho+q)s} \max_{c_s}\left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda^G e^{\rho s} H_s X_s^G c_s\right) ds + \lambda^G \int_0^\infty e^{-qs} H_s \left(X_s^G - 1\right) dG_s\right) + \lambda^G W_0\\ &= E\int_0^\infty e^{-(\rho+q)s}\left(\frac{\left(e^{\rho s}\lambda^G H_s X_s^G\right)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda^G e^{\rho s} H_s \left(e^{\rho s}\lambda^G H_s X_s^G\right)^{-\frac{1}{\gamma}}\right) ds + \lambda^G W_0 \qquad (67)\end{aligned}$$

Moreover, since the process G_t enforces $c_t \geq \xi$, equation (20) implies that $\lambda^G \leq \xi^{-\gamma}$. Next take an arbitrary $\lambda > 0$. Since

$$c_t = \left(e^{\rho t}\lambda^G H_t X_t^G\right)^{-\frac{1}{\gamma}}$$

is an optimal consumption process, it exhausts the "budget constraint" of the consumer so that:

$$E\int_0^\infty e^{-(\rho+q)s} e^{\rho s} H_s \left(e^{\rho s}\lambda^G H_s X_s^G\right)^{-\frac{1}{\gamma}} = W_0 - D_0 + E\int_0^\infty e^{-qs} H_s dG_s$$

Using (13), this implies that:

$$E\int_0^\infty e^{-(\rho+q)s} e^{\rho s} H_s \left(e^{\rho s}\lambda^G H_s X_s^G\right)^{-\frac{1}{\gamma}} = W_0$$

This furthermore implies that (67) can be rewritten as:

$$V(W_0) = E\int_0^\infty e^{-(\rho+q)s}\left(\frac{\left(e^{\rho s}\lambda^G H_s X_s^G\right)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda e^{\rho s} H_s \left(e^{\rho s}\lambda^G H_s X_s^G\right)^{-\frac{1}{\gamma}}\right) ds + \lambda W_0 \qquad (68)$$

Next define X_t^* as in equation (27), and let the process N_t be given as:

$$N_t = \frac{\lambda^G}{\lambda} \frac{X_t^G}{X_t^*}$$

Using N_t one can rewrite equation (68) as

$$V(W_0) = E \int_0^\infty e^{-(\rho+q)s} \left(\frac{\left(e^{\rho s} \lambda H_s X_s^* N_s\right)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda e^{\rho s} H_s \left(e^{\rho s} \lambda H_s X_s^* N_s\right)^{-\frac{1}{\gamma}} \right) ds + \lambda W_0 \tag{69}$$

Since $\lambda^G X_t^G$ is a decreasing process that starts at λ^G and always stays below $\xi^{-\gamma}$, the Skorohod equation[29] implies that there exists another decreasing process $\lambda^G X_t^{*G}$ that also starts at λ^G and stays below $\xi^{-\gamma}$, with the property

$$\lambda^G X_t^G \leq \lambda^G X_t^{*G} \tag{70}$$

This process is given by:

$$X_t^{*G} = \min \left[1, \frac{\xi^{-\gamma} / \lambda^G}{\max_{0 \leq s \leq t} \left(e^{\rho s} H_s\right)} \right]$$

Note that X_t^{*G} is identical to X_t^* with the exception that λ replaces λ^G. Using (70) and the definition of N_t yields:

$$N_t = \frac{\lambda^G}{\lambda} \frac{X_t^G}{X_t^*} \leq \frac{\lambda^G}{\lambda} \frac{X_t^{*G}}{X_t^*} \tag{71}$$

Using (71) and (69) leads to:

$$V(W_0) \leq E \int_0^\infty e^{-(\rho+q)s} A(s) ds + \lambda W_0 \tag{72}$$

where:

$$A(s) = \max_{N_s \leq Q_s} \left(\frac{\left(e^{\rho s} \lambda H_s X_s^* N_s\right)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda e^{\rho s} H_s \left(e^{\rho s} \lambda H_s X_s^* N_s\right)^{-\frac{1}{\gamma}} \right) \tag{73}$$

and

$$Q_s = \max \left[1, \frac{\lambda^G}{\lambda} \frac{X_s^{*G}}{X_s^*} \right]$$

To study the maximization problem of equation (73) it is useful to compute the derivative of the expression inside round brackets with respect to N_t. Performing this computation and combining terms gives:

$$\frac{\partial A(s)}{\partial N_s} = -\frac{1}{\gamma} \left(e^{\rho s} \lambda H_s X_s^* N_s\right)^{1-\frac{1}{\gamma}} N_s^{-\frac{1}{\gamma}} \left(1 - \frac{1}{N_s X_s^*} \right) \tag{74}$$

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[29] For the Skorohod equation see Karatzas and Shreve (1991).

At this stage it is useful to consider two cases separately. The first case is $\lambda > \lambda^G$. In this case, it is straightforward to show that:

$$Q_s = 1$$

Hence in maximizing $A(s)$, one can constrain attention to values of $N_s \leq 1$. An examination of (74) reveals that $\frac{\partial A(s)}{\partial N_s} \geq 0$ for all $N_s \leq 1$ and all X_s^*, since $X_s^* \leq 1$. Hence the solution to (73) is $N_s = 1$ when $\lambda > \lambda^G$.

In the case where $\lambda < \lambda^G$ it is also true that the optimal N_s in (73) is equal to one. To see this, observe that:

$$Q_s = \begin{cases} \frac{\lambda^G}{\lambda}\frac{X_s^{*G}}{X_s^*} & \text{when } X_s^* = 1 \\ 1 & \text{when } X_s^* < 1 \end{cases}$$

Using this observation in (74) reveals that the optimal choice for N_s is always equal to 1.[30]

The above reasoning shows that the optimal solution of (73) is given by $N_s = 1$. Returning to (72), this implies that:

$$V\left(W_0\right) \leq E\int_0^\infty e^{-(\rho+q)s}\left(\frac{\left(e^{\rho s}\lambda H_s X_s^*\right)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda e^{\rho s} H_s \left(e^{\rho s}\lambda H_s X_s^*\right)^{-\frac{1}{\gamma}} ds\right) + \lambda W_0$$

Since this bound holds for arbitrary $\lambda \in (0, \xi^{-\gamma}]$ and arbitrary $G_t \in \mathcal{G}$, it also holds for the $\lambda \in (0, \xi^{-\gamma}]$ that minimizes the right hand side of the above equation and the $G_t \in \mathcal{G}$ that maximizes the right hand side. Hence (66) follows. ■

The next part of the proof is to show that equation (28) holds. A first step is to show that (28) provides an upper bound to $J\left(W_0\right)$:

Lemma 3 *The value function of problem 2 is bounded above by:*

$$J\left(W_0\right) \leq$$
$$\min_{\lambda \in (0,\xi^{-\gamma}]}\left[E\left(\int_0^\infty e^{-(\rho+q)s}\frac{\left(\lambda e^{\rho s}H_s X_s^*\right)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda\int_0^\infty e^{-qs}H_s\left(\lambda e^{\rho s}H_s X_s^*\right)^{-\frac{1}{\gamma}} + \lambda W_0\right)\right] \tag{75}$$

Proof. The proof of this Lemma follows identical steps to the proof of the previous Lemma. To see this, take an arbitrary triplet $< \widehat{\lambda}, X_t, c_t >$ that satisfies equations (24)-(26) of Problem 2. Then for any $\lambda > 0$ one obtains:

$$J\left(W_0\right) = E\left(\int_0^\infty e^{-(\rho+q)s}\frac{\left(\widehat{\lambda}e^{\rho s}H_s X_s\right)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda\int_0^\infty e^{-qs}H_s\left(\widehat{\lambda}e^{\rho s}H_s X_s\right)^{-\frac{1}{\gamma}} + \lambda W_0\right)$$

---

[30] To see this distinguish cases. When $X_s^* = 1$, then solving $\frac{\partial A(s)}{\partial N_s} = 0$ gives $N_s = 1 \leq Q_s$. Hence N_s is the unique interior solution. When $X_s^* < 1$, then $\frac{\partial A(s)}{\partial N_s} > 0$ for all $N_s \leq Q_s = 1$. Hence the solution is given by the corner $N_s = Q_s = 1$.

Notice that this equation is identical to equation (68), with the exception that λ^G is replaced by $\widehat{\lambda}$ and X_t^G is replaced by X_t. Since the equations following (68) hold for any λ^G, X_t^G they also hold for $\widehat{\lambda}$,X_t. Accordingly, by repeating the same steps, one can arrive at (75). ■

The next step in the proof of the proposition is to show that the inequality in (75) holds with equality for the optimal policy. The following Lemma presents a step in this direction:

Lemma 4 *Let $F(\lambda)$ be given by:*

$$F(\lambda) = E\left(\int_0^\infty e^{-(\rho+q)s} \frac{(\lambda e^{\rho s} H_s X_s^*)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda \int_0^\infty e^{-qs} H_s \left(\lambda e^{\rho s} H_s X_s^*\right)^{-\frac{1}{\gamma}}\right) \tag{76}$$

Then

$$F(\lambda) = -\frac{K\xi^{1-\gamma}}{\gamma\phi\left(\phi-1\right)}\left(\frac{\lambda}{\xi^{-\gamma}}\right)^{\phi} + K\frac{\gamma}{1-\gamma}\lambda^{1-\frac{1}{\gamma}} \tag{77}$$

Assume moreover that (8) is met. Then

$$\min_{\lambda\in(0,\xi^{-\gamma}]}\left[F(\lambda)+\lambda W_0\right] = \min_{\lambda>0}\left[F(\lambda)+\lambda W_0\right] \tag{78}$$

and (75) can be rewritten as:

$$J\left(W_0\right) \leq \min_{\lambda>0}\left[F(\lambda)+\lambda W_0\right]$$

Moreover, letting λ^* be given as:

$$\lambda^* = \arg\min_{\lambda>0}\left[F(\lambda)+\lambda W_0\right]$$

implies that:

$$E_0\left[\int_0^\infty e^{-qs} H_s \left(\lambda^* e^{\rho s} H_s X_s^*\right)^{-\frac{1}{\gamma}}\right] = W_0$$

and accordingly $c_s^* = \left(\lambda^* e^{\rho s} H_s X_s^*\right)^{-\frac{1}{\gamma}}$ is a feasible consumption to the central planner of problem 2.

Proof. To save notation, let

$$Z_t = \lambda e^{\rho s} H_s X_s^* \tag{79}$$

and note that $Z_0 = \lambda$, and that $Z_t \in (0, \xi^{-\gamma}]$ by the definition of X_t^* in equation (27). Equation (76) can now be rewritten as:

$$F(\lambda) = E\left[\int_0^\infty e^{-(\rho+q)s}\frac{1}{1-\gamma}\left(Z_s\right)^{1-\frac{1}{\gamma}} ds - \int_0^\infty e^{-(\rho+q)s}\frac{Z_s^{1-\frac{1}{\gamma}}}{X_s^*} ds\right] \tag{80}$$

It will be convenient to compute the two terms inside equation (80) separately. Define first:

$$G\left(Z_t\right) = E\left[\int_t^{\infty} e^{-(\rho+q)s}\frac{1}{1-\gamma}\left(Z_s\right)^{1-\frac{1}{\gamma}}ds|Z_t\right] \tag{81}$$

To compute $G\left(Z_t\right)$, it is easiest to let τ be the first hitting time of Z_t to the level $\varepsilon > 0$:

$$\tau = \inf_{s\geq t}\left\{Z_s = \varepsilon\right\}$$

and then compute the expression:

$$G^{\varepsilon}\left(Z_t\right) = E\left[\int_t^{\tau} e^{-(\rho+q)s}\frac{1}{1-\gamma}\left(Z_s\right)^{1-\frac{1}{\gamma}}ds|Z_t\right] \tag{82}$$

To compute (82) apply first Ito's Lemma to (79) to obtain:

$$\frac{dZ_t}{Z_t} = (\rho - r)\,dt - \kappa dB_t + \frac{dX_t^*}{X_t^*}$$

Next, I shall construct a function $G^{\varepsilon}(Z)$ that satisfies the ODE:

$$\frac{\kappa^2}{2}G_{ZZ}^{\varepsilon}Z^2 - G_Z^{\varepsilon}Z\left(\rho - r\right) - (\rho+q)G^{\varepsilon} + \frac{1}{1-\gamma}\left(Z\right)^{1-\frac{1}{\gamma}} = 0 \tag{83}$$

subject to the boundary conditions:

$$\begin{aligned} G_Z^{\varepsilon}\left(\xi^{-\gamma}\right) &= 0 && (84)\\ G^{\varepsilon}(\varepsilon) &= 0 && (85)\end{aligned}$$

(83) is a linear ordinary differential equation with general solution:

$$G^{\varepsilon}\left(Z\right) = C_1 Z^{\chi} + C_2 Z^{\phi} + K\frac{1}{1-\gamma}Z^{1-\frac{1}{\gamma}}$$

where C_1, C_2 are arbitrary constants, K is given in equation (10), $\phi > 0$ in (9), and χ is given by:

$$\chi = \frac{-\left(\rho - r - \frac{\kappa^2}{2}\right) - \sqrt{\left(\rho - r - \frac{\kappa^2}{2}\right)^2 + 2\left(\rho+q\right)\kappa^2}}{\kappa^2} < 0 \tag{86}$$

To satisfy (84), (85) C_1 and C_2 must be chosen so that:

$$\begin{aligned} \chi C_1\left(\xi^{-\gamma}\right)^{\chi} + \phi C_2\left(\xi^{-\gamma}\right)^{\phi} - \frac{1}{\gamma}K\left(\xi^{-\gamma}\right)^{1-\frac{1}{\gamma}} &= 0\\ C_1\varepsilon^{\chi} + C_2\varepsilon^{\phi} + K\frac{1}{1-\gamma}\varepsilon^{1-\frac{1}{\gamma}} &= 0\end{aligned}$$

Solving this system yields:

$$C_2 = \frac{K\left[\frac{1}{\gamma\chi}\left(\xi^{-\gamma}\right)^{1-\frac{1}{\gamma}-\chi}\varepsilon^{\chi} - \frac{1}{1-\gamma}\varepsilon^{1-\frac{1}{\gamma}}\right]}{\frac{\phi}{\chi}\left(\xi^{-\gamma}\right)^{\phi-\chi}\varepsilon^{\chi} - \varepsilon^{\phi}}$$

and

$$C_1 = -C_2 \varepsilon^{\phi - \chi} - K \frac{1}{1-\gamma} \varepsilon^{1-\frac{1}{\gamma}-\chi}$$

It remains now to verify that $G^{\varepsilon}(Z_t)$ satisfies (82). To this end, apply Ito's Lemma to $e^{-(\rho+q)t} G^{\varepsilon}(Z_t)$ to obtain for any time $T \wedge \tau$:

$$\begin{aligned} e^{-(\rho+q)T} G^{\varepsilon}(Z_{T\wedge\tau}) - e^{-(\rho+q)t} G^{\varepsilon}(Z_t) &= \int_t^{T\wedge\tau} \left(\frac{\kappa^2}{2} G^{\varepsilon}_{ZZ} Z_s^2 - G^{\varepsilon}_Z Z_s \left(\rho - r\right) - (\rho+q) G^{\varepsilon} \right) e^{-(\rho+q)s} ds \\ &\quad - \int_t^{T\wedge\tau} e^{-(\rho+q)s} \kappa G^{\varepsilon}_Z Z_s dB_s \\ &\quad + \int_t^{T\wedge\tau} e^{-(\rho+q)s} G^{\varepsilon}_Z \left(\xi^{-\gamma}\right) \xi^{-\gamma} \frac{dX_s^*}{X_s^*} \end{aligned}$$

Using (83) in the first line of the above equation along with (84) in the third line, letting $T \to \infty$ along with (85) and using the monotone convergence theorem gives:

$$G^{\varepsilon}(Z_t) = E_t \left[\int_t^{\tau} e^{-(\rho+q)(s-t)} \frac{1}{1-\gamma} \left(Z_s\right)^{1-\frac{1}{\gamma}} ds + \int_t^{\tau} e^{-\rho(s-t)} \kappa G_Z Z_s dB_s \right] \tag{87}$$

Since $G_Z Z$ is bounded between t and τ, the second term in the above expression is a martingale and hence obtain (81). Next, letting $\varepsilon \to 0$, it is straightforward to show that:

$$C_2 = \frac{K\left[\frac{1}{\gamma\chi}\left(\xi^{-\gamma}\right)^{1-\frac{1}{\gamma}-\chi} - \frac{1}{1-\gamma}\varepsilon^{1-\frac{1}{\gamma}-\chi}\right]}{\frac{\phi}{\chi}\left(\xi^{-\gamma}\right)^{\phi-\chi} - \varepsilon^{\phi-\chi}} \to K \frac{1}{\gamma\phi}\left(\xi^{-\gamma}\right)^{1-\frac{1}{\gamma}-\phi}$$

since $\varepsilon^{\phi-\chi} \to 0$ and $\varepsilon^{1-\frac{1}{\gamma}-\chi} \to 0$. By a similar argument it is easy to show that $C_1 \to 0$ and hence:

$$\lim_{\varepsilon\to 0} G^{\varepsilon}(Z) = G(Z) = \frac{1}{\phi}\frac{1}{\gamma} K \xi^{1-\gamma} \left(\frac{Z}{\xi^{-\gamma}}\right)^{\phi} + K \frac{1}{1-\gamma} Z^{1-\frac{1}{\gamma}} \tag{88}$$

Now (81) is a consequence of the monotone convergence theorem.

It remains to compute the expression

$$N\left(Z_t, X_t^*\right) = E_t \left(\int_t^{\infty} e^{-(\rho+q)(s-t)} \frac{Z_s^{1-\frac{1}{\gamma}}}{X_s^*} ds \right)$$

Using a similar logic as above, the next step is to search for a function N that satisfies:

$$\begin{aligned} \frac{\kappa^2}{2} N_{ZZ} Z^2 - N_Z Z\left(\rho - r\right) - (\rho+q) N + \frac{(Z)^{1-\frac{1}{\gamma}}}{X^*} &= 0 \\ N_Z\left(\xi^{-\gamma}, X^*\right) \frac{\xi^{-\gamma}}{X^*} + N_X\left(\xi^{-\gamma}, X^*\right) &= 0 \end{aligned}$$

One can check that such a function exists and is given by:

$$N(Z, X^*) = \frac{1}{(\phi-1)} \frac{1}{\gamma} \frac{K\left(\xi^{-\gamma}\right)^{1-\frac{1}{\gamma}}}{X^*} \left(\frac{Z}{\xi^{-\gamma}}\right)^{\phi} + K \frac{Z^{1-\frac{1}{\gamma}}}{X^*} \tag{89}$$

One can now use the same steps as for the function $G(Z_t)$ to verify that:

$$N\left(Z_t, X_t^*\right) = E_t\left(\int_t^{\infty} e^{-(\rho+q)(s-t)} \frac{Z_s^{1-\frac{1}{\gamma}}}{X_s^*} ds\right) \tag{90}$$

It is now possible to compute $F(\lambda)$ which is given by:

$$\begin{aligned} F(\lambda) &= G(\lambda) - N\left(\lambda, 1\right) = \\ &= -\frac{K\xi^{1-\gamma}}{\gamma\phi\left(\phi-1\right)}\left(\frac{\lambda}{\xi^{-\gamma}}\right)^{\phi} + K\frac{\gamma}{1-\gamma}\lambda^{1-\frac{1}{\gamma}} \end{aligned} \tag{91}$$

To show the second part of the proposition, observe that (90), (79) and (89) imply that

$$\begin{aligned} \frac{N\left(\lambda, 1\right)}{\lambda} &= \frac{1}{\lambda}E_0\left(\int_0^{\infty} e^{-(\rho+q)s} \frac{Z_s^{1-\frac{1}{\gamma}}}{X_s^*} ds\right) = E_0\left(\int_0^{\infty} e^{-qs} H_s \left(\lambda e^{\rho s} H_s X_s^*\right)^{-\frac{1}{\gamma}} ds\right) = \\ &= \frac{K\xi^{1-\gamma}}{\left(\phi-1\right)\gamma}\frac{1}{\gamma}\left(\frac{\lambda}{\xi^{-\gamma}}\right)^{\phi}\frac{1}{\lambda} + K\lambda^{-\frac{1}{\gamma}} \end{aligned} \tag{92}$$

Moreover, computing $F'(\lambda)$ in (91) yields:

$$F'(\lambda) = -\frac{K\xi^{1-\gamma}}{\left(\phi-1\right)\gamma}\frac{1}{\gamma}\left(\frac{\lambda}{\xi^{-\gamma}}\right)^{\phi}\frac{1}{\lambda} - K\lambda^{-\frac{1}{\gamma}} \tag{93}$$

Combining (92) and (93) yields:

$$\begin{aligned} F'(\lambda) &= -\frac{N\left(\lambda, 1\right)}{\lambda} = \\ &= -E_0\left(\int_0^{\infty} e^{-qs} H_s \left(\lambda e^{\rho s} H_s X_s^*\right)^{-\frac{1}{\gamma}} ds\right) \end{aligned} \tag{94}$$

Using the formula for $F(\lambda)$, equation (75) can be expressed as:

$$\min_{\lambda>0}\left\{F(\lambda) + \lambda W_0\right\}$$

which leads to the first order condition for the minimizing λ^* :

$$F'\left(\lambda^*\right) = -W_0 \tag{95}$$

Using (94) leads to:

$$W_0 = E_0\left(\int_0^{\infty} e^{-qs} H_s \left(\lambda^* e^{\rho s} H_s X_s^*\right)^{-\frac{1}{\gamma}} ds\right) = E_0\left(\int_0^{\infty} e^{-qs} H_s c_s^* ds\right)$$

This last equation implies that λ^*, X_t^* and the associated consumption process $c_t^* = \left(\lambda^* e^{\rho t} H_t X_t^*\right)^{-\frac{1}{\gamma}}$ satisfy (24) and (26). To show that the choice $\langle \lambda^*, X_t^*, c_t^* \rangle$ constitutes a feasible triplet, it remains to show that it also satisfies (25). By construction of X_t^* this will be the case as long as $\lambda^* < \xi^{-\gamma}$. This will indeed be the case as long as W_0 satisfies (8). To see this, note that $\xi^{-\gamma}$ is the unique solution of (95) when W_0 is given by

$$W_0 = \frac{\frac{1}{\gamma}+\phi-1}{\phi-1}K\xi$$

Moreover, equation (93) implies that:

$$\begin{aligned} F''(\lambda) &= -K\left(\xi^{-\gamma}\right)^{1-\frac{1}{\gamma}}\frac{1}{\gamma}\left(\frac{1}{\xi^{-\gamma}}\right)^{\phi}\lambda^{\phi-2}+\frac{1}{\gamma}K\lambda^{-\frac{1}{\gamma}-1} \\ &= \frac{1}{\gamma}K\lambda^{-\frac{1}{\gamma}-1}\left[1-\left(\frac{\lambda}{\xi^{-\gamma}}\right)^{\phi+\frac{1}{\gamma}-1}\right] > 0 \end{aligned} \tag{96}$$

The above equation shows that $F^{'}(\lambda)$ is an increasing function of λ for $0<\lambda<\xi^{-\gamma}$ and hence the solution λ^* of equation (95) is a decreasing function of W_0. Hence, as long as W_0 satisfies (8), then $\lambda^* < \xi^{-\gamma}$. Since the interior solution λ^* is smaller than $\xi^{-\gamma}$, equation (78) follows. ■

Combining the above Lemma with (75) implies that:

$$\begin{aligned} J(W_0) &\leq \min_{\lambda>0}\left[F(\lambda)+\lambda W_0\right] = F(\lambda^*)+\lambda^* W_0 = \\ &= E\left(\int_0^{\infty} e^{-(\rho+q)s}\frac{\left(\left(\lambda^* e^{\rho s}H_s X_s^*\right)^{-\frac{1}{\gamma}}\right)^{1-\gamma}}{1-\gamma}\right) \\ &= E\left(\int_0^{\infty} e^{-(\rho+q)s}\frac{(c_s^*)^{1-\gamma}}{1-\gamma}\right) \leq J\left(W_0\right) \end{aligned}$$

The last inequality follows because $c_s^* = \left(\lambda^* e^{\rho s}H_s X_s^*\right)^{-\frac{1}{\gamma}}$ is a feasible consumption process for problem for problem 2 and $J\left(W_0\right)$

is the value function of the problem. The above three lines imply that equation (75) holds with equality as long as one chooses the optimal solution in the statement of the proposition. This concludes the proof of Proposition 2.

B Remaining Proofs

Proof of Proposition 3. The proof of this Proposition is just a special case of Section 6 in He and Pages (1993) and hence I give only a sketch and refer the reader to He and Pages (1993) for details.

To start define:

$$\widetilde{V}(\lambda) = \min_{X_s\in\mathcal{D}} E\left[\int_0^{\infty} e^{-(\rho+q)s}\max_{c_s}\left(\frac{c_s^{1-\gamma}}{1-\gamma}-\lambda e^{\rho s}H_s X_s c_s\right)ds+\lambda\int_0^{\infty} e^{-qs}H_s X_s y_0 ds\right] \tag{97}$$

By equation (13) and equation (19) of Proposition 1

$$V(W_0) = \min_{\lambda > 0} \left[\widetilde{V}(\lambda) + \lambda \left(W_0 - \frac{y_0}{r} \right) \right] \tag{98}$$

since

$$y_0 E \int_0^{\infty} H_s ds = \frac{y_0}{r}$$

Next, for an arbitrary decreasing process X_t let Z_t be defined as:

$$Z_t = \lambda e^{\rho s} H_s X_s$$

Note that $Z_0 = \lambda$. Applying Ito's Lemma to Z_t gives:

$$\frac{dZ_t}{Z_t} = (\rho - r)\, dt - \kappa dB_t + \frac{dX_t}{X_t} \tag{99}$$

With this definition of Z_t one can solve the maximization problem inside (97) and rewrite $\widetilde{V}(\lambda)$ as

$$\widetilde{V}(Z_0) = \min_{X_s \in \mathcal{D}} E \left[\int_0^{\infty} e^{-(\rho+q)s} \left(\frac{\gamma}{1-\gamma} Z_s^{1-\frac{1}{\gamma}} + y_0 Z_s \right) ds \right] \tag{100}$$

From this point on, one can use similar arguments to He and Pages (1993) and treat (100) as a singular stochastic control problem over the set of decreasing processes X_t. As He and Pages (1993) show, the optimal solution is to always decrease X_t appropriately, so as to keep Z_t in the interval $(0, \overline{Z}]$. $\overline{Z}$ is a free boundary that is determined next.

Using this conjecture for the optimal policy one can now proceed as He and Pages (1993) to establish that $\widetilde{V}(Z)$ satisfies the ordinary differential equation:

$$\frac{\kappa^2}{2} \widetilde{V}_{ZZ} Z^2 + (\rho - r)\, \widetilde{V}_Z Z - (\rho + q) \widetilde{V} + \frac{\gamma}{1-\gamma} Z^{1-\frac{1}{\gamma}} + y_0 Z = 0 \text{ for all } Z \in (0, \overline{Z}]$$

The general solution to this equation is:

$$\widetilde{V}(Z) = C_1 Z^{\phi} + C_2 Z^{\chi} + K \frac{\gamma}{1-\gamma} Z^{1-\frac{1}{\gamma}} + \frac{y_0}{r+q} Z \tag{101}$$

where K is given in (10), ϕ in (9) and χ in (86) and C_1, C_2 are arbitrary constants. As in He and Pages (1993) one can set $C_2 = 0$ (since $\chi < 0$). Hence it remains to determine C_1 and the free boundary $\overline{Z}$. As most singular stochastic control problems, one can employ a "smooth pasting" and "high contact" principle, namely by determining C_1 and $\overline{Z}$ so that:

$$\widetilde{V}_Z \left(\overline{Z} \right) = 0 \tag{102}$$

$$\widetilde{V}_{ZZ} \left(\overline{Z} \right) = 0 \tag{103}$$

Using the general solution in (101) along with $C_2 = 0$ and plugging into the equations (102) and (103) gives the system of equations

$$\begin{aligned}\phi C_1 \overline{Z}^{\phi-1} - K\overline{Z}^{-\frac{1}{\gamma}} + \frac{y_0}{r+q} &= 0 \\ \phi\left(\phi-1\right) C_1 \overline{Z}^{\phi-2} + \frac{1}{\gamma} K \overline{Z}^{-\frac{1}{\gamma}-1} &= 0\end{aligned}$$

Solving this system for C_1 and $\overline{Z}$ gives:

$$\overline{Z}^{-\frac{1}{\gamma}} = \frac{1}{K}\frac{y_0}{r+q}\left(\frac{\phi-1}{\frac{1}{\gamma}+\phi-1}\right) \tag{104}$$

$$C = -\frac{\frac{1}{\gamma}\frac{y_0}{r+q}}{\phi\overline{Z}^{\phi-1}\left[\frac{1}{\gamma}+\phi-1\right]} \tag{105}$$

The next steps to verify that the conjectured policy is indeed optimal are identical to He and Pages (1993) and are left out.

To conclude the proof , note that sofar the calculations were true for an arbitrary y_0. To determine the y_0 that will safeguard that $c_t \geq \xi$ observe that:

$$c_t = Z^{-\frac{1}{\gamma}}$$

by equation (20). Since the optimal policy is to control X_t so as to "keep" Z_t in the interval $(0, \overline{Z}]$ it follows that the minimum level of consumption is given by $\overline{Z}^{-\frac{1}{\gamma}}$. Hence, in order to guarantee condition $c_t \geq \xi$ it suffices to determine y_0 so that:

$$\xi = \overline{Z}^{-\frac{1}{\gamma}} = \frac{1}{K}\frac{y_0}{r+q}\left(\frac{\phi-1}{\frac{1}{\gamma}+\phi-1}\right)$$

Solving for y_0 gives:

$$y_0 = \xi(r+q)K\frac{\frac{1}{\gamma}+\phi-1}{\phi-1}$$

One can now substitute that level of y_0 into (105), (104) and use the resulting expressions to obtain from (101) the following expression for $\widetilde{V}(Z)$:

$$\widetilde{V}(Z) = -\frac{K\xi^{1-\gamma}}{\gamma\phi\left(\phi-1\right)}\left(\frac{Z}{\xi^{-\gamma}}\right)^{\phi} + K\frac{\gamma}{1-\gamma}Z^{1-\frac{1}{\gamma}} + \frac{y_0}{r+q}Z$$

Evaluating this expression at $Z_0 = \lambda$ and using equation (98) gives the value function of the agent as

$$\begin{aligned}V(W_0) &= \min_{\lambda>0}\left[\widetilde{V}(\lambda) + \lambda\left(W_0 - \frac{y_0}{r+q}\right)\right] = \\ &= \min_{\lambda>0}\left[-\frac{K\xi^{1-\gamma}}{\gamma\phi\left(\phi-1\right)}\left(\frac{\lambda}{\xi^{-\gamma}}\right)^{\phi} + K\frac{\gamma}{1-\gamma}\lambda^{1-\frac{1}{\gamma}} + \lambda W_0\right]\end{aligned}$$

This last equation is precisely equation (29) of proposition 2, which shows that the "constant income" policy of the current proposition attains the upper bound of proposition 2. ■

Proof of Proposition 4. The proof of this proposition proceeds in steps. The first two Lemmas establish that the proposed transfer policy will make it possible for an agent who follows the optimal consumption process of proposition 2 to satisfy the intertemporal budget constraint. The proof then continues to show that the wealth process associated with the optimal consumption process of proposition 2 along with the portfolio process (35) will lead to non-negative levels of wealth at all times. Finally, it is shown that the consumption policy of proposition 2 along with the portfolio choice (35) are optimal for an agent who is faced with transfers given by (34) and attain the upper bound of proposition 2.

Lemma 5 *Let K and ϕ be given by (10) and (9) and for any $0 < \lambda < \xi^{-\gamma}$ let:*

$$Z_t = \lambda e^{\rho s} H_s X_s^*$$

Then:

$$\int_0^\infty E_t \left(\int_t^\infty e^{-q(s-t)} H_s X_s^* dG_s - \int_t^\infty e^{-q(s-t)} H_s X_s^* Z_s^{-\frac{1}{\gamma}} ds \right) dX_t^* = 0 \tag{106}$$

Proof of Lemma 5. . It will simplify notation to let:

$$\eta = -K\xi \left(\phi - 1 + \frac{1}{\gamma} \right) \tag{107}$$

The first step is to compute

$$\frac{E_t \int_t^\infty e^{-qs} H_s X_s^* dG_s}{e^{-qt} H_t X_t^*} = \eta \frac{E_t \int_t^\infty e^{-qs} H_s dX_s^*}{e^{-qt} H_t X_t^*} \tag{108}$$

Applying integration by parts and using the definition of Z_t gives:

$$E_t \left(\int_t^T e^{-qs} H_s dX_s^* \right) = \frac{1}{\lambda} \left[-e^{-(\rho+q)t} Z_t + E_t \left(\int_t^T (r+q) \, e^{-(\rho+q)s} Z_s ds \right) \right] \tag{109}$$

Using (109) in equation (108) gives:

$$\frac{E_t \int_t^\infty e^{-qs} H_s X_s^* dG_s}{e^{-qt} H_t X_t^*} = \eta \left[(r+q) \frac{E_t \left(\int_t^T e^{-(\rho+q)(s-t)} Z_s ds \right)}{Z_t} - 1 \right] \tag{110}$$

By using a logic similar to equations (83)-(87) it can be shown that:

$$E_t \left(\int_t^T e^{-(\rho+q)(s-t)} Z_s ds \right) = -\frac{1}{\phi} \frac{\xi^{-\gamma}}{r+q} \left(\frac{Z_t}{\xi^{-\gamma}} \right)^{\phi} + \frac{1}{r+q} Z_t \tag{111}$$

where ϕ is defined in equation (9). Plugging back (111) into (110) gives:

$$\frac{E_t \int_t^\infty e^{-qs} H_s X_s^* dG_s}{e^{-qt} H_t X_t^*} = -\frac{\eta}{\phi} \left(\frac{Z_t}{\xi^{-\gamma}} \right)^{\phi-1} \tag{112}$$

To conclude the proof, note that equations (81) and (88) imply that:

$$\frac{E_t\left(\int_t^{\infty} e^{-qs} H_s X_s^* Z_s^{-\frac{1}{\gamma}} ds\right)}{e^{-qt} H_t X_t^*} = \frac{E_t\left(\int_t^{\infty} e^{-(\rho+q)(s-t)} Z_s^{1-\frac{1}{\gamma}} ds\right)}{Z_t} = \frac{\frac{1}{\phi}\frac{1-\gamma}{\gamma} K \xi^{1-\gamma} \left(\frac{Z_t}{\xi^{-\gamma}}\right)^{\phi} + K Z_t^{1-\frac{1}{\gamma}}}{Z_t} \tag{113}$$

Combining (113) with (112) gives:

$$\frac{E_t\left(\int_t^{\infty} e^{-qs} H_s X_s^* dG_s - \int_t^{\infty} e^{-qs} H_s X_s^* Z_s^{-\frac{1}{\gamma}} ds\right)}{e^{-qt} H_t X_t^*} =$$

$$= -\frac{\eta}{\phi^+}\left(\frac{Z_t}{\xi^{-\gamma}}\right)^{\phi^+ - 1} - \frac{\frac{1}{\phi^+}\frac{1-\gamma}{\gamma} K \xi^{1-\gamma} \left(\frac{Z_t}{\xi^{-\gamma}}\right)^{\phi^+} + K Z_t^{1-\frac{1}{\gamma}}}{Z_t}$$

Since $dX_t^* \neq 0$ when and only when $Z_t = \xi^{-\gamma}$, equation (106) amounts to checking that:

$$-\frac{\eta}{\phi^+} - \left(\frac{1}{\phi^+}\frac{1-\gamma}{\gamma} + 1\right) K \xi = 0$$

which follows easily from the definition of η. ■

Lemma 6 *Let Z_s^* be as in the statement of the proposition 4 and let G_t be as in (34). Then the consumption policy:*

$$c_s^* = \left(Z_s^*\right)^{-\frac{1}{\gamma}} \tag{114}$$

satisfies:

$$E \int_0^{\infty} H_s X_s^* c_s^* ds = W_0 + \int_0^{\infty} H_s \left(X_s^* - 1\right) dG_s \tag{115}$$

Proof of Lemma 6. Taking any $\lambda \in (\,0, \xi^{-\gamma}]$,using the definition of X_t^*, and equation (106), the same reasoning behind (56) leads to:

$$E\left(\int_0^{\infty} e^{-(\rho+q)s} \max_{c_s}\left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s} H_s X_s^* c_s\right) ds + \lambda \int_0^{\infty} e^{-qs} H_s \left(X_s^* - 1\right) dG_s\right) + \lambda W_0 = \tag{116}$$

$$= E\left[\int_0^{\infty} e^{-(\rho+q)s} \frac{\gamma}{1-\gamma} \left(e^{\rho s} \lambda H_s X_s^*\right)^{\frac{\gamma-1}{\gamma}} ds + \int_0^{\infty} e^{-(\rho+q)s} \left(e^{\rho s} \lambda H_s X_s^*\right)^{1-\frac{1}{\gamma}} \left(1 - \frac{1}{X_s^*}\right) ds\right] + \lambda W_0 \tag{117}$$

Hence the λ^* that minimizes (29) (and hence minimizes [117]) also minimizes (116). But since λ minimizes (116), the same argument as in He and Pages (1993) (Proof of Theorem 1) leads to (115). ■

Proof of Proposition 4 continued. Lemma 6 has demonstrated that the consumption policy (114) satisfies the intertemporal budget constraint (115). It remains to show that this consumption policy along with the portfolio policy (35) will lead to a process for financial wealth that satisfies $W_t \geq 0$. To that end let η be given as in (107) and define:

$$W^*\left(Z_t\right) = -K\left(\xi^{-\gamma}\right)^{-\frac{1}{\gamma}} \left(\frac{Z_t}{\xi^{-\gamma}}\right)^{\phi-1} + K Z_t^{-\frac{1}{\gamma}} \tag{118}$$

It is straightforward to verify the following facts about $W^*(Z_t)$:

$$\frac{\kappa^2}{2} Z^2 W^*_{ZZ} + \left(\rho - r + \kappa^2\right) Z W^*_Z - (r+q) W + (Z)^{-\frac{1}{\gamma}} = 0 \tag{119}$$

$$W^*\left(\xi^{-\gamma}\right) = 0, W^*(Z) \geq 0 \text{ for all } Z \in (0, \xi^{-\gamma}] \tag{120}$$

$$W^*_Z\left(\xi^{-\gamma}\right) = -K\xi\left(\phi - 1 + \frac{1}{\gamma}\right)\left(\xi^{-\gamma}\right)^{-1} = \frac{\eta}{\xi^{-\gamma}} \tag{121}$$

The next step is to verify that $W^*(Z_t)$ is the stochastic process for the financial wealth of the agent. To see this, use the definition of c^*_s (equation [114]) along with the definitions of dG_t, W^*_t (equations [34] and [118] respectively) and apply Ito's Lemma to obtain:

$$\begin{aligned}
& d\left(\int_0^t c^*_s ds - \int_0^t dG_s + W^*_t\right) = \\
&\\
= \quad & c^*_t dt - \eta \frac{dX^*_t}{X^*_t} + W^*_Z dZ_t + \frac{\kappa^2}{2} W^*_{ZZ} Z^2_t dt \\
= \quad & \left(c^*_t - Z_t^{-\frac{1}{\gamma}}\right) dt + \left[W^*_Z\left(\xi^{-\gamma}\right)\xi^{-\gamma} - \eta\right]\frac{dX^*_t}{X^*_t} + (r+q) W^*_t dt - \kappa^2 Z_t W^*_Z dt - \kappa W^*_Z Z_t dB_t = \\
= \quad & (r+q) W^*_t dt - \kappa^2 Z_t W^*_Z dt - \frac{\kappa}{\sigma} W^*_Z Z_t \left(\frac{dP_t}{P_t} - \mu dt\right) \\
= \quad & (r+q) W^*_t dt - \kappa^2 Z_t W^*_Z dt - \frac{\kappa}{\sigma} W^*_Z Z_t \left(\frac{dP_t}{P_t} - (\mu - r)\, dt - r dt\right) = \\
= \quad & q W^*_t dt + r\left(W^*_t + \frac{\kappa}{\sigma} W^*_Z Z_t\right) dt - \frac{\kappa}{\sigma} W^*_Z Z_t \frac{dP_t}{P_t} = \\
= \quad & q W^*_t dt + r\left(W^*_t - \pi^*_t\right) dt + \pi^*_t \frac{dP_t}{P_t}
\end{aligned}$$

Integrating gives

$$\int_0^t c^*_s ds + W^*_t = W_0 - D_0 + \int_0^t dG_s + \int_0^t q W^*_s dt + \int_0^t r\left(W^*_t - \pi^*_t\right) dt + \int_0^t \pi^*_t \frac{dP_t}{P_t}$$

Hence the process W^*_t satisfies the equation (15) for an agent who chooses a consumption policy given by (114) and a portfolio policy given by (35). Accordingly, it is the financial wealth process that is associated with that policy pair. Moreover, by equation (120) the financial wealth process is non-negative. Accordingly, the policies given by (114) and (35) are feasible for an agent who is faced with the transfer process (34).

Verifying the optimality of the stated policy pair is straightforward. According to proposition 1:

$$V(W_0) = \min_{\lambda>0,\ X_s \in \mathcal{D}} \left[\begin{array}{c} E\left(\int_0^\infty e^{-(\rho+q)s} \max_{c_s}\left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s} H_s X_s c_s\right) ds + \lambda \int_0^\infty e^{-qs} H_s X_s dG_s\right) \\ +\lambda\left(W_0 - D_0\right) \end{array}\right] \leq Q(W_0)$$

where:

$$Q(W_0) = \min_{\lambda>0} \left[\begin{array}{c} E\left(\int_0^\infty e^{-(\rho+q)s} \max_{c_s}\left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s} H_s X^*_s c_s\right) ds + \lambda \int_0^\infty e^{-qs} H_s X^*_s dG_s\right) \\ +\lambda\left(W_0 - D_0\right) \end{array}\right]$$

One can use now Lemma 6 to illustrate that the consumption policy (114) leads to a payoff for the agent equal to $Q(W_0)$ which is an upper bound to the value function of the agent $V(W_0)$. Since the consumption policy (114) is also feasible, the payoff associated with that policy also provides a lower bound to the value function $V(W_0)$. Hence this policy must be optimal, since the payoff associated with it is equal to the value function.

Finally, the easiest way to show that

$$D_0 = K\xi \frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} \left(\frac{\lambda^*}{\xi^{-\gamma}} \right)^{\phi - 1}$$

is to observe that the intertemporal budget constraint implies that:

$$E_{\tau_0} \left(\int_{\tau_0}^{\infty} e^{-q(s-\tau_0)} \frac{H_s}{H_{\tau_0}} c_s^* ds \right) = E_{\tau_0} \left(\int_{\tau_0}^{\infty} e^{-q(s-\tau_0)} \frac{H_s}{H_{\tau_0}} dG_s \right)$$

where τ_0 is the first time that $X_{\tau_0} \geq 1$ (or equivalently the first time that $W_{\tau_0} = 0$ and $\lambda^* e^{\rho \tau_0} H_{\tau_0} = \xi^{-\gamma}$) . A few manipulations can be used to show that

$$E_{\tau_0} \left(\int_{\tau_0}^{\infty} e^{-q(s-\tau_0)} \frac{H_s}{H_{\tau_0}} c_s^* ds \right) = \frac{N\left(\xi^{-\gamma}, 1\right)}{\xi^{-\gamma}} = K\xi \frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1}$$

where N is defined and computed in (89) and (90). Finally, since there are no transfers between 0 and τ_0 :

$$\begin{aligned} D_0 &= E\left(e^{-q\tau_0} H_{\tau_0}\right) K\xi \frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} = \frac{1}{\lambda^*} E\left(e^{-(\rho+q)\tau_0} \lambda^* e^{\rho\tau_0} H_{\tau_0}\right) K\xi \frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} = \\ &= \frac{\xi^{-\gamma}}{\lambda^*} E\left(e^{-(\rho+q)\tau_0}\right) K\xi \frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} = \left(\frac{\lambda^*}{\xi^{-\gamma}}\right)^{\phi-1} K\xi \frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} \end{aligned}$$

where the proof of $E\left(e^{-(\rho+q)\tau_0}\right) = \left(\frac{\lambda^*}{\xi^{-\gamma}}\right)^{\phi}$ is identical to the one given in Oksendal (1998), Chapter 10. ■

Proof of Proposition 5. . Take any transfer process G_t such that the resulting consumption process of the agent satisfies $c_t \geq \xi$. Proposition 1 implies then that there exists a cumulative multiplier process X_t^G and a constant λ^G such that:

$$c_t = \left(\lambda^G e^{\rho t} H_t X_t^G\right)^{-\frac{1}{\gamma}} \geq \xi$$

Letting:

$$X_t^* = \min\left[1, \frac{\xi^{-\gamma}/\lambda^G}{\max_{0 \leq s \leq t}\left(e^{\rho s} H_s\right)}\right]$$

and:

$$P = E\left(\int_0^{\infty} e^{-qs} H_s c_s ds\right)$$

gives:

$$P = E\left(\int_0^{\infty} e^{-qs} H_s \left(\lambda^G e^{\rho s} H_s X_s^G\right)^{-\frac{1}{\gamma}} ds\right) \geq E\left(\int_0^{\infty} e^{-qs} H_s \left(\lambda^G e^{\rho s} H_s X_s^*\right)^{-\frac{1}{\gamma}} ds\right) \tag{122}$$

since[31] $X_s^*\left(\lambda^G\right) \geq X_s^G$. Equation (92) implies that:

$$E\left(\int_0^\infty e^{-qs} H_s \left(\lambda^G e^{\rho s} H_s X_s^*\right)^{-\frac{1}{\gamma}} ds\right) = \frac{K\xi^{1-\gamma}}{(\phi-1)} \frac{1}{\gamma} \left(\frac{\lambda^G}{\xi^{-\gamma}}\right)^{\phi} \frac{1}{\lambda^G} + K\left(\lambda^G\right)^{-\frac{1}{\gamma}}$$

Combining (94) and (96) implies that the right hand side of the above equation is decreasing in λ^G whenever $\lambda^G \leq \xi^{-\gamma}$. Since $c_0 = \left(\lambda^G\right)^{-\frac{1}{\gamma}} \geq \xi$ this implies furthermore:

$$\begin{aligned} E\left(\int_0^\infty e^{-qs} H_s \left(\lambda^G e^{\rho s} H_s X_s^*\right)^{-\frac{1}{\gamma}} ds\right) &\geq \frac{K\xi^{1-\gamma}}{(\phi-1)} \frac{1}{\gamma} \frac{1}{\xi^{-\gamma}} + K\xi = K\xi\left(1 + \frac{1}{\phi-1}\frac{1}{\gamma}\right) \\ &= K\xi\left(\frac{\frac{1}{\gamma}+\phi-1}{\phi-1}\right) \end{aligned} \quad (123)$$

Combining (122) and (123) yields the conclusion of the theorem. ■

Proof of Proposition 6. First note that by raising the tax rate χ by an additional percentage point in each period prior to retirement the government can raise the agents minimum assets by:

$$\omega = Y\int_{-T}^{0} e^{-(r+q)s} ds = Y\frac{e^{(r+q)T}-1}{r+q}$$

By an argument similar to Proposition 1, the agent's value function at birth (time $-T$) can be rewritten as:

$$F = \min_{\widetilde{X}_s, \lambda>0} E_{(-T)} \left[\begin{array}{c} \int_{-T}^{0} e^{-(\rho+q)(s+T)} \max_{c_s}\left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho(s+T)} \widetilde{X}_s \frac{H_s}{H_{(-T)}} c_s\right) ds \\ +\lambda(1-\chi)Y\int_{-T}^{0} e^{-q(s+T)} \frac{H_s}{H_{(-T)}} \widetilde{X}_s ds + \max_{W_{0^+}\geq 0}\left(e^{-(\rho+q)T} J(W_{0^+}+\chi\omega) - \lambda \widetilde{X}_0 e^{-qT} \frac{H_0}{H_{(-T)}} W_{0^+}\right) \end{array} \right] \quad (124)$$

where $J\left(W_{0^+}+\chi\omega\right)$ is given in proposition 2 and $\widetilde{X}_s$ is a decreasing process starting at $\widetilde{X}_{(-T)} = 1$. Let the expected value of the expression inside the square brackets be denoted as $U(\widetilde{X}_s, \lambda)$, so that:

$$F^{(\chi)} = \min_{\widetilde{X}_s, \lambda} U(\widetilde{X}_s, \lambda; \chi)$$

Differentiating $U(\widetilde{X}_s, \lambda; \chi)$ with respect to χ gives:

$$U_\chi = E_{(-T)}\left[1_{\left\{\lambda \widetilde{X}_0 \frac{H_0}{H_{(-T)}} < \xi^{-\gamma}\right\}} e^{-(\rho+q)T} J'(W_{0^+}+\chi\omega)\omega - \lambda Y \int_{-T}^{0} e^{-q(s+T)} \frac{H_s}{H_{(-T)}} \widetilde{X}_s ds\right] \quad (125)$$

Whenever $\lambda \widetilde{X}_0 \frac{H_0}{H_{(-T)}} \geq \xi^{-\gamma}$, so that the constraint $W_{0^+} \geq 0$ does not bind, one can use the first order condition from the second maximization problem inside the square brackets of (124) to obtain

$$J'(W_{0^+}+\chi\omega) = \lambda \widetilde{X}_0 e^{\rho T} \frac{H_0}{H_{(-T)}}$$

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[31] This is an implication of the Skorohod equation. See Karatzas and Shreve (1991).

This allows one to rewrite expression (125) as

$$\begin{aligned} U_\chi &= E_{(-T)}\left[\left(1_{\left\{\lambda\widetilde{X}_0\frac{H_0}{H_{(-T)}}<\xi^{-\gamma}\right\}}\lambda\widetilde{X}_0 e^{-qT}\frac{H_0}{H_{(-T)}}\omega-\lambda Y\int_{-T}^{0}e^{-q(s+T)}\frac{H_s}{H_{(-T)}}\widetilde{X}_s ds\right)\right] \\ &\le E_{(-T)}\left[\left(\lambda\widetilde{X}_0 e^{-qT}\frac{H_0}{H_{(-T)}}\omega-\lambda Y\int_{-T}^{0}e^{-q(s+T)}\frac{H_s}{H_{(-T)}}\widetilde{X}_s ds\right)\right] \\ &= \lambda\delta e^{-qT}\omega-\lambda\delta E_{(-T)}\left(\int_{-T}^{0}e^{-q(s+T)}Y\frac{H_s}{H_{(-T)}}\frac{\widetilde{X}_s}{\delta}ds\right) \end{aligned} \tag{126}$$

where:

$$\delta = E_{(-T)}\left(\widetilde{X}_0\frac{H_0}{H_{(-T)}}\right)$$

Furthermore,

$$\begin{aligned} E_{(-T)}\left(\int_{-T}^{0}e^{-q(s+T)}Y\frac{H_s}{H_{(-T)}}\frac{\widetilde{X}_s}{\delta}ds\right) &= \int_{-T}^{0}e^{-q(s+T)}Y\frac{E_{(-T)}\left(H_s\widetilde{X}_s\right)}{E_{(-T)}\left(H_0\widetilde{X}_0\right)}ds = \\ &= e^{rT}\int_{-T}^{0}e^{-(r+q)(s+T)}Y\frac{E_{(-T)}\left(e^{r(s+T)}\frac{H_s}{H_{(-T)}}\widetilde{X}_s\right)}{E_{(-T)}\left(e^{rT}\frac{H_0}{H_{(-T)}}\widetilde{X}_0\right)}ds \\ &\ge Ye^{rT}\int_{-T}^{0}e^{-(r+q)(s+T)}ds = \omega e^{-qT} \end{aligned} \tag{127}$$

where the inequality follows from the fact that $e^{rs}H_s$ is a martingale while $\widetilde{X}_s$ is a decreasing process, so that $\widetilde{X}_s \ge \widetilde{X}_0$ for all $s \in [-T, 0]$. Combining (126) and (127) leads to $U_x \le 0$.

Hence, letting $\chi^{\min}$ denote the minimum tax rate that will satisfy (8) as given by (43), it follows that $U(\widetilde{X}_s,\lambda;\chi^{\min}) > U(\widetilde{X}_s,\lambda;\chi)$ for all $\chi \in \left(\chi^{\min}, 1\right)$. This furthermore implies that:

$$F^{(\chi^{\min})} = U(\widetilde{X}_s^{\chi^{\min}}, \lambda^{\chi^{\min}}; \chi^{\min}) \ge U(\widetilde{X}_s^{\chi^{\min}}, \lambda^{\chi^{\min}}; \chi) \ge U(\widetilde{X}_s^{\chi}, \lambda^{\chi}; \chi) = F^{(\chi)}$$

where $\widetilde{X}_s^{\chi}, \lambda^{\chi}$ denote the minimizers of U given χ and similar for $\widetilde{X}_s^{\chi^{\min}}, \lambda^{\chi^{\min}}$. Hence it is never optimal to set the tax rate above $\chi^{\min}$. ■

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