

Continuous Record Laplace-based Inference about the Break Date in Structural Change Models*

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Abstract

Building upon the continuous record asymptotic framework recently introduced by [Casini and Perron \(2020a\)](#) for inference in structural change models, we propose a Laplace-based (Quasi-Bayes) procedure for the construction of the estimate and confidence set for the date of a structural change. It is defined by an integration rather than an optimization-based method. A transformation of the least-squares criterion function is evaluated in order to derive a proper distribution, referred to as the Quasi-posterior. For a given choice of a loss function, the Laplace-type estimator is the minimizer of the expected risk with the expectation taken under the Quasi-posterior. Besides providing an alternative estimate that is more precise—lower mean absolute error (MAE) and lower root-mean squared error (RMSE)—than the usual least-squares one, the Quasi-posterior distribution can be used to construct asymptotically valid inference using the concept of Highest Density Region. The resulting Laplace-based inferential procedure is shown to have lower MAE and RMSE, and the confidence sets strike a better balance between empirical coverage rates and average lengths of the confidence sets relative to traditional long-span methods, whether the break size is small or large.

JEL Classification: C12, C13, C22

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1 Introduction

In recent work [Casini and Perron \(2020a\)](#), henceforth CP, developed a continuous record asymptotic framework for inference about the break date in a linear time series regression model with a single shift in some regression parameters while [Casini and Perron \(2020b\)](#) introduced the Generalized Laplace (GL) estimation and inference as an alternative to least-squares in structural change models. In this paper we use the GL method under the continuous record asymptotic framework of CP to tackle some of the problems in this literature. To illustrate the issues involved, consider the simple model $Y_{kh} = (\delta_1^0 + \delta^0 \mathbf{1}_{\{k > T_b^0\}})Z_{kh} + e_{kh}$. We have access to $T + 1$ observations ($k = 0, \dots, T$) at equidistant time intervals h over a fixed time span $N = Th$. The parameter δ^0 is referred to as the magnitude of the break. The statistical problem is to 1) estimate the break date T_b^0 with highest precision possible and, 2) construct confidence sets for this quantity that possess correct empirical coverage rates and short average lengths in finite samples. So far with respect to 1) little work has been done to consider alternatives to OLS in the context of the linear model. With respect to 2) several methods have been investigated, some of which work well with small breaks, while others work well with large ones, none of which deliver adequate coverage rates and “decent” lengths of the confidence sets over all break sizes. We refer to [Casini and Perron \(2019\)](#), [Chang and Perron \(2018\)](#) and the discussion later for an investigation on the drawbacks of popular existing methods, such as those proposed in [Bai \(1997\)](#) and [Elliott and Müller \(2007\)](#). In contrast, CP developed an alternative asymptotic framework based on a continuous record of observations over a fixed time horizon $[0, N]$. This involves letting the sample size T grow to infinity by requiring the sampling $h \downarrow 0$ at the same rate so that $N = Th$ remains fixed. The limiting distribution depends on structural parameters but feasible inference can be developed.

We consider the asymptotics with $h \downarrow 0$ as an alternative asymptotic experiment that can deliver better approximations for discrete-time applications and not necessarily as a framework that is useful only for high-frequency data. [Figures 1-2](#) present plots of the density of the distribution of the least-squares estimate of the break date for the simple model discussed above with a parameter shift associated to a regressor Z_{kh} specified to follow an ARMA(1,1) process and i.i.d. Gaussian disturbances e_{kh} . The distributions presented are: the exact finite-sample distribution, [Bai’s \(1997\)](#) classical large- N limit distribution, CP’s continuous record limit distribution and its feasible version (which uses plug-in estimates for the true parameter values). The continuous record limiting distributions provide an impressively accurate approximation to the finite-sample distribution for both small (cf. [Figure 1](#)) and large break sizes (cf. [Figure 2](#)) and does so for different break locations (at one-fourth and half-sample in, respectively, the left and right plots). Additionally, it is evident that the classical shrinkage large- N asymptotic distribution does not offer an accurate approximation, especially when the break magnitude is small (cf. [Figure 1](#)).

[Bai’s \(1997\)](#) method for the construction of the confidence intervals for T_b^0 relies on standard

asymptotic arguments with the span increasing such that each regime increases proportionately. This coupled with the standard mixing assumption implies that the limit distribution depends only on a neighborhood around the true value. This is in stark contrast to the finite-sample distribution which, especially for small breaks, is influenced by the location of the break and the properties of the processes over the whole sample. Hence, it should not be surprising that the confidence intervals from Bai’s (1997) method display empirical coverage rates often below the nominal level when the break is not large. CP documented the highly non-standard features of the density and consequently proposed an alternative inference procedure based on the concept of Highest Density Region (HDR). Their method was shown to offer the best balance between empirical coverage rates and average lengths of the confidence sets among existing methods.

We develop additional inference procedures for the break date in a linear regression model based on the continuous record asymptotic framework and using the Generalized Laplace estimation method. The aim is to provide both a more precise estimate and methods to construct confidence sets with good finite-sample properties. The idea behind the Laplace (or Quasi-Bayes) estimator goes back to Laplace (1774). For a recent application, see Chernozhukov and Hong (2003) for regular problems oriented toward microeconomic applications [see also Forneron and Ng (2018) for a review and comparisons]. Laplace-type estimators rely on statistical criterion functions and integration-based estimation rather than optimization. They are constructed as integral transformations of extremum criterion functions and can be computed using Markov Chain Monte Carlo methods (MCMC). The integral transformation of the criterion function is meant to provide an approximation to the likelihood approach. Hence, they are often referred to as Quasi-Bayesian estimators. Since no parametric likelihood is available, such a transformation turns the objective function into a proper density—referred to as the Quasi-posterior—over the parameter of interest. Our goal is to consider Laplace estimators and HDR methods to construct the confidence sets, as alternatives to the least-squares estimator coupled with classical inference procedure, which have better finite sample properties, namely reduced Mean Absolute Error (MAE) and Root-Mean Squared Error (RMSE), and confidence sets with accurate coverage rates and short lengths. Of particular importance, we aim to achieve this goal whatever the size of the break, a notoriously difficult problem in the structural change literature.

The use of Laplace-type estimators in Chernozhukov and Hong (2003) was mainly motivated by a need to circumvent the curse of dimensionality inherent in the computation of some classical extremum estimators. Furthermore, they focused on regular problems for which a quadratic expansion of the criterion function is available. Our implementation stems from different concerns. First, our motivation does not arise from a practical need of reducing the curse of dimensionality inherent to computation. Indeed, the least-squares estimates are simple to compute even in models with multiple changes [cf. Hawkins (1976) and Bai and Perron (2003) for an efficient algorithm based on the principle of dynamic programming]. Using the Laplace-type estimator, we aim to have better

estimates than those based on the least-squares principle; e.g., reduced MAE and RMSE. This should not come at the expense of less adequate confidence sets. Hence, we use the Laplace-type estimator in conjunction with the continuous record limiting distribution and the HDR method to construct confidence sets. A second crucial distinction is that estimating the date of a structural change is a non-standard statistical problem. The limiting distribution of the break point estimator is related to the location of the maximum of a two-sided Gaussian process with drift and thus quite different from the asymptotic distribution of regular estimators. More importantly, the objective function does not admit a quadratic expansion, which adds technical complexities. Although the definition of the Laplace estimator in our context is similar to [Chernozhukov and Hong \(2003\)](#), its asymptotic properties are different in that the contribution of information provided by the prior function does not vanish asymptotically due to the non-regularities of the structural change problem. Hence, the estimator does not share the same Quasi-Bayesian characterization used by [Chernozhukov and Hong \(2003\)](#) in regular problems. In addition, our estimator also differs from the Expectation-Maximization (EM) algorithm [see [Dempster, Laird, and Rubin \(1977\)](#)] because we compute the expected risk for a given loss function and not just the expected value of some log-likelihood function. However, what our approach shares with the EM algorithm is that when the “Quasi-prior” defined below is used, then our estimator also relies on a two-step procedure.

The Laplace-type estimation is explored as follows. We begin with the least-squares estimation of the break date and present the feasible limit distribution developed by CP under a continuous record framework. We treat the corresponding density function as our “prior” and call it the “Quasi-prior”.¹ We compute a transformation of the least-squares criterion function and combine it with the Quasi-prior in order to derive a proper distribution which we refer to as the “Quasi-posterior”. We then use simple computational methods based on integration to define the so-called “Generalized Laplace” (GL) estimator. Asymptotically valid inferences are constructed using HDR-type methods. The main idea behind the usefulness of our method can be explained as follows. First, note that the least-squares objective function is quite flat with respect to the candidates break dates T_b . However, the distribution of the least-squares estimate is quite informative in that it assigns sharply different density mass across T_b (c.f. the multi-modal feature). Hence, working with the objective function weighted by a Quasi-prior set equal to the density of the least-square estimate yields estimates with better properties as shown in the simulations. Using a least-absolute deviation loss function, the Laplace estimate with our chosen quasi-prior is shown to have substantially lower MAE and RMSE in finite-samples compared to that obtained using least-squares. As in other methods, our procedure relies on a trimming parameter $0 < \epsilon < 1$ which prevents the estimator to locate the break date in the first and last $100\epsilon\%$ of the sample. To

¹This distribution should not be viewed as a prior in Bayesian sense. Indeed, it should be simply interpreted as a weight function. Furthermore, it should be noted that this step implicitly relies on an optimization procedure which may not be present under the setting of [Chernozhukov and Hong \(2003\)](#).

achieve good results it is necessary that the trimming parameter should not be chosen too high because otherwise the estimate might tend to overestimate (resp. underestimate) the break date if it is in the first (resp. second) half of the sample. With regards to the GL estimator, the smaller the trimming the larger is the information used by the procedure. Since its construction relies on the overall behavior of the criterion function we find that $\epsilon = 0.05$ performs well for different locations of the break date. Our proposed inference results in confidence sets for the break date with more accurate coverage rates and often accompanied by a non-trivial reduction in the width of the confidence set relative to popular methods.

Recently [Baek \(2019\)](#) proposed to modify the least-squares objective function by applying some weights so as to reduce the multimodality effect. The latter estimator displays lower RMSE but has often higher bias relative to the original LS estimator. The latter may be due to the fact that the weighting is not data-dependent while our approach fully exploits the information from the criterion function since the weighting or quasi-prior is data-dependent.

This paper relates to two recent papers, namely [CP \(2020a, 2020b\)](#). CP developed the continuous record asymptotics that we use in this paper and proposed confidence sets for the break date based on the continuous record asymptotic distribution. Hence, it contains the essential ingredients for our proposed GL method, while at the same time offering alternative inference procedures. [Casini and Perron \(2020b\)](#) analyzed the GL method under classical asymptotics and focused on the theoretical relationship between the asymptotic distribution of frequentist and Bayesian estimators of the break point. They showed that depending on some input parameter, the GL estimator exhibits a dual limit distribution—the shrinkage asymptotic distribution of [Bai and Perron \(1998\)](#), or a Bayes-type asymptotic distribution [cf. [Ibragimov and Has'minskiĭ \(1981\)](#)]. Moreover, the results in the former extend to models with multiple breaks and models with trending regressors which are not covered in this paper. Finally, some of the results in this paper allow for long-memory which is ruled out by [Casini and Perron \(2020b\)](#). See also [Phillips \(1987\)](#) and [Perron \(1991\)](#) for early applications of continuous record asymptotics in the context of unit roots.

Recent works have used continuous-time asymptotics for structural change and nonstationarity models more generally. Besides the already discussed approach of CP, [Jiang, Wang, and Yu \(2018\)](#) studied the finite-sample bias of a break point estimator for a univariate diffusion with constant volatility and a change-point in the drift while [Jiang, Wang, and Yu \(2017\)](#) focused on the break point estimator for the Ornstein-Uhlenbeck process. Their approach is different from CP in that their results are less general and feasible inference for the break point is not discussed. Finally, [Chambers and Taylor \(2019\)](#) considered both deterministic one-time and continuous stochastic parameter change in a continuous-time autoregressive model while [Casini \(2018\)](#) introduced continuous-time asymptotics to test for forecast failure.

The paper is organized as follows. Section 2 introduces the setup. Section 3 summarizes results from CP needed for subsequent analyses. In Section 4, we develop the large-sample properties of

the GL estimators. We verify their accuracy in Section 5. Section 6 describes the inference methods proposed, which are evaluated via simulations Section 7. Section 8 offers brief concluding remarks. All technical derivations are contained in a supplement [Casini and Perron (2020c)].

2 The Statistical Setting

Section 2.1 introduces the model and some assumptions. We also provide a discussion of our assumptions and the relationship of our framework with that of the long-span (shrinkage) asymptotics considered by Bai (1997), Bai and Perron (1998) and Perron and Qu (2006). The least-squares estimator is defined in Section 2.3. The following notations are used in the sequel. All limits are taken as $T \rightarrow \infty$ with the span N kept fixed, where $N = Th$ with h , the sampling interval so that $h \downarrow 0$ at the same rate as $T \rightarrow \infty$. \mathbb{R} denotes the set of real numbers. For two vectors a and b , we write $a \leq b$ if the inequality holds component-wise. We denote the transpose of a matrix A by A' . The (i, j) elements of A are denoted by $A^{(i,j)}$. We use $\|\cdot\|$ to denote the Euclidean norm of a linear space i.e., $\|x\| = (\sum_{i=1}^p x_i^2)^{1/2}$ for $x \in \mathbb{R}^p$. For a matrix A we use the vector-induced norm, i.e., $\|A\| = \sup_{x \neq 0} \|Ax\| / \|x\|$. For a sequence of matrices $\{A_T\}$, we write $A_T = o_{\mathbb{P}}(1)$ if each of its elements is $o_{\mathbb{P}}(1)$ and likewise for $O_{\mathbb{P}}(1)$. The symbol $[\cdot]$ denotes the largest smaller integer function while \otimes is used for the product of σ -fields. A sequence $\{u_{kh}\}_{k=1}^T$ is i.i.d. (resp., i.n.d.) if the u_{kh} are independent and identically (resp., non-identically) distributed. We use $\xrightarrow{\mathbb{P}}$, and \Rightarrow to denote, respectively, convergence in probability (under some measure \mathbb{P}), and weak convergence, while $\xrightarrow{\mathcal{L}^s}$ is used to denote stable convergence in law. We write $\mathcal{MN}(a, b)$ for the mixed normal distribution with parameters (a, b) . The space $\mathbb{M}_p^{\text{càdlàg}}$ collects all $p \times p$ positive definite real-valued matrices whose elements are càdlàg. We use the superscript $+$ in \mathbb{M}_p if these matrices are required to be positive definite. \mathbf{FV}^c denotes the class of continuous adapted finite variation processes and $\mathbf{M}_{\text{loc}}^c$ denotes the class of continuous local martingale processes with finite positive definite conditional variance. For semimartingales $\{S_t\}_{t \geq 0}$ and $\{V_t\}_{t \geq 0}$, we denote their covariation process by $[S, V]_t$ and its predictable counterpart by $\langle S, V \rangle_t$. We anticipate that in our setting the latter two processes will be equivalent. The symbol “ \triangleq ” refers to definitional equivalence.

2.1 The Model

A standard discrete-time partial structural change model with a single break is given by:

$$\begin{aligned} Y_t &= D_t' \varrho^0 + Z_t' \delta_1^0 + u_t, & (t = 0, \dots, T_b^0) \\ Y_t &= D_t' \varrho^0 + Z_t' \delta_2^0 + u_t, & (t = T_b^0 + 1, \dots, T), \end{aligned} \quad (2.1)$$

where Y_t is the dependent variable, D_t and Z_t are, respectively, $p \times 1$ and $q \times 1$ vectors of regressors and e_t is an unobservable disturbance. The statistical problem is to estimate the unknown parameters ϱ^0 , δ_1^0 , δ_2^0 and the break date T_b^0 . A structural change occurs at T_b^0 because by assumption $\delta_1^0 \neq \delta_2^0$. The break magnitude is $\delta^0 = \delta_2^0 - \delta_1^0$. The inference problem is to construct estimates and confidence sets for the break date T_b^0 when $T + 1$ observations on (Y_t, D_t, Z_t) are available.

We introduce the high-frequency setting which serves as a probabilistic background on which our asymptotic arguments are developed. There is a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ on which the continuous time process $X \triangleq (Y, D', Z)'$ is defined, where $Y \triangleq \{Y_t\}_{t \geq 0}$, $D \triangleq \{D_t\}_{t \geq 0}$, $Z \triangleq \{Z_t\}_{t \geq 0}$ are assumed to be Itô semimartingales. We observe $T + 1$ realizations of Y_t , D_t and Z_t at equidistant discrete times $t = 0, h, 2h, \dots, N$, where $N \triangleq Th$ is the length of the fixed time span $[0, N]$. Note that in general N is not identified and could be normalized to one. However, we keep a generic N throughout to allow a better intuitive understanding of the results. Under the continuous record asymptotic scheme, we let the sample size T grow by shrinking the sampling interval h to zero so that the span N remains fixed.

For each h , $D_{kh} \in \mathbb{R}^p$ and $Z_{kh} \in \mathbb{R}^q$ are random vector step functions changing at times $0, h, 2h, \dots, Th$. The continuous-time unobservable error sequence is the *càdlàg* adapted process $\{e_t^*\}$. The regressors D_{kh} and Z_{kh} include locally-integrable semimartingales (extended to allow for predictable processes in Section 2.2). We assume that the discretized processes D_{kh} and Z_{kh} are adapted to the increasing and right-continuous filtration $\{\mathcal{F}_t\}$. By the Doob-Meyer decomposition we can write,² for $k = 1, \dots, T$: $\Delta_h D_k \triangleq \mu_{D,k} h + \Delta_h M_{D,k}$ and $\Delta_h Z_k \triangleq \mu_{Z,k} h + \Delta_h M_{Z,k}$, where the drifts $\mu_{D,t} \in \mathbb{R}^p$, $\mu_{Z,t} \in \mathbb{R}^q$ are \mathcal{F}_{t-h} -measurable, and $M_{D,k} \in \mathbb{R}^p$, $M_{Z,k} \in \mathbb{R}^q$ are continuous local martingales with \mathbb{P} -a.s. finite conditional covariance matrices $\mathbb{E}(\Delta_h M_{D,t} \Delta_h M'_{D,t} | \mathcal{F}_{t-h}) = \Sigma_{D,t-h} \Delta t$ and $\mathbb{E}(\Delta_h M_{Z,t} \Delta_h M'_{Z,t} | \mathcal{F}_{t-h}) = \Sigma_{Z,t-h} \Delta t$. Exact assumptions will be given below. To the continuous-time process $\{e_t^*\}$ corresponds a disturbance sequence $\{\Delta_h e_t^*\}$ such that $\{\Delta_h e_t^*, \mathcal{F}_t\}$ is a continuous local martingale difference sequence with finite conditional variance given by $\mathbb{E}[(\Delta_h e_t^*)^2 | \mathcal{F}_{t-h}] = \sigma_{e,t-h}^2 \Delta t$ \mathbb{P} -a.s. Under this formulation, the discretized model is

$$\Delta_h Y_k \triangleq \begin{cases} (\Delta_h D_k)' \varrho^0 + (\Delta_h Z_k)' \delta_1^0 + \Delta_h e_k^*, & (k = 1, \dots, \lfloor T \lambda_0 \rfloor) \\ (\Delta_h D_k)' \varrho^0 + (\Delta_h Z_k)' \delta_2^0 + \Delta_h e_k^*, & (k = \lfloor T \lambda_0 \rfloor + 1, \dots, T) \end{cases} \quad (2.2)$$

where $\lambda_0 \in (0, 1)$, $\varrho^0 \in \mathbb{R}^p$, $\delta_1^0, \delta_2^0 \in \mathbb{R}^q$, and $\delta^0 \triangleq \delta_2^0 - \delta_1^0$. It holds that $\delta^0 \neq 0$ so that a structural change in the parameter associated with $\Delta_h Z_k$ has occurred at time $\lfloor N \lambda_0 \rfloor$. Under our setting, we need to distinguish between the actual break date $N_b^0 = N \lambda_0$ (on a calendar time) and the index of the discrete-time observation associated with the break point: $T_b^0 \triangleq \lfloor T \lambda_0 \rfloor = \lfloor N_b^0 / h \rfloor$. Under

²For any process X we denote the “increments” of X by $\Delta_h X_k = X_{kh} - X_{(k-1)h}$. We sometimes use Δt in place of h in order to conveniently make our sums look like integrals.

the classical long-span setting this is not necessary since $h = 1$ or $N = T$.

The specification implicitly assumes the following continuous-time data-generating process,

$$D_t = D_0 + \int_0^t \mu_{D,s} ds + \int_0^t \sigma_{D,s} dW_{D,s}, \quad Z_t = Z_0 + \int_0^t \mu_{Z,s} ds + \int_0^t \sigma_{Z,s} dW_{Z,s}, \quad (2.3)$$

where $\mu_{D,t}$ (resp., $\mu_{Z,t}$) is the infinitesimal conditional mean of D_t (resp., Z_t) and takes value in \mathbb{R}^p (resp., \mathbb{R}^q); $\sigma_{D,t}$ and $\sigma_{Z,t}$ are the instantaneous covariance processes taking values in $\mathbb{M}_p^{\text{càdlàg}}$ and $\mathbb{M}_q^{\text{càdlàg}}$, respectively; W_D (resp., W_Z) is a p (resp., q)-dimensional standard Wiener process; and D_0 and Z_0 are \mathcal{F}_0 -measurable random vectors. The process e_t^* is a continuous local martingale orthogonal (in martingale sense) to both D_t and Z_t , i.e., $\langle e, D \rangle_t = \langle e, Z \rangle_t = 0$ for all t . We consider only processes with continuous sample paths, as stated in the following assumption.

Assumption 2.1. D, Z, e and $\Sigma^0 \triangleq \{\Sigma_{\cdot,t}, \sigma_{e,t}\}_{t \geq 0}$ are continuous Itô semimartingales.

Assumption 2.1 rules out processes with discontinuous sample paths from our analysis. Hence, our results are not expected to provide good approximations for applications involving high-frequency data for which jumps are likely to be important; e.g., Andersen, Fusari, and Todorov (2016), Bandi and Renò (2016) and Hounyo, Liu, and Varneskov (2020) for financial-oriented applications and Li and Xiu (2016) for a GMM setup; for a textbook account, see Aït-Sahalia and Jacod (2014). Another important difference from the high-frequency statistics centers on the “mean effect” which is captured here by the drift process. This poses several challenges because the drift is not identified under a continuous record asymptotics. CP dealt with this issue by introducing a so-called small-dispersion model, for which the parameters affecting the limit distribution can be consistently estimated and all theoretical results are shown to apply to this case as well (c.f. Section 2.2 below).

Assumption 2.2. The model (2.2)-(2.3) satisfies the following: (i) the càdlàg processes $\sigma_{D,t}$ and $\sigma_{Z,t}$ are locally bounded; (ii) $\mu_{D,t}$ and $\mu_{Z,t}$ are locally bounded. $\int_0^t \mu_{D,s} ds$ and $\int_0^t \mu_{Z,s} ds$ belong to \mathbf{FV}^c ; (iii) $\int_0^t \sigma_{D,s} dW_{D,s}, \int_0^t \sigma_{Z,s} dW_{Z,s} \in \mathbf{M}_{\text{loc}}^c$. Further, they possess \mathbb{P} -a.s. finite positive definite conditional variances defined by $\Sigma_{D,t} = \sigma_{D,t} \sigma'_{D,t}$ and $\Sigma_{Z,t} = \sigma_{Z,t} \sigma'_{Z,t}$, which for all $t < \infty$ satisfy, for $j = 1, \dots, p$: $\int_0^t \Sigma_{D,s}^{(j,j)} ds < \infty$ and $\int_0^t \Sigma_{Z,s}^{(j,j)} ds < \infty$ where $\Sigma_{D,t}^{(j,r)}$ denotes the (j,r) -th element of the process $\Sigma_{D,t}$. Furthermore, for every $j = 1, \dots, p, r = 1, \dots, q$, and $k = 1, \dots, T$, $h^{-1} \int_{(k-1)h}^{kh} \Sigma_{D,s}^{(j,j)} ds$ and $h^{-1} \int_{(k-1)h}^{kh} \Sigma_{Z,s}^{(r,r)} ds$ are finite and bounded away from zero, uniformly in k and h ; (iv) $e^* \triangleq \{e_t^*\}_{t \geq 0}$ is a continuous local martingale satisfying $e_t^* \triangleq \int_0^t \sigma_{e,s} dW_{e,s}$ with $0 < \sigma_{e,t}^2 < \infty$ for all $t \geq 0$, where W_e is a Wiener process. Also, $\langle e, D \rangle_t = \langle e, Z \rangle_t = 0, t \geq 0$.

Parts (i)-(iii) contains regularity conditions imposed in the high-frequency financial statistics literature [cf. Barndorff-Nielsen and Shephard (2004), Li, Todorov, and Tauchen (2017) and Li and Xiu (2016)]. Part (iv) specifies the disturbance process to possess continuous sample paths

and to be contemporaneously orthogonal to the regressors. Assumptions 2.1-2.2 imply that the processes in our model are diffusion-type processes if one further assumes that the volatilities are deterministic. We shall not impose the latter condition. As a consequence, $X = (Y, D', Z)'$ is a member of the continuous stochastic volatility semimartingale class. We model volatility as a latent factor since it does not pose any substantial impediment for the development of our theoretical results and they are more general including, for example, nonstationarity and long-memory.

Assumption 2.3. $N_b^0 = \lfloor N\lambda_0 \rfloor$, for some $\lambda_0 \in (0, 1)$.

Assumption 2.3 dictates the asymptotic framework adopted and implies that the break date occurs at the observation-index $T_b^0 = T\lambda_0$, where $T_b^0 = N_b^0/h$. Under our framework it implies that the pre- and post-break segments of the sample remain fixed. The usual assumption under the classical large- N asymptotics implies that the time horizons before and after the break date grow proportionately, which, along with the mixing assumption implies that only a small neighborhood around the true break date is relevant asymptotically, thereby ruling out the possibility to discern any asymmetric feature of the asymptotic distribution simply caused by the location of the break date. The continuous record asymptotic framework preserves information about the data span and the location of the break, and the mixing and ergodic assumptions are not needed. CP showed that the theoretical results derived for conducting inference about the break date in model (2.2) are applicable to classical structural change models for which a long-span setting is usually adopted. It is convenient to use the following re-parametrization. Let $y_{kh} = \Delta_h Y_{kh}$, $x_{kh} = (\Delta_h D'_k, \Delta_h Z'_k)'$, $z_{kh} = \Delta_h Z_k$, $e_{kh} = \Delta_h e_k^*$, and $\beta^0 = ((\varrho^0)', (\delta_1^0)')'$. Then, (2.2) is

$$\begin{aligned} y_{kh} &= x'_{kh} \beta^0 + e_{kh}, & (k = 1, \dots, T_b^0) \\ y_{kh} &= x'_{kh} \beta^0 + z'_{kh} \delta^0 + e_{kh}, & (k = T_b^0 + 1, \dots, T). \end{aligned} \quad (2.4)$$

We further define the full column rank $(p+q) \times q$ known matrix H such that $z_{kh} = H'x_{kh}$. We can then state model (2.4) in matrix format, which will be used for the derivations. Let $Y = (y_h, \dots, y_{Th})'$, $X = (x_h, \dots, x_{Th})'$, $e = (e_h, \dots, e_{Th})'$, $X_1 = (x_h, \dots, x_{T_b^0 h}, 0, \dots, 0)'$, $X_2 = (0, \dots, 0, x_{(T_b^0+1)h}, \dots, x_{Th})'$ and $X_0 = (0, \dots, 0, x_{(T_b^0+1)h}, \dots, x_{Th})'$. Also, $Z_1 = X_1 H$, $Z_2 = X_2 H$ and $Z_0 = X H$. Then (2.4) is equivalent to

$$Y = X\beta^0 + Z_0\delta^0 + e. \quad (2.5)$$

2.2 The Extended Model with Predictable Processes

The assumptions on D_t and Z_t specify that these processes are continuous Itô semimartingales of the form (2.3). This precludes predictable processes, which are often of interest in applications; e.g., a constant and/or a lagged dependent variable. Technically, these require a separate treatment

since the coefficients associated with predictable processes are not identified under a fixed-span asymptotic setting. CP considered the following extended model:

$$\Delta_h Y_k \triangleq \begin{cases} \mu_{1,h}h + \alpha_{1,h}Y_{(k-1)h} + (\Delta_h D_k)' \varrho^0 + (\Delta_h Z_k)' \delta_{Z,1}^0 + \Delta_h e_k^*, & (k = 1, \dots, \lfloor T\lambda_0 \rfloor) \\ \mu_{2,h}h + \alpha_{2,h}Y_{(k-1)h} + (\Delta_h D_k)' \varrho^0 + (\Delta_h Z_k)' \delta_{Z,2}^0 + \Delta_h e_k^*, & (k = \lfloor T\lambda_0 \rfloor + 1, \dots, T) \end{cases} \quad (2.6)$$

for some given initial value Y_0 . We specify the parameters associated with the constant and the lagged dependent variable as increasing as $h \downarrow 0$ in order for some fixed true parameter to remain in the asymptotics. This is done by specifying: $\mu_{1,h} \triangleq \mu_1^0 h^{-1/2}$, $\mu_{2,h} \triangleq \mu_2^0 h^{-1/2}$, $\mu_{\delta,h} \triangleq \mu_{2,h} - \mu_{1,h}$, $\alpha_{1,h} \triangleq \alpha_1^0 h^{-1/2}$, $\alpha_{2,h} \triangleq \alpha_2^0 h^{-1/2}$ and $\alpha_{\delta,h} \triangleq \alpha_{2,h} - \alpha_{1,h}$. Our framework is then similar to the small-diffusion setting which has been extensively studied in the statistics literature [cf. [Ibragimov and Has'minskiĭ \(1981\)](#), [Laredo \(1990\)](#) and [Sorensen and Uchida \(2003\)](#)]. The results to be discussed below go through with modifications using the results in CP. In particular, a two-step procedure is needed to estimate the parameters as discussed in Section A.2 in CP. The model and results can be trivially extended to allow more general forms of predictable processes.

2.3 The Least-Squares Estimator of the Break Date

CP considered the break date least-squares (LS) estimator $\widehat{N}_b^{\text{LS}} = \widehat{T}_b^{\text{LS}} h$ defined as the minimizer of the sum of squares residuals $S_h(\theta, N_b)$ from (2.4), where $\theta \triangleq (\beta', \delta')'$. The parameter vector θ can be concentrated out of the criterion function to yield an optimization problem over N_b only:

$$\widehat{\theta}_h^{\text{LS}}(N_b) = \underset{\theta}{\operatorname{argmin}} S_h(\theta, N_b), \quad \widehat{N}_b^{\text{LS}} = \underset{h q \leq N_b \leq N}{\operatorname{argmin}} S_h(\widehat{\theta}_h^{\text{LS}}(N_b), (N_b)).$$

Using a correspondence between the sum of squared residuals and the sup-Wald statistic, we have:

$$\underset{h q \leq N_b \leq N}{\operatorname{argmin}} S_h(\widehat{\theta}_h^{\text{LS}}(N_b), N_b) = \underset{h q \leq N_b \leq N}{\operatorname{argmax}} \widehat{\delta}^{\text{LS}}(N_b)' (Z_2' M_X Z_2) \widehat{\delta}^{\text{LS}}(N_b), \quad (2.7)$$

where $M_X \triangleq I - X(X'X)^{-1}X'$ and $\widehat{\delta}^{\text{LS}}(N_b)$ is the LS estimator of δ^0 obtained by regressing Y on X and Z_2 . This follows since $Q_h(\widehat{\theta}_h^{\text{LS}}(N_b), N_b) \triangleq \widehat{\delta}^{\text{LS}}(N_b)' (Z_2' M_X Z_2) \widehat{\delta}^{\text{LS}}(N_b)$ is the numerator of a modified sup-Wald statistic. The GL estimator will depend on the criterion function $Q_h(\theta(N_b), N_b)$ evaluated at each possible break date. Hence, we need the following assumptions.

Assumption 2.4. $\theta^0 = ((\beta^0)', (\delta^0)')' \in \Theta \subset \mathbb{R}^{\dim(\theta)}$, a compact parameter space.

Assumption 2.5. There exists an l_0 such that for all $l > l_0$, the matrices $(lh)^{-1} \sum_{k=1}^l x_{kh} x'_{kh}$, $(lh)^{-1} \sum_{k=T-l+1}^T x_{kh} x'_{kh}$, $(lh)^{-1} \sum_{k=T_b^0-l+1}^{T_b^0} x_{kh} x'_{kh}$, and $(lh)^{-1} \sum_{k=T_b^0+1}^{T_b^0+l} x_{kh} x'_{kh}$ have minimum eigenvalues bounded away from zero in probability.

Assumption 2.6. Let $\bar{Q}_h(\theta^0, N_b) \triangleq \mathbb{E}[Q_h(\theta^0, N_b) - Q_h(\theta^0, N_b^0)]$. There exists a N_b^0 such that $\bar{Q}_h(\theta^0, N_b^0) > \sup_{(\theta^0, N_b) \notin \mathbf{B}} \bar{Q}_h(\theta^0, N_b)$, for every open set \mathbf{B} that contains (θ^0, N_b^0) .

Assumption 2.5 is in the same spirit as A2 in Bai and Perron (1998). It requires that there be enough variation around the break point so that it can be identified. Multiplying the increment x_{kh} by the factor $h^{-1/2}$ allows one to normalize x_{kh} so that the assumption is implied by a weak law of large numbers. Assumption 2.6 is a conventional uniqueness condition.

3 Asymptotic Results for Least-squares Estimation

In this section, we review continuous record asymptotic results about the break date LS estimator from CP. They provide intuition on the forthcoming asymptotic results about the Laplace estimator since both have a common LS criterion function. The typical asymptotic framework for structural change problems relies on a shrinking shifts assumption. Under a continuous record, we also require a small shifts assumption to analyze the limit distribution of the Laplace estimator. We compare the asymptotics of CP with that of Bai (1997) and Elliott and Müller (2007) in Section 3.2.

3.1 Continuous Record Asymptotics

Assumption 3.1. Let $\delta_h = h^{1/4}\delta^0$ and assume that for all $t \in (N_b^0 - \epsilon, N_b^0 + \epsilon)$, with $\epsilon \downarrow 0$ and $T^{1-\kappa}\epsilon \rightarrow B < \infty$, $0 < \kappa < 1/2$, $\mathbb{E}[(\Delta_h e_t^*)^2 | \mathcal{F}_{t-h}] = \sigma_{h,t-h}^2 \Delta t$ \mathbb{P} -a.s, where $\sigma_{h,t} \triangleq \sigma_h \sigma_{e,t}$, $\sigma_h \triangleq h^{-1/4}\bar{\sigma}$ and $\bar{\sigma} \triangleq \int_0^N \sigma_{e,s}^2 ds$.

The first part states that the shift parameter converges to zero at a controlled rate. The second allows for a higher degree of uncertainty around the change-point by requiring $\{\Delta_h e_t^*\}$ to oscillate more as $h \downarrow 0$. The latter neither prevents nor facilitates the identification of the break, i.e., it plays no role for the global properties of the estimator, namely the consistency of \widehat{N}_b and $\widehat{\theta}$ as well as the asymptotic distribution of $\widehat{\theta}$, though not that of \widehat{N}_b .

Under Assumption 3.1, the rate of convergence of the LS estimator is $T^{1-\kappa}$, with $0 < \kappa < 1/2$. This rate is fast and therefore the volatility of the errors is scaled up around the change-point so that the objective function behaves as if it were a standard diffusion process. Note that the $T^{-(1-\kappa)}$ -neighborhood in which the errors have higher variance arises from the $T^{1-\kappa}$ -rate of convergence of $\widehat{\lambda}_b$. Note also that the rate of convergence $T^{1-\kappa}$ is sufficiently fast to guarantee a \sqrt{T} -consistent estimation of the slope parameters. The results proved in CP are stated in the following propositions. Let $\Sigma^* \triangleq \{\mu_{\cdot,t}, \Sigma_{\cdot,t}, \sigma_{e,t}\}_{t \geq 0}$, and $Z_\Delta \triangleq (0, \dots, 0, z_{(T_b+1)h}, \dots, z_{T_b^0 h}, 0, \dots, 0)$ if

$T_b < T_b^0$ and $Z_\Delta \triangleq (0, \dots, 0, z_{(T_b^0+1)h}, \dots, z_{T_b h}, 0, \dots, 0)$ if $T_b > T_b^0$. Define

$$\Delta_h \tilde{e}_t \triangleq \begin{cases} \Delta_h e_t^*, & t \notin (N_b^0 - \epsilon, N_b^0 + \epsilon) \\ h^{1/4} \Delta_h e_t^*, & t \in (N_b^0 - \epsilon, N_b^0 + \epsilon) \end{cases}. \quad (3.1)$$

Proposition 3.1. *Under Assumptions 2.2-2.6 and 3.1: (i) $\widehat{N}_b \xrightarrow{\mathbb{P}} N_b^0$; (ii) for every $\epsilon > 0$ there exists a $K > 0$ such that for all large T , $\mathbb{P}\left(T^{1-\kappa} \left| \widehat{N}_b - N_b^0 \right| > K \|\delta^0\|^{-2} \bar{\sigma}^2\right) < \epsilon$; and (iii) for $\kappa \in (0, 1/4]$, as $T \rightarrow \infty$, $\left(\sqrt{T/N} (\widehat{\beta} - \beta^0), \sqrt{T/N} (\widehat{\delta} - \delta_h)\right)' = O_{\mathbb{P}}(1)$.*

The derivation of the continuous record asymptotic distribution uses a change of time scale $s \mapsto \psi_h^{-1} t$ where $\psi_h = h^{1-\kappa}$. Under fixed-shifts, Proposition 3.2 in CP shows that $\widehat{N}_b - N_b^0 = O_p(T^{-1})$, i.e., \widehat{N}_b is in a shrinking neighborhood of N_b^0 , which however shrinks too fast and impedes the development of a feasible limit theory. Hence, the need to analyze the objective function in a small neighborhood of the true break date under this “fast time scale”. See Section 4.3 for details.

Proposition 3.2. *Under Assumptions 2.2-2.6 and 3.1, and under the “fast time scale”,*

$$N \left(\widehat{\lambda}_b - \lambda_0 \right) \xrightarrow{\mathcal{L}} \underset{v \in \left[-\frac{N_b^0}{\|\delta^0\|^{-2} \bar{\sigma}^2}, \frac{N - N_b^0}{\|\delta^0\|^{-2} \bar{\sigma}^2} \right]}{\operatorname{argmax}} \mathcal{V}(v), \quad (3.2)$$

where $\mathcal{V}(v) \triangleq -(\delta^0)' \langle Z_\Delta, Z_\Delta \rangle (v) \delta^0 + 2(\delta^0)' \mathcal{W}(v)$, $\langle Z_\Delta, Z_\Delta \rangle (v)$ is the predictable quadratic variation process of Z_Δ and the process $\mathcal{W}(v)$ is, conditionally on the σ -field \mathcal{F} , a two-sided centered Gaussian martingale with independent increments.

To save space, we do not report the explicit expression for the covariance of $\mathcal{W}(v)$; see Theorem 4.1 in CP. Under stationary regimes, Proposition 3.3 presents the corresponding result.

Assumption 3.2. *The process Σ^0 is (possibly time-varying) deterministic; $\{z_{kh}, e_{kh}\}$ is second-order stationary within each regime. For $k = 1, \dots, T_b^0$, $\mathbb{E}(z_{kh} z'_{kh} | \mathcal{F}_{(k-1)h}) = \Sigma_{Z,1} h$, $\mathbb{E}(\tilde{e}_{kh}^2 | \mathcal{F}_{(k-1)h}) = \sigma_{e,1}^2 h$ and $\mathbb{E}(z_{kh} z'_{kh} \tilde{e}_{kh}^2 | \mathcal{F}_{(k-1)h}) = \Omega_{\mathcal{W},1} h^2$ while for $k = T_b^0 + 1, \dots, T$, $\mathbb{E}(z_{kh} z'_{kh} | \mathcal{F}_{(k-1)h}) = \Sigma_{Z,2} h$, $\mathbb{E}(\tilde{e}_{kh}^2 | \mathcal{F}_{(k-1)h}) = \sigma_{e,2}^2 h$ and $\mathbb{E}(z_{kh} z'_{kh} \tilde{e}_{kh}^2 | \mathcal{F}_{(k-1)h}) = \Omega_{\mathcal{W},2} h^2$.*

Let $W_i^*(s)$, $i = 1, 2$, be two independent standard Wiener processes defined on $[0, \infty)$, starting at the origin when $s = 0$. Let

$$\mathcal{V}^*(s) = \begin{cases} -\frac{|s|}{2} + W_1^*(s), & \text{if } s < 0 \\ -\frac{(\delta^0)' \Sigma_{Z,2} \delta^0 |s|}{(\delta^0)' \Sigma_{Z,1} \delta^0} + \left(\frac{(\delta^0)' \Omega_{\mathcal{W},2}(\delta^0)}{(\delta^0)' \Omega_{\mathcal{W},1}(\delta^0)} \right)^{1/2} W_2^*(s), & \text{if } s \geq 0. \end{cases}$$

Proposition 3.3. *Under Assumptions 2.2-2.6 and 3.1-3.2, and under the “fast time scale”,*

$$\frac{\left((\delta^0)' \Sigma_{Z,1} \delta^0\right)^2}{(\delta^0)' \Omega_{\mathcal{Y},1} \delta^0} N \left(\widehat{\lambda}_b - \lambda_0\right) \Rightarrow \underset{s \in \left[-\frac{N_b^0}{\|\delta^0\|^{-2\sigma^2}} \frac{\left((\delta^0)' \Sigma_{Z,1} \delta^0\right)^2}{(\delta^0)' \Omega_{\mathcal{Y},1}(\delta^0)}, \frac{N - N_b^0}{\|\delta^0\|^{-2\sigma^2}} \frac{\left((\delta^0)' \Sigma_{Z,1} \delta^0\right)^2}{(\delta^0)' \Omega_{\mathcal{Y},1} \delta^0} \right]}{\text{argmax}} \mathcal{Y}^*(s). \quad (3.3)$$

3.2 Comparison with Other Approaches

As shown in Figures 1-2, the structural change problem is characterized by a high degree of uncertainty when the break magnitude is not large. The classical shrinkage asymptotics of Bai (1997), with δ_T converging to zero at a rate slower than $O(T^{1/2})$, underestimates the degree of uncertainty and, as the figures show, it provides a poor approximation to the finite-sample behavior of the LS estimator. CP argued that this issue is also responsible for the poor coverage probabilities of the associated confidence intervals when the break size is small. This asymptotic distribution does not capture important features of the finite-sample distribution such as multimodality, asymmetry and shape changing with the magnitude of δ^0 . Underestimating the true uncertainty leads to confidence intervals that are too short and consequently undercover.

The goal is to find an asymptotic experiment that delivers a good approximation and leads to inference that is reliable in practice. Elliott and Müller [cf. Elliott and Müller (2007) and Elliott, Müller, and Watson (2015)] proposed an alternative framework where $\delta_T = O(T^{-1/2})$ so that δ_T goes faster to zero. This can be referred to as a weak identification. Although it increases the uncertainty in the problem, the rate at which δ_T goes to zero is too fast in the sense that statistical uncertainty is too high so that $\widehat{\lambda}_b = \widehat{T}_b/T$, $\widehat{\delta}_1$ and $\widehat{\delta}_2$ become inconsistent for λ_0 , δ_1^0 and δ_2^0 , respectively. This can be problematic and indeed, their inference suffers from the opposite problem in that confidence intervals for \widehat{T}_b can be too large and thus uninformative [Casini and Perron (2019), Chang and Perron (2018) and CP]. Furthermore, inconsistency for the regression coefficients is unappealing since researchers often are ultimately interested in making inference about the regression coefficients and not just about T_b^0 .

Although the asymptotic experiments considered so far all impose conditions on the break magnitude δ^0 , CP pointed out that what matters is not just δ^0 . Consider a location model with a change δ^0 in the mean and independent errors. What describe the uncertainty in the model is the ratio δ^0/σ where σ is the volatility of the errors. Instead of controlling just δ , one can rather control the signal-to-noise ratio δ^0/σ . CP proposed to let δ^0 go to zero at a not too fast rate while letting σ increase to infinity in a neighborhood of T_b^0 . That is $(\delta_T/\sigma_t) \rightarrow 0$ at rate $O(T^{-1/2})$ in a neighborhood of T_b^0 . This is the same rate Elliott and Müller used for $\delta_T \rightarrow 0$. However, the difference here is that all the parameters in the models remain consistent. See CP for details.

4 Laplace-based Estimation

We first formally define the Generalized Laplace (GL) estimator and introduce assumptions needed for the derivation of its large-sample properties. Section 4.2 describes the asymptotic framework adopted. The main results about the limit distribution of the estimate of λ_0 are presented in Section 4.3. Section 4.4 explains how to construct the GL estimator in practice.

4.1 The Estimator and Additional Assumptions

Using the properties of orthogonal LS projections, θ can be concentrated out from the objective function $Q_h(\theta, N_b)$, allowing us to dispense with the dependence of $Q_h(\theta, N_b)$ on θ and use $Q_h(N_b)$ hereafter. The parameter of interest is N_b , and the Quasi-posterior density $p_h(N_b)$ is given by,

$$p_h(N_b) \triangleq \frac{\exp(Q_h(N_b)) \pi(N_b)}{\int_{\Gamma^0} \exp(Q_h(N_b)) \pi(N_b) dN_b}, \quad (4.1)$$

which constitutes a proper distribution over the parameter space $\Gamma^0 \triangleq (0, N)$. The Quasi-prior $\pi(\cdot)$ is a weight function or simply a prior probability density. For example, $\pi(N_b) = d\mathbb{Q}(N_b)/d\text{Leb}(N_b)$ for some probability distribution \mathbb{Q} , with $\text{Leb}(\cdot)$ the Lebesgue measure. The function $\pi(\cdot)$ satisfies weak regularity conditions. Following Chernozhukov and Hong (2003) we restrict attention to convex loss functions $l_h(\cdot)$. Common examples include, (i) $l_h(r) = a_h |r|^m$, the polynomial loss function with the squared loss function obtained when $m = 2$, and the absolute deviation loss with $m = 1$; (ii) $l_h(r) = a_h (\tau - \mathbf{1}(r \leq 0)) r$, the check loss function where $\{a_h\}$ is a positive sequence with $a_h \rightarrow \infty$. Given the Quasi-posterior density, we can define the expected risk under $p_h(\cdot)$, for the loss $l_h(\cdot)$, as $\mathcal{R}_{l,h}(s) \triangleq \mathbb{E}_{p_h} [l_h(s - \widetilde{N}_b)]$, where \widetilde{N}_b is a random variable with distribution p_h and \mathbb{E}_{p_h} denotes expectation taken under p_h . Then,

$$\mathcal{R}_{l,h}(s) = \int_{\Gamma^0} l_h(s - N_b) \left(\frac{\exp(Q_h(N_b)) \pi(N_b)}{\int_{\Gamma^0} \exp(Q_h(N_b)) \pi(N_b) dN_b} \right) dN_b. \quad (4.2)$$

The GL estimator $\widehat{N}_b^{\text{GL}}$ is a decision rule that is least unfavorable given the information provided by the Quasi-posterior p_h according to the loss function $l_h(\cdot)$. That is, $\widehat{N}_b^{\text{GL}}$ minimizes the expected risk function in (4.2): $\widehat{N}_b^{\text{GL}} \triangleq \text{argmin}_{s \in \Gamma^0} \mathcal{R}_{l,h}(s)$. Observe that if $p_h(N_b)$ were a true posterior, then $\widehat{N}_b^{\text{GL}}$ would naturally be viewed as a Bayesian estimator for the loss function $l_h(\cdot)$ and prior π . This suggests an interpretation of the Laplace-type estimator as a Quasi-Bayesian estimator.

In our setting, one can treat the density of the continuous record limit distribution of $\widehat{N}_b^{\text{LS}}$ as the ‘‘Quasi-prior’’ π . As discussed below, the resulting Quasi-posterior in (4.1) provides useful information for inference about the parameter N_b^0 beyond that already included in the objective function $Q_h(N_b)$. Inference procedures based on the median of the Quasi-posterior density obtained

when using the absolute loss function is a reasonable choice as we shall show. The GL estimation provides alternative inference methods that may be combined with the (frequentist) continuous record asymptotic framework, through the choice of π . Note that in order to construct such GL estimator with the continuous record Quasi-prior, one needs an estimate of the density of the continuous record distribution, computed as in CP; see Section 4.4 for details.

To understand why the GL estimation can be useful, consider Figure 1, which shows that when the magnitude of the break is small both the finite-sample and continuous record distribution display highly non-standard features. First, there are three modes; two are near the start and end of sample while the mode at the origin corresponds to the estimated break point. This multimodality signifies that there is a substantial tendency for the LS estimator to locate the break date towards the tails rather than close to the true break date. Second, the asymmetry in the density—which is always present unless the true break date is at mid-sample—implies that the span and actual location of the break matters for the precision of the estimator. CP documented that such features are still present to a lesser extent for moderate break sizes, although they disappear when they are large (cf. Figure 2). As opposed to simply relying on $Q_h(N_b)$ which is quite flat for small breaks, the GL estimation combines information from the continuous record density—through $\pi(N_b)$ —with information from the distribution of the criterion function to yield the Quasi-posterior. Given the highly non-standard features of the finite-sample distribution, the Quasi-posterior contains more accurate information about the uncertainty of the change-point. Hence, estimation and inference based on the latter may have better properties as we shall show.

Assumption 4.1. L denotes the set of functions $l : \mathbb{R} \rightarrow \mathbb{R}_+$ that satisfy: (i) $l(r)$ is defined on \mathbb{R} , with $l(r) \geq 0$ and $l(r) = 0$ if and only if $r = 0$; (ii) $l(r)$ is continuous at $r = 0$ but is not identically zero; (iii) $l(\cdot)$ is convex and $l(r) \leq 1 + |r|^m$ for some $m > 0$.

Assumption 4.2. The function $\pi : \mathbb{R} \rightarrow \mathbb{R}_+$ is a continuous, uniformly positive density function satisfying $\pi^0 \triangleq \pi(N_b^0) > 0$, and $\pi^0 < C_\pi$ for some finite C_π . Furthermore, $\pi(N_b) = 0$ for all $N_b \notin \Gamma^0$, and π is twice continuously differentiable in N_b at N_b^0 .

Definition 4.1. The random variable ξ_l^0 is uniquely defined by

$$\Psi_l^*(\xi_l^0) \triangleq \inf_s \Psi_l^*(s) = \inf_s \int_{\Gamma^*} l(s-v) \left(e^{\mathcal{Y}(v)} \pi(N_b^0 + v/\vartheta) / \int_{\Gamma^*} e^{\mathcal{Y}(w)} \pi(N_b^0 + w/\vartheta) dw \right) dv$$

where $\vartheta \triangleq \|\delta^0\|/\sigma^2$; Γ^* and the process $\mathcal{Y}(\cdot)$ are specified below.

Lemma 4.1. For any $\eta > 0$, there exists an $\epsilon > 0$, such that

$$\liminf_{h \downarrow 0} \mathbb{P} \left[\sup_{|N_b - N_b^0| \geq \eta} Q_h(N_b) - Q_h(N_b^0) \leq -\epsilon \right] = 1. \quad (4.3)$$

The conditions in Assumption 4.1 are similar in spirit to those in Bickel and Yahav (1969) and Chernozhukov and Hong (2003). A convex loss function is usually employed in applications. The restriction imposed in part (iii) is not essential. What one needs is that the growth of the function $l(r)$ as $|r| \rightarrow \infty$ is slower than that of functions of the form $\exp(\epsilon|r|)$, $\epsilon > 0$. Further, note that part (iii) implicitly assumes that $l(\cdot)$ has been scaled by some constant for the inequality to hold. Assumption 4.2 on the Quasi-prior is rather mild. As anticipated above, one may take $\pi(\cdot)$ as the density of the limit process appearing in Proposition 3.2. The uniqueness of ξ_l^0 follows from the fact that there is a single break. Lemma 4.1 guarantees the identification of the parameters.

4.2 Asymptotic Framework for the Generalized Laplace Estimation

Our framework builds upon an expansion of the criterion function derived by CP under a continuous record. This is useful because it concentrates out the regression parameters and allows us to frame our theory in terms of a single local parameter $u = r_h(N_b - N_b^0)$, while the regression parameters are kept fixed at their true values; r_h is some sample size-dependent sequence which converges to infinity and whose properties will be specified below. This makes $Q_h(\theta(N_b), N_b) = \delta'(N_b)(Z_2' M_X Z_2) \delta(N_b)$ a function only of the localized break date $u = r_h(N_b - N_b^0)$. Because there is a single break in the model, it follows that the unknown parameter $N_b^0 \in \Gamma^0 \subset (0, N)$ is the unique maximizer of $Q_{0,h}(\theta(N_b), N_b) = \mathbb{E}[Q_h(\theta(N_b), N_b) - Q_h(\theta(N_b^0), N_b^0)]$. Further, let $\bar{Q}_h(\theta(N_b), N_b) \triangleq Q_h(\theta(N_b), N_b) - Q_h(\theta(N_b^0), N_b^0)$ and $G_h(\theta(N_b), N_b) \triangleq \bar{Q}_h(\theta(N_b), N_b) - Q_{0,h}(\theta(N_b), N_b)$. These expansions of the criterion function shows that G_h , \bar{Q}_h and $Q_{0,h}$ do not depend on $\theta(N_b)$ but only on θ^0 . Hence, since $\theta^0 \in \mathbf{S} \subset \mathbb{R}^{p+q} \times \mathbb{R}^q$ is given, we shall omit it from the arguments of G_h , \bar{Q}_h and $Q_{0,h}$ in what follows.³ The GL estimator $\widehat{N}_b^{\text{GL}}$ can equivalently be defined as the minimizer of

$$\Psi_{l,h}(s) = \int_{\Gamma^0} l(s - N_b) \frac{\exp(\gamma_h \bar{Q}_h(N_b)) \pi(N_b)}{\int_{\Gamma^0} \exp(\gamma_h \bar{Q}_h(N_b)) \pi(N_b) dN_b} dN_b, \quad (4.4)$$

where $\{\gamma_h\}$ is a sequence which normalizes the sample criterion. Conditions on $\{\gamma_h\}$ will be stated below. The main theoretical result of this section concerns the large-sample properties of the GL estimator, which we derive as follows. We first show the convergence of the marginal distributions of the sample function $\Psi_{l,h}(s)$ to the marginal distributions of the random function $\Psi_l^*(s) = \int_{\Gamma^*} l(s - v) \exp(\mathcal{V}(v)) \pi(N_b^0 + v/\vartheta) / \left(\int_{\Gamma^*} \exp(\mathcal{V}(w)) \pi(N_b^0 + w/\vartheta) dw \right) du$, where $\pi_h(v)$ is defined below. Let $\mathbf{K} \triangleq \{s \in \mathbb{R} : |s| \leq K < \infty\}$. Next, we show that the family of probability measures in $\mathbb{C}_b(\mathbf{K})$ (the space of bounded continuous function from \mathbf{K} into \mathbb{R}), generated by the contractions of $\Psi_{l,h}^*(s)$ on \mathbf{K} , is dense. As a final step, we analyze the oscillations of the minimum points of the sample criterion $\Psi_{l,h}^*(s)$. Given $u = r_h(N_b - N_b^0)$, we let $\pi_h(u) \triangleq \pi(N_b^0 + u/r_h)$,

³The exact expressions for G_h , \bar{Q}_h and $Q_{0,h}$ are provided at the beginning of Section A.2 of the supplement.

$\tilde{Q}_h(u) \triangleq \bar{Q}_h(N_b^0 + u/r_h)$, $\tilde{G}_h(u) \triangleq G_h(N_b^0 + u/r_h)$ and $\tilde{Q}_{0,h}(u) \triangleq Q_{0,h}(N_b^0 + u/r_h)$. Using the expression for u , we can apply a simple substitution in (4.4) which results in,

$$\Psi_{l,h}(s) = \int_{\Gamma_h} l(s-u) \frac{\exp\left(\gamma_h\left(\tilde{G}_h(u) + \tilde{Q}_{0,h}(u)\right)\right) \pi_h(u) du}{\int_{\Gamma_h} \exp\left(\gamma_h\left(\tilde{G}_h(w) + \tilde{Q}_{0,h}(w)\right)\right) \pi_h(w) dw}, \quad (4.5)$$

where $\Gamma_h \triangleq \{u \in \mathbb{R} : N_b^0 + u/r_h \in \Gamma^0\}$. The local parameter u introduced above depends on the normalizing factor $\{r_h\}$. We set $r_h \triangleq T^{1-\kappa}/\vartheta N$, $\kappa \in (0, 1/2)$ with $\vartheta \triangleq \|\delta^0\|^2/\bar{\sigma}^2$ so as to have⁴

$$u \triangleq \vartheta^{-1} T^{1-\kappa} (\lambda_b - \lambda_0). \quad (4.6)$$

The factor $T^{1-\kappa}$ is the rate at which $N\hat{\lambda}_b^{\text{LS}}$ converges to N_b^0 under the continuous record asymptotic setting. Note that the local parameter u is allowed to vary over the entire real line. On the range $\{|u| < K\}$, N_b approaches N_b^0 at the rate $T^{1-\kappa}$. Thus, we will deduce a convergence result for the normalized criterion function $\gamma_h \tilde{Q}_h(u)$ toward a tight Gaussian process. For other values of u , we show that the tails of the Quasi-posterior are negligible. For the latter, one uses properties of the Gaussian component of the limit process $\mathcal{V}(\cdot)$, and show that it cannot diverge faster than the (negative) drift component. Since the Quasi-posterior is an exponential transform of the centered objective function, the tails of some expansion of the criterion function diverge (after rescaling) to minus infinity. This follows because the objective function can only be maximized for values of the parameter sufficiently close to N_b^0 , and so its re-centered version is always negative and bounded away from zero if N_b is far from N_b^0 . The normalizing sequence $\{\gamma_h\}$ then makes the exponential transformation negligible for such N_b far from N_b^0 .

4.3 Asymptotic Distribution of the Generalized Laplace Estimator

The main result of this section is Theorem 4.1, which presents the limit distribution of the GL estimator for a general loss function $l(\cdot)$. As part of the proof, we show the weak convergence of $\tilde{Q}_h(u)$ on the space of bounded functions from compact sets $\mathbf{B} \subset \mathbb{R}$ into \mathbb{R} , denoted by $\mathbb{D}_b(\mathbf{B})$. As a matter of notation let $\tilde{\mathcal{W}}(\theta, u)$ denote an arbitrary sample process with bounded paths evaluated at (θ, u) with $u \in \mathbb{R}$. For each fixed θ , we write $\tilde{\mathcal{W}}(\theta, u) \Rightarrow \mathcal{W}(\theta, u)$ on the space $\mathbb{D}_b(\mathbf{B})$ if the process $\tilde{\mathcal{W}}(\theta, \cdot)$ converges in law under the Skorohod metric to a process $\mathcal{W}(\theta, \cdot)$ defined on $\mathbb{D}_b(\mathbf{B})$. To simplify notation we omit the argument θ from the limit process. For general loss functions, the Laplace estimator is defined implicitly as the solution of a convex optimization problem. The theorem presents the limit distribution of the estimator under the change of time scale, discussed above. Formally, the limiting distribution is derived under a change of time scale $s \mapsto \psi_h^{-1}s$. Then, $t \triangleq \psi_h^{-1}s$ is the index on the new time scale. Hence, the sample criterion function

⁴ $N_b = N\lambda_b$ and so theoretical results about λ_b translate immediately to N_b (up to a constant N).

$\tilde{Q}_h(v) = \tilde{G}_h(v) + \tilde{Q}_{0,h}(v)$ will be shown to vary on $\Gamma^* \triangleq \{v \in \mathbb{R} : -\vartheta N_b^0 \leq v \leq \vartheta(N - N_b^0)\}$ where $\vartheta = \|\delta^0\|^2 \bar{\sigma}^{-2}$ and the optimization problem is then also defined with respect to Γ^* , namely via,

$$\Psi_{l,h}^*(s) \triangleq \int_{\Gamma^*} l(s-u) \frac{\exp(\gamma_h(\tilde{Q}_h(v))) \pi_h(v) dv}{\int_{\Gamma^*} \exp(\gamma_h(\tilde{Q}_h(w))) \pi_h(w) dw}. \quad (4.7)$$

We next impose conditions on $\zeta_k \triangleq h^{-1} z_{kh} e_{kh}$ and γ_h to derive the required limit distribution.

Assumption 4.3. *Uniformly in $r \in [0, 1]$, $(T_b^0)^{-1/2} \sum_{k=1}^{\lfloor rT_b^0 \rfloor} \zeta_k \Rightarrow \mathscr{W}_1(r)$, $(T - T_b^0)^{-1/2} \sum_{k=T_b^0+1}^{T_b^0 + \lfloor r(T-T_b^0) \rfloor} \zeta_k \Rightarrow \mathscr{W}_2(r)$ where $\mathscr{W}_i(\cdot)$ is a multivariate Gaussian process on $[0, 1]$ with zero mean and covariance $\mathbb{E}[\mathscr{W}_i(u), \mathscr{W}_i(s)] = \min\{u, s\} \Omega_{\mathscr{W},i}$ ($i = 1, 2$), $\Omega_{\mathscr{W},1} \triangleq \lim_{T \rightarrow \infty} \mathbb{E}[(T_b^0)^{-1/2} \sum_{k=1}^{T_b^0} \zeta_k]^2$ and $\Omega_{\mathscr{W},2} \triangleq \lim_{T \rightarrow \infty} \mathbb{E}[(T - T_b^0)^{-1/2} \sum_{k=T_b^0+1}^T \zeta_k]^2$. For any $0 < r_0 < 1$ with $r_0 \neq \lambda_0$, $T^{-1} \sum_{k=\lfloor r_0 T \rfloor+1}^{\lfloor \lambda_0 T \rfloor} h^{-1} z_{kh} z'_{kh} \xrightarrow{\mathbb{P}} (\lambda_0 - r_0) \Sigma_{Z,1}$, and $T^{-1} \sum_{k=\lfloor \lambda_0 T \rfloor+1}^{\lfloor r_0 T \rfloor} h^{-1} z_{kh} z'_{kh} \xrightarrow{\mathbb{P}} (r_0 - \lambda_0) \Sigma_{Z,2}$ with λ_- and λ_+ , the minimum and maximum eigenvalues of the last two matrices with $0 < \lambda_- \leq \lambda_+ < \infty$.*

Condition 1. As $T \rightarrow \infty$, $\gamma_h/T^{3/2-\kappa} \rightarrow \kappa_\gamma$ where $\kappa_\gamma > 0$ is some constant.

Theorem 4.1. *Let $l \in \mathbf{L}$ and Assumptions 2.1-2.6, 3.1-3.2, 4.1-4.3 as well as Condition 1 hold. Then, under the ‘‘fast time scale’’ as $T \rightarrow \infty$, we have $N(\hat{\lambda}_b^{\text{GL}} - \lambda_0) \Rightarrow \xi_l^0$, where ξ_l^0 is defined in Definition 4.1 with $\Gamma^* = (-\vartheta N_b^0, \vartheta(N - N_b^0))$ and \mathscr{V} from Proposition 3.2.*

Corollary 4.1. *Under the squared loss function $l_h(r) = a_h |r|^2$,*

$$N(\hat{\lambda}_b^{\text{GL}} - \lambda_0) \Rightarrow \int_{\Gamma^*} v \frac{\exp(\mathscr{V}(v)) \pi(N_b^0 + v/\vartheta)}{\int_{\Gamma^*} \exp(\mathscr{V}(w)) \pi(N_b^0 + w/\vartheta) dw} dv,$$

in $\mathbb{D}_b(\mathbb{R})$ with $\mathscr{V}(v)$ defined in Proposition 3.2. Under a least-absolute loss function, $N(\hat{\lambda}_b^{\text{GL}} - \lambda_0)$ converges to the median of $\exp(\mathscr{V}(v)) \pi(N_b^0 + v/\vartheta) / \int_{\Gamma^} \exp(\mathscr{V}(w)) \pi(N_b^0 + w/\vartheta) dw$.*

Let $\rho \triangleq ((\delta^0)' \langle Z, Z \rangle_1 \delta^0)^2 / ((\delta^0)' \Omega_{\mathscr{W},1} \delta^0)$. After applying the usual change in variables [cf. Bai (1997), Bai and Perron (1998) and CP], we obtain a limit distribution expressed directly in terms of quantities that can be estimated.

Corollary 4.2. *Let $l \in \mathbf{L}$ and Assumptions 2.1-2.6, 3.1-3.2, 4.1-4.3 and Condition 1 hold. Then, under the ‘‘fast time scale’’ and as $T \rightarrow \infty$, $N(\hat{\lambda}_b^{\text{GL}} - \lambda_0) \Rightarrow \xi_l^0$, with ξ_l^0 uniquely defined by*

$$\Psi_l^*(\xi_l^0) = \inf_s \Psi_l^*(s) = \inf_s \int_{\Gamma_\rho^*} l(s-v) \frac{\exp(\mathscr{V}^*(v)) \pi(N_b^0 + v/\vartheta\rho)}{\int_{\Gamma_\rho^*} \exp(\mathscr{V}^*(w)) \pi(N_b^0 + w/\vartheta\rho) dw} dv$$

where $\Gamma_\rho^ = (-\vartheta\rho N_b^0, \vartheta\rho(N - N_b^0))$ with $\mathscr{V}^*(\cdot)$ defined in Proposition 3.3.*

Theorem 4.1 states that the asymptotic distribution of the GL estimator—under the new “fast time scale” asymptotic framework—is an integral-ratio of functions of Gaussian processes. An advantage is that the knowledge of κ , which determines the rate of convergence on the original time scale, is not needed to conduct inference.

We can compare this limiting distribution with that of the Bayesian change-point estimator of Ibragimov and Has’minskii (1981) [see their equation (2.17) on p. 338]. They considered maximum likelihood and Bayesian estimators of the change-point in a simple diffusion process. We note a few differences. First, the region of integration is Γ^* instead of \mathbb{R} . Second, due to the change of time scale, the Quasi-prior $\pi(\cdot)$ enters the limiting distribution, a useful property in view of the features of the finite-sample distributions. It reflects the fact that the uncertainty in a change-point problem is often high enough that the information provided by the prior can influence the limiting behavior. Finally, note that the GL estimator conserves a classical (frequentist) interpretation.

4.4 Construction of the GL Estimate

The definition of the GL estimate involves a component, $\exp(Q_h(N_b)) / \int_{\Gamma^0} \exp(Q_h(N_b)) dN_b$, which is immediately available while the Quasi-prior $\pi(N_b)$ needs to be replaced by a consistent estimate. This requires to obtain by simulations an estimate of the density of the continuous record limiting distribution in (3.3). We follow CP. Theorem 3.2 shows that the limiting distribution of the LS break point estimator is related to the distribution of the location of the maximum of the process $\mathcal{V}(s) = \mathcal{W}(s) - \Lambda(s)$, or after a change in variable, of the process,

$$\mathcal{V}^*(s) = \begin{cases} -\frac{|s|}{2} + W_1^*(-s), & \text{if } s < 0 \\ -\frac{s}{2}\phi_Z + \phi_e W_2^*(s), & \text{if } s \geq 0, \end{cases} \quad (4.8)$$

where W_i^* ($i = 1, 2$) are independent standard Wiener processes, $\phi_Z \triangleq (\delta^0)' \Sigma_{Z,2} \delta^0 / (\delta^0)' \Sigma_{Z,1} \delta^0$ and $\phi_e \triangleq (\delta^0)' \Omega_{\mathcal{W},2} \delta^0 / (\delta^0)' \Omega_{\mathcal{W},1} \delta^0$. Further, for every $t, s \in \mathbb{R}_+$, let $\Sigma^0(t, s) \triangleq \mathbb{E}(\mathcal{W}(t), \mathcal{W}(s))$. To conduct inference, one needs estimates for N_b^0 , ρ , $\rho \vartheta N_b^0$, ϕ_Z and ϕ_e . With the normalization $N = 1$, $\hat{\lambda}_b^{\text{LS}} = \hat{T}_b^{\text{LS}}/T$ is a natural estimate of λ_0 . Consistent estimates of ϕ_Z and ϕ_e are given by

$$\begin{aligned} \hat{\phi}_Z &= \frac{\hat{\delta}' (T - \hat{T}_b^{\text{LS}})^{-1} \sum_{k=\hat{T}_b^{\text{LS}}+1}^T z_{kh} z'_{kh} \hat{\delta}}{\hat{\delta}' (\hat{T}_b^{\text{LS}})^{-1} \sum_{k=1}^{\hat{T}_b^{\text{LS}}} z_{kh} z'_{kh} \hat{\delta}} \\ \hat{\phi}_e &= \frac{\hat{\delta}' (T - \hat{T}_b^{\text{LS}})^{-1} \sum_{k=\hat{T}_b^{\text{LS}}+1}^T \hat{e}_{kh}^2 z_{kh} z'_{kh} \hat{\delta}}{\hat{\delta}' (\hat{T}_b^{\text{LS}})^{-1} \sum_{k=1}^{\hat{T}_b^{\text{LS}}} \hat{e}_{kh}^2 z_{kh} z'_{kh} \hat{\delta}}, \end{aligned}$$

where $\hat{\delta}$ and \hat{e}_{kh} are the LS estimator of δ_h and the residuals. Let $\vartheta = \|\delta^0\|^2 \bar{\sigma}^{-2} \rho$ and

$$\hat{\vartheta} = \hat{\rho} \|\hat{\delta}\|^2 \left(T^{-1} \sum_{k=1}^T \hat{e}_{kh}^2 \right)^{-1} \frac{\left(\hat{\delta}' \left(\hat{T}_b^{\text{LS}} \right)^{-1} \sum_{k=1}^{\hat{T}_b^{\text{LS}}} z_{kh} z'_{kh} \hat{\delta} \right)^2}{\hat{\delta}' \left(\hat{T}_b^{\text{LS}} \right)^{-1} \sum_{k=1}^{\hat{T}_b^{\text{LS}}} \hat{e}_{kh}^2 z_{kh} z'_{kh} \hat{\delta}}$$

$$\hat{\rho} = \frac{\left(\hat{\delta}' \left(\hat{T}_b^{\text{LS}} \right)^{-1} \sum_{k=1}^{\hat{T}_b^{\text{LS}}} z_{kh} z'_{kh} \hat{\delta} \right)^2}{\hat{\delta}' \left(\hat{T}_b^{\text{LS}} \right)^{-1} \sum_{k=1}^{\hat{T}_b^{\text{LS}}} \hat{e}_{kh}^2 z_{kh} z'_{kh} \hat{\delta}}.$$

We have $\hat{\vartheta}/h \xrightarrow{p} \vartheta$ and $\hat{\rho}/h \xrightarrow{p} \rho$. The final step is to derive numerically the empirical counterpart of (4.8), i.e., to estimate the density of the continuous record limiting distribution (cf. Proposition 5.1 in CP). The estimate is consistent under the conditions of Theorem 4.1 and under fixed shifts.

5 Small-Sample Properties of the GL Estimators

We conduct a Monte Carlo study to assess the small-sample accuracy of the GL estimator. We consider a least-absolute loss function with a continuous record prior (GL-CR) as well as a method based on the following iterative procedure which exploits Theorem 4.1. This variant, labeled GL-CR-Iter, uses the median of the feasible density of the continuous record distribution evaluated at the GL-CR estimate instead of at $\widehat{N}_b^{\text{LS}}$. Its justification is as follows. First, when the break size is small, the objective function is quite flat and highly variable. Hence, taking the median of the Quasi-prior instead of $p_h(N_b)$ is likely to involve less variability and more precise estimates. When the break size is large, $p_h(N_b)$ and the continuous record Quasi-prior have a similar shape with a peak at the same estimate. In principle, this iterative procedure may be based on any estimate \widehat{N}_b such that $\widehat{N}_b = N_b^0 + o_p(1)$. Furthermore, it can be viewed as an iterative version of the GL estimator as defined by (4.4) with $\gamma_h \rightarrow 0$. We also consider the least-squares (OLS) and the GL estimator under a least-absolute loss function with a uniform prior (GL-Uni). The least-squares estimator relies on a trimming parameter ϵ . Common choices in applied work are $\epsilon = 0.15$, 0.10 and 0.05, which preclude locating the break in the first and last 100 ϵ % of the sample. Since the GL estimator uses the least-squares criterion function, it also involves the same trimming. Moreover, since it takes into account its whole distribution, the trimming parameter plays a relatively more important role. We recommend to set ϵ not too high, otherwise the GL estimator tends to be too much concentrated toward the middle of the sample for small breaks. Thus, we set $\epsilon = 0.05$. We compare the mean absolute error (MAE), standard deviation (Std), root-mean-squared error (RMSE), and the 25% and 75% quantiles.

We consider discrete-time DGPs which take the following form:

$$y_t = D_t \varrho^0 + Z_t \delta_1^0 + Z_t \delta^0 \mathbf{1}_{\{t > T_b^0\}} + e_t, \quad t = 1, \dots, T, \quad (5.1)$$

with $T = 100$. Three versions of (5.1) are investigated: M1 is a partial structural change model with $\{Z_t\}$ a zero-mean stationary Gaussian AR(1) with autoregressive coefficient 0.3 and unit innovation variance, $D_t = 1$ for all t , $\varrho^0 = 1$ and $\{e_t\}$ *i.i.d.* $\mathcal{N}(0, 1.21)$ disturbances independent of $\{Z_t\}$; M2 involves a break in the mean which corresponds to $Z_t = 1$ for all t , D_t absent, and zero-mean stationary Gaussian AR(1) disturbances $\{e_t\}$ with AR coefficient 0.6 and innovation variance 0.49; M3 is a model with a lagged dependent variable with $D_t = y_{t-1}$, $Z_t = 1$, $e_t \sim i.i.d. \mathcal{N}(0, 0.5)$, $\varrho^0 = 0.6$ and $Z_t \delta^0 \mathbf{1}_{\{t > T_b^0\}}$ replaced by $Z_t (1.4 \varrho^0 \delta^0 \mathbf{1}_{\{t > T_b^0\}})$. We set $\delta_1^0 = 1$ for all DGPs except in M3 where $\delta_1^0 = 0$. We consider $\lambda_0 = 0.3$ and 0.5 , and $\delta^0 = 0.3, 0.4, 0.6$ and 1 . By symmetry, the case $\lambda_0 = 0.7$ is omitted to avoid repetitions.

The results are presented in Tables 1-3. We document that the LS estimator displays a large absolute bias (large MAE) when the size of the break is small, which increases as the true break point moves away from mid-sample. The GL estimators successfully reduce the absolute bias; when the break point is about mid-sample the reduction in MAE is roughly 50% when the size of the break is small or moderate. It is interesting to note that the distributions of GL-CR and GL-CR-Iter are more concentrated around mid-sample than the distribution of the OLS estimator which locates nontrivial mass in the tails. This explains the large reduction in MAE when the break date is about mid-sample. When the break date is close to the tails the GL estimators still perform better than OLS, though the margin is smaller. This feature arises because for small breaks the objective function is quite flat with a small peak at the least-squares estimate while the resulting Quasi-posterior inherits some trimodality from the continuous record asymptotic distribution. Since higher mass is located close to the least-squares estimate—which corresponds to the middle mode—and less in the tails, the GL estimator tends to concentrate toward mid-sample. Therefore, it is useful to consider choosing a smaller trimming parameter so as not to miss the information contained in the tails of the distribution and to avoid the GL concentrating too much toward mid-sample. The smaller is ϵ , the greater is the information used by the GL method to locate the break date. When the size of the break is large (i.e., $\delta^0 = 1$, bottom panel), although the absolute bias of the LS estimator becomes relatively small, the GL estimators still have lower MAE. Notably, the GL-type estimators display smaller variances than the LS estimator which leads to smaller RMSE. Thus, the GL estimate based on the CR prior dominates the LS estimator in both MAE and RMSE sense. Finally, the GL estimator that does not use the continuous record prior (i.e., GL-Uni), has a performance similar to the LS estimator since the Quasi-posterior contains essentially the same information about the objective function. Hence, using the continuous record prior is indeed important. This suggests that our asymptotic theory in Theorem 4.1 provides

a useful approximation since it states that the prior adds useful information even in the limit. Comparing GL-CR and GL-CR-Iter, we note that their performance is similar, though the former often seems to be slightly more precise especially when the break is near mid-sample.

We now discuss how the GL estimator achieves more precision relative to LS. As shown in Figures 1-2 the finite-sample distribution of the LS estimator displays trimodality when the magnitude of the break is small. This occurs because when the evidence for a break is weak, the LS estimator has a tendency to locate the break in the tails as confirmed by the quantiles which show for example that the 75% quantile for the LS estimator is much larger than for the GL estimators. This is a quite undesirable property given that the break is assumed to be, e.g., in the middle 70% of the sample if $\epsilon = 0.15$. The GL estimator does not share this property because it tends to concentrate more mass in the middle 70% of the sample. This works better when the true break date is about mid-sample. When it is near the tails, however, the choice $\epsilon = 0.05$ avoids the distribution being too concentrated near mid-sample and be more evenly spread. What is important is that as δ^0 increases we should expect the empirical distribution to move toward—and eventually concentrates about—the true break date. This is indeed achieved by the GL estimators.

6 Inference Methods based on the GL Estimate

In this section, we show how one can use the asymptotic properties of the GL estimator in order to extend the inference procedures for the break date presented in CP. We discuss several methods relying on the concept of highest density region. We begin with a description of this concept in Section 6.1. We present the new methods in Section 6.2. Section 6.3 provides the relevant theory.

6.1 Highest Density Regions

CP proposed to construct confidence sets using the concept of highest density region (HDR). The limit probability distribution of the break date estimator is given by (3.3), which is non-pivotal and depends on nuisance parameters. However, it can be simulated as described in Section 4.4. The next step is to derive numerically the regions of high density which can be used to construct the confidence sets for T_b^0 . The method based on HDR is especially useful when the distribution of the estimate is multi-modal and/or asymmetric. When the density is symmetric and uni-modal, the HDR method reduces to the conventional one to construct confidence sets, i.e., symmetric and based on standard errors. CP discussed an algorithm to obtain the HDR following Hyndman (1996) [for more recent developments see Samworth and Wand (2010) and Mason and Polonik (2008, 2009)]. Choose some $0 < \alpha < 1$ and let $\hat{\mathbb{P}}_{T_b}^{\text{CR}}$ denote the empirical counterpart of the limit distribution in Proposition 3.3 and $\hat{\mu}^{\text{CR}}$ its corresponding density function.

Definition 6.1. Highest Density Region: Assume that the density function $f_Y(y)$ of a random variable Y defined on a probability space $(\Omega_Y, \mathcal{F}_Y, \mathbb{P}_Y)$ and taking values on the measurable space $(\mathcal{Y}, \mathcal{B})$ is continuous and bounded. The $(1 - \alpha)$ 100% HDR is a subset $\mathbf{S}(\kappa_\alpha)$ of \mathcal{Y} defined as $\mathbf{S}(\kappa_\alpha) = \{y : f_Y(y) > \kappa_\alpha\}$ where κ_α is the largest constant that satisfies $\mathbb{P}_Y(Y \in \mathbf{S}(\kappa_\alpha)) \geq 1 - \alpha$.

Definition 6.2. OLS-CR HDR-based Confidence Sets for T_b^0 : A $(1 - \alpha)$ 100% HDR-based confidence set for T_b^0 is a subset of $\{1, \dots, T\}$ given by $C(\text{cv}_\alpha) = \{T_b \in \{1, \dots, T\} : T_b \in \mathbf{S}(\text{cv}_\alpha)\}$, where $\mathbf{S}(\text{cv}_\alpha) = \{T_b : \hat{\mu}^{\text{CR}} > \text{cv}_\alpha\}$ with cv_α satisfying $\sup_{\text{cv}_\alpha \in \mathbb{R}_+} \hat{\mathbb{P}}_{T_b}^{\text{CR}}(T_b \in \mathbf{S}(\text{cv}_\alpha)) \geq 1 - \alpha$.

The confidence set $C(\text{cv}_\alpha)$ has a frequentist interpretation even though the concept of HDR is often encountered in Bayesian analyses since it associates naturally to the derived posterior distribution, especially when the latter is multi-modal. Another feature of the confidence set $C(\text{cv}_\alpha)$ is that it may consist of the union of several disjoint intervals when the size of the shift is small. In summary, one needs to carry out the following steps to construct the HDR-based confidence sets when using the LS estimate, labeled OLS-CR.

Algorithm 1. OLS-CR HDR-based Confidence sets for T_b^0

(1) Estimate by LS the date of the break and the regression coefficients from model (2.5); (2) Replace the break date N_b^0 by $\widehat{N}_b^{\text{LS}}$, and the other quantities appearing in (3.3) by corresponding estimates as explained in Section 4.4; (3) Simulate the limit distribution $\widehat{\mathbb{P}}_{T_b}^{\text{CR}}$ from (3.3); (4) Compute the HDR of the empirical distribution $\widehat{\mathbb{P}}_{T_b}^{\text{CR}}$ and include the point T_b in the level $(1 - \alpha)$ confidence set $C(\text{cv}_\alpha)$ if T_b satisfies the conditions in Definition 6.2.

6.2 Confidence Sets based on $\widehat{N}_b^{\text{GL}}$ with $\pi(N_b) = \hat{\mu}^{\text{CR}}(N_b)$

We use the definition in (4.1) with the Quasi-prior $\pi(N_b)$ given by the continuous record asymptotic density $\hat{\mu}^{\text{CR}}$. We introduce the concept of Highest Quasi-posterior Density (HQPD) regions, defined analogously to HDR in Definition 6.1 with $p_h(N_b)$ being the object of interest. We then construct the confidence sets for T_b^0 as follows. We label this method by GL-CR.

Algorithm 2. GL-CR HQPD-based Confidence sets for T_b^0

(1) Estimate by least-squares N_b^0 and the regression coefficients from model (2.4); (2) Replace the break date N_b^0 by $\widehat{N}_b^{\text{LS}}$, and the other quantities appearing in (3.3) by corresponding estimates as explained in Section 4.4; (3) Simulate the limit distribution $\widehat{\mathbb{P}}_{T_b}^{\text{CR}}$ from (3.3) and set the Quasi-prior $\pi(N_b)$ equal to the probability density of $\widehat{\mathbb{P}}_{T_b}^{\text{CR}}$; (4) Construct the Quasi-posterior (4.1); (5) Obtain numerically the limit probability distribution of $\widehat{T}_b^{\text{GL}}$ given in Corollary 4.2 and label it $\widehat{\mathbb{P}}_{T_b}^{\text{GL}}$; (6) Compute the HQPD region of the probability distribution $\widehat{\mathbb{P}}_{T_b}^{\text{GL}}$ and include the point T_b in the level $(1 - \alpha)$ % confidence set $C_{\text{GL-CR}}(\text{cv}_\alpha)$ if T_b satisfies the conditions in Definition 6.2.

The confidence set $C_{\text{GL-CR}}(c\nu_\alpha)$ may also be interpreted as a Quasi-Bayes confidence set. Step 5 involves the derivation of the empirical counterpart of the limiting distribution of the GL estimator. In step 3, we use the continuous record Quasi-prior. Any prior $\pi(N_b)$ satisfying Assumption 4.2 could in principle be used, in which case the first three steps would not be needed. As discussed in Section 5, Theorem 4.1 also offers the possibility to construct alternative methods based on an iterative procedure as follows.

Algorithm 3. GL-CR-Iter HDR Confidence sets for T_b^0

(1) Carry out steps 1-3 in Algorithm 2; (2) Construct the Quasi-posterior in (4.1) and compute the estimate $\widehat{N}_b^{\text{GL}}$; (3) Replace the break date N_b^0 by $\widehat{N}_b^{\text{GL}}$, and the other quantities appearing in (3.3) by estimates as explained in Section 4.4; (4) Simulate the empirical counterpart of the probability distribution from Proposition 3.3 using step 3 labeled $\widehat{\mathbb{P}}_{T_b}^{\text{CR-GL}}$; (5) Compute the HDR of the empirical distribution $\widehat{\mathbb{P}}_{T_b}^{\text{CR-GL}}$ and include T_b in the level $(1 - \alpha)$ confidence set $C_{\text{GL-CR-Iter}}(c\nu_\alpha)$ if it satisfies the conditions in Definition 6.2.

6.3 Theoretical Results on Inference Methods

As explained above one generally needs estimates of some population quantities appearing in the continuous record asymptotic distribution in (3.2) and in the asymptotic distribution of the GL estimator. They can be constructed as explained in Section 4.4.

Assumption 6.1. Let $\widehat{\Sigma}_{i,h}(\cdot)$ be based on the estimates $\widehat{N}_{b,h}$, $\widehat{\delta}_h$, $\widehat{\phi}_{Z,h}$, and $\widehat{\phi}_{e,h}$ discussed in Section 4.4. For all $t, s \in \mathbb{R}$ and any $c > 0$, $\widehat{\Sigma}_{i,h}(\cdot)$ satisfies, for $i = 1, 2$: (i) $\widehat{\Sigma}_{i,h}(t, s) = \Sigma_i^0(t, s) + o_{\mathbb{P}}(1)$; (ii) $\widehat{\Sigma}_{i,h}(t, s) = \widehat{\Sigma}_{i,h}(t, t)$ if $t < s$ and $\widehat{\Sigma}_{i,h}(t, s) = \widehat{\Sigma}_{i,h}(s, s)$ if $t > s$; (iii) $\widehat{\Sigma}_{i,h}(ct, ct) = c\widehat{\Sigma}_{i,h}(t, t)$; (iv) $\mathbb{E}\{\sup_{t=1} \widehat{\Sigma}_{i,h}^2(t, t)\} = O(1)$.

Assumption 6.1-(i) requires the availability of certain consistent estimators. Let $\{\widehat{\mathcal{W}}_h\}$ be a (sample-size dependent) sequence of two-sided Gaussian processes with covariance $\widehat{\Sigma}_h$. By Assumption 6.1-(i), the limit law of $\{\widehat{\mathcal{W}}_h\}$ is the same as the law of \mathcal{W} . Construct the process $\widehat{\mathcal{V}}_h$ by replacing the population quantities in \mathcal{V}^* and replacing \mathcal{W} by $\widehat{\mathcal{W}}_h$. Parts (ii)-(iv) are technical conditions needed for the integrability of $\exp(\widehat{\mathcal{V}}_h(\cdot))$ and to prove the tightness of $\{\widehat{\mathcal{W}}_h\}$. Introduce the following random sample quantity:

$$\widehat{\Psi}_{l,h}(s) \triangleq \int_{\widehat{\Gamma}^*} l(s-v) \frac{\exp(\widehat{\mathcal{V}}_h(v)) \pi(\widehat{N}_b^{\text{GL}} + v)}{\int_{\widehat{\Gamma}^*} \exp(\widehat{\mathcal{V}}_h(w)) \pi(\widehat{N}_b^{\text{GL}} + w) dw} dv$$

where $\widehat{\Gamma}^*$ uses the estimates in Assumption 6.1 instead of the true values. Further, let $\widehat{\xi}_{l,h}$ be the absolute minimum point of $\widehat{\Psi}_{l,h}(s)$. Most inference methods introduced in the previous sub-section involve a numerical simulation of $\widehat{\Psi}_{l,h}$ or the moments or functions of the Quasi-posterior. The following theorem shows that $\widehat{\xi}_{l,h}$ converges in distribution to ξ_l^0 .

Theorem 6.1. *Let $l \in \mathbf{L}$. Under Assumptions 2.1-2.6, 4.1-4.3 and 6.1, the distribution of $\widehat{\xi}_{l,h}$ is first-order equivalent to the distribution of ξ_l^0 in Theorem 4.1.*

By Theorem 6.1, $\widehat{\xi}_{l,h}$ can be used to conduct statistical inference. For example, in order to use the GL HQDR-based method, step 5 of Algorithm 2 requires the derivation of the empirical counterpart of ξ_l^0 . Theorem 6.1 shows the validity of methods based on numerically evaluating $\widehat{\xi}_{l,h}$. This is achieved by simulating $\widehat{\Psi}_{l,h}$ using of the estimators in Assumption 6.1. Since $\widehat{\xi}_{l,h} \Rightarrow \xi_l^0$, inference based on the probability distribution of $\widehat{\xi}_{l,h}$ is asymptotically valid.

7 Small-Sample Evaluation of the GL Confidence Sets

In this section, we compare the performance of the proposed confidence sets about T_b^0 with existing methods. We consider Bai’s (1997) approach based on long-span shrinkage asymptotic arguments, Elliott and Müller’s (2007) approach based on inverting Nyblom’s (1989) statistic, the ILR method proposed by Eo and Morley (2015) based on the results in Qu and Perron (2007) and the recent HDR method proposed in CP based on the continuous record asymptotics with the least-squares estimate (OLS-CR).⁵ For brevity, we refer the readers to CP and Chang and Perron (2018) for a review and comprehensive evaluation of the first three methods. A brief summary is as follows. The empirical coverage probabilities of the confidence intervals obtained from Bai’s (1997) method are often below the nominal level when the size of the break is small. This feature is not present for the other methods. Elliott and Müller’s (2007) approach is by far the best one among existing methods in achieving an exact coverage rate that is the closest to the nominal level. However, this comes at the cost of lengths of the confidence sets which are always substantially larger relative to the other methods. This holds true across all break magnitudes. In addition, in models with serially correlated errors or lagged dependent variables, the length of the confidence set approaches the whole sample as the magnitude of the break increases. The ILR has coverage rates often above the nominal level and an average length significantly longer than the OLS-CR method at least when the magnitude of the shift is small or moderate. Further, the ILR does not work well when the errors are heteroskedastic and the regressor display serial correlation.

The recent OLS-CR method proposed in CP was shown to provide adequate empirical coverage probabilities over a wide range of data-generating mechanisms and across all break sizes and/or location of the break. The average length of the confidence sets is always shorter than that obtained with Elliott and Müller’s (2007) method. The OLS-CR method delivers confidence sets with lengths only slightly larger than Bai’s (1997) and the differences get smaller as the size of the break increases. Overall, the OLS-CR method strikes a better balance between accurate coverage

⁵Recently, Elliott et al. (2015) proposed a new test for structural breaks aimed at improving upon EM’s approach. However, they did not propose methods for the inversion of such test to construct confidence intervals for the break date. Hence, we cannot evaluate their method.

probabilities and length of the confidence sets. The numerical analysis in this section documents that the confidence sets derived from the GL inference are even more precise in terms of coverage probability than those from the OLS-CR method and the average length is always substantially shorter than that from Elliott and Müller’s (2007).

We consider the same DGPs as in Section 5. To construct the OLS-CR method we follow the steps outlined in the previous section (see also CP for more details on the procedure for models with predictable processes). For model M2, to estimate the long-run variance we use Andrews’ (1991) method along with Andrews and Monahan’s (1992) AR(1) pre-whitened two-stage procedure to select the bandwidth. We consider the version $\hat{U}_T(T_m)$.neq of Elliott and Müller (2007) that allows for heteroskedastic regimes; using the restricted version when applicable leads to similar results.

The least-squares estimation method is employed with a trimming parameter $\epsilon = 0.15$ and we use the required degrees of freedom adjustment for the statistic \hat{U}_T of Elliott and Müller (2007). The OLS-CR method and the methods proposed here do not involve any trimming and the confidence sets can potentially include any observation. The significance level is set at $\alpha = 0.05$, and the break date occurs at $\lfloor T\lambda_0 \rfloor$, where $\lambda_0 = 0.2, 0.35, 0.5$. The results for the 95% nominal coverage rates are presented in Tables 4-6. Each column reports the exact coverage rate and average length for a given break size δ^0 . The last row of each panel includes the rejection probability of a 5%-level sup-Wald test using the asymptotic critical value of Andrews (1993); it serves as a statistical measure about the magnitude of the break.

Overall, the simulation results confirm previous findings about the performance of existing methods. Bai’s (1997) method has a coverage rate below the nominal coverage level when the size of the break is small. For example, in model M2 for which there is high serial correlation in the disturbances, it fails to display a coverage rate above 90% even for moderate break sizes. In contrast, the method of Elliott and Müller (2007) overall yields very accurate empirical coverage rates. However, the average length of the confidence intervals is systematically much larger than those from all other methods across all DGPs, break sizes and break locations.⁶

The OLS-CR method provides good coverage rates for all break magnitudes and break locations. This holds even when there is high serial correlation in the errors (cf. M2). Furthermore, the confidence sets have average lengths significantly shorter than those from Elliott and Müller (2007). Thus, the OLS-CR method strikes overall a good balance between accurate coverage rates and length of the confidence sets. As observed in CP, this method tends to display an empirical coverage rate slightly below 95% for particular DGPs when the break size is small.

For the GL-based methods, we consider GL-CR and GL-CR-Iter implemented with the least-

⁶This problem is more severe when the errors are serially correlated or the model includes lagged dependent variables. Regarding the former, this in part may be due to issues with Newey and West HAC-type estimators when there are structural breaks [see Casini (2018, 2019), Casini and Perron (2019), Chang and Perron (2018), Crainiceanu and Vogelsang (2007), Deng and Perron (2006), Fossati (2018), Juhl and Xiao (2009), Kim and Perron (2009), Martins and Perron (2016), Perron and Yamamoto (2019) and Vogeslang (1999)].

absolute deviation loss function. We do not report results with a uniform prior because as documented Section 5, it has poor finite-sample properties with respect to MAE and RMSE for small breaks. In general, the family of methods based on the GL estimator display better features compared to the OLS-CR method. GL-CR provides shorter length than GL-CR-Iter and OLS-CR, and it displays decent coverage rates. GL-CR-Iter provides coverage rates and lengths similar to OLS-CR, yet the coverage rates are more accurate than GL-CR. Thus, we find that the GL-CR-Iter is better than GL-CR if the primary goal is accurate coverage of the confidence sets.

In summary, the simulation results suggest that the GL-CR-Iter method using the GL estimates is reliable as it provides accurate coverage rates close to the nominal level and average lengths of the confidence sets shorter relative to existing methods developed under large- N asymptotics.

8 Concluding Remarks

Building upon the continuous record asymptotic framework of [Casini and Perron \(2020a\)](#), we propose a Generalized Laplace (GL) procedure for the construction of the estimates and confidence sets for the date of a structural change. It is defined as the minimizer of the expected risk with the expectation taken under the Quasi-posterior, where the latter is constructed by applying a simple transformation to the least-squares criterion function. Our motivation stems from the non-standard properties of the finite-sample distribution of the least-squares break point estimator. The advantage of the GL procedure is that it combines information from the LS estimate of the break point, the objective function and the continuous record distribution theory. The GL estimate is more precise than the usual LS estimate, lower MAE and RMSE, especially when the true break date is about middle sample. In order to achieve this result, it is advisable to choose a small trimming parameter (e.g., $\epsilon = 0.05$). We also present inference methods that use the concept of HDR; the resulting confidence sets for the break date strike a better balance between empirical coverage rates and average lengths of the confidence sets relative to traditional long-span methods. Overall, among the GL procedures, we find that GL-CR provides more precise estimates for the break date while GL-CR-Iter results in more accurate coverage rates.

References

- Aït-Sahalia, Y., Jacod, J., 2014. High-frequency financial econometrics. Princeton University Press.
- Andersen, T.G., Fusari, N., Todorov, V., 2016. Short-term market risk implied by weekly options. *Journal of Finance* 72 (3), 1335–1386.
- Andrews, D.W.K., 1991. Heteroskedasticity and autocorrelation consistent covariance matrix estimation. *Econometrica* 59, 817–858.
- Andrews, D.W.K., 1993. Tests for parameter instability and structural change with unknown change-point. *Econometrica* 61 (4), 821–56.
- Andrews, D.W.K., Monahan, J.C., 1992. An improved heteroskedasticity and autocorrelation consistent covariance matrix estimator. *Econometrica* 60 (4), 953–966.
- Baek, Y., 2019. Estimation of structural break point in linear regression models. arXiv preprint arXiv 1811.03720.
- Bai, J., 1997. Estimation of a change-point in multiple regression models. *The Review of Economics and Statistics* 79 (4), 551–563.
- Bai, J., Perron, P., 1998. Estimating and testing linear models with multiple structural changes. *Econometrica* 66 (1), 47–78.
- Bai, J., Perron, P., 2003. Computation and analysis of multiple structural changes. *Journal of Applied Econometrics* 18, 1–22.
- Bandi, F. M., Renò, R., 2016. Price and volatility co-jumps. *Journal of Financial Economics* 119 (1), 107–146.
- Barndorff-Nielsen, O.E., Shephard, N., 2004. Econometric analysis of realised covariation: high frequency based covariance, regression and correlation in financial economics. *Econometrica* 72 (3), 885–925.
- Bickel, P.J., Yahav, J.A., 1969. Some contributions to the asymptotic theory of Bayes solutions. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 11 (4), 257–276.
- Casini, A., 2018. Tests for forecast instability and forecast failure under a continuous record asymptotic framework. arXiv preprint arXiv:1803.10883.
- Casini, A., 2019. Theory of evolutionary spectra for heteroskedasticity and autocorrelation robust inference in possibly misspecified and nonstationary models. Unpublished manuscript, Department of Economics and Finance, University of Rome Tor Vergata.
- Casini, A., Perron, P., 2019. Structural breaks in time series. In: *Oxford Research Encyclopedia of Economics and Finance*. Oxford University Press.
- Casini, A., Perron, P., 2020a. Continuous record asymptotics for structural change models. arXiv preprint arXiv:1803.10881.
- Casini, A., Perron, P., 2020b. Generalized Laplace inference in multiple change-points models. arXiv preprint arXiv:1803.10871.
- Casini, A., Perron, P., 2020c. Supplement to continuous record Laplace-based inference in structural change models. Unpublished manuscript, Department of Economics and Finance, University of Rome Tor Vergata.
- Chambers, M.J., Taylor, A.M.R., 2019. Deterministic parameter change models in continuous and discrete time. *Journal of Time Series Analysis*.
- Chang, S.Y., Perron, P., 2018. A comparison of alternative methods to construct confidence intervals for the estimate of a break date in linear regression models. *Econometric Reviews* 37 (6), 577–601.
- Chernozhukov, V., Hong, H., 2003. An MCMC approach to classical estimation. *Journal of Eco-*

- nometrics 115 (2), 293–346.
- Crainiceanu, C.M., Vogelsang, T.J., 2007. Nonmonotonic power for tests of a mean shift in a time series. *Journal of Statistical Computation and Simulation* 77 (6), 457–476.
- Dempster, A.P., Laird, N.M., Rubin, D.B., 1977. Maximum likelihood from incomplete data via the EM algorithm. *Journal of the Royal Statistical Society. Series B* 39 (1), 1–38.
- Deng, A., Perron, P., 2006. A comparison of alternative asymptotic frameworks to analyse a structural change in a linear time trend. *Econometrics Journal* 9 (3), 423–447.
- Elliott, G., Müller, U.K., 2007. Confidence sets for the date of a single break in linear time series regressions. *Journal of Econometrics* 141 (2), 1196–1218.
- Elliott, G., Müller, U.K., Watson, M.W., 2015. Nearly optimal tests when a nuisance parameter is present under the null hypothesis. *Econometrica* 83 (2), 771–811.
- Eo, Y., Morley, J., 2015. Likelihood-ratio-based confidence sets for the timing of structural breaks. *Quantitative Economics* 6 (2), 463–497.
- Forneron, J.J., Ng, S., 2018. The ABC of simulation estimation with auxiliary statistics. *Journal of Econometrics* 205 (1), 112–139.
- Fossati, S., 2018. Testing for state-dependent predictive ability. Unpublished manuscript, Department of Economics, University of Alberta.
- Hawkins, D., 1976. Point estimation of the parameters of piecewise regression models. *Journal of Applied Statistics* 25 (1), 51–57.
- Hounyo, U., Liu, Z., Varneskov, R.T., 2020. Bootstrapping Laplace Transforms of Volatility. Unpublished manuscript.
- Hyndman, R.J., 1996. Computing and graphing highest density regions. *The American Statistician* 50 (2), 120–126.
- Ibragimov, A., Has'minskii, R.Z., 1981. *Statistical estimation: asymptotic theory*. Springer-Verlag New York.
- Jiang, L., Wang, X., Yu, S., 2017. In-fill asymptotic theory for structural break point in autoregression: A unified theory. Unpublished manuscript, School of Economics, Singapore Management University.
- Jiang, L., Wang, X., Yu, S., 2018. New distribution theory for the estimation of structural break point in mean. *Journal of Econometrics* 205 (1), 156–176.
- Juhl, T., Xiao, Z., 2009. Testing for changing mean with monotonic power. *Journal of Econometrics* 148 (1), 14–24.
- Kim, D., Perron, P., 2009. Assessing the relative power of structural break tests using a framework based on the approximate Bahadur slope. *Journal of Econometrics* 149 (1), 26–51.
- Laplace, P.S., 1774. *Memoir on the probability of causes of events*. *Mémoires de Mathématique et de Physique Tome Sixième*, (English translation by S. M. Stigler 1986. *Statist. Sci.*, 1(19):364–378).
- Laredo, C., 1990. A sufficient condition for asymptotic sufficiency of incomplete observations of a diffusion process. *Annals of Statistics* 18 (3), 1158–1171.
- Li, J., Todorov, V., Tauchen, G., 2017. Adaptive estimation of continuous-time regression models using high-frequency data. *Journal of Econometrics* 200 (1), 36–47.
- Li, J., Xiu, D., 2016. Generalized method of integrated moments for high-frequency data. *Econometrica* 84 (4), 1613–1633.
- Martins, L., Perron, P., 2016. Improved tests for forecast comparisons in the presence of instabilities. *Journal of Time Series Analysis* 37 (5), 650–659.
- Mason, D.M., Polonik, W., 2008. Asymptotic normality of plug-in level set estimates, extended

version.

- Mason, D.M., Polonik, W., 2009. Asymptotic normality of plug-in level set estimates. *Annals of Applied Probability* 19 (3), 1108–1142.
- Newey, W.K., West, K.D., 1987. A simple positive semidefinite, heteroskedastic and autocorrelation consistent covariance matrix. *Econometrica* 55, 703–708.
- Nyblom, J., 1989. Testing for the constancy of parameters over time. *Journal of the American Statistical Association* 89 (451), 223–230.
- Perron, P., 1991. A continuous-time approximation to the unstable first-order autoregressive process: the case without an intercept. *Econometrica* 59 (1), 211–236.
- Perron, P., Qu, Z., 2006. Estimating restricted structural change model. *Journal of Econometrics* 134 (2), 373–399.
- Perron, P., Yamamoto, Y., 2019. Testing for changes in forecast performance. *Journal of Business and Economic Statistics* forthcoming.
- Phillips, P.C.B., 1987. Time series regression with a unit root. *Econometrica* 55 (2), 277–301.
- Qu, Z., Perron, P., 2007. Estimating and Testing Structural Changes in Multivariate Regressions. *Econometrica* 75 (2), 459–502.
- Samworth, R.J., Wand, M.P., 2010. Asymptotics and optimal bandwidth selection for highest density region estimation. *Annals of Statistics* 38 (3), 1767–1792.
- Sorensen, M., Uchida, M., 2003. Small-diffusion asymptotics for discretely sampled stochastic differential equations. *Bernoulli* 9 (6), 1051–1069.
- Vogelsang, T., 1999. Sources of nonmonotonic power when testing for a shift in mean of a dynamic time series. *Journal of Econometrics* 88 (2), 283–299.

A Appendix

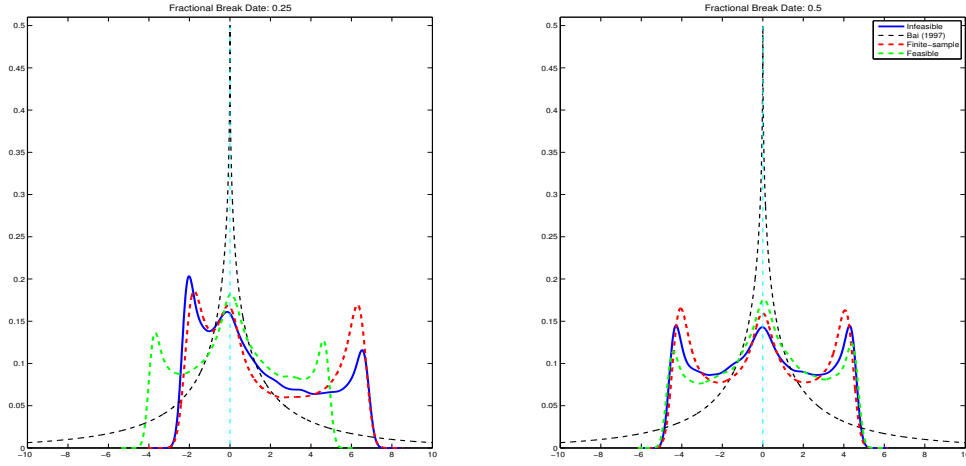


Figure 1: The probability density of the least-squares estimator $\widehat{T}_b - T_b^0$ for the model $y_t = \mu^0 + Z_t \delta_1^0 + Z_t \delta^0 \mathbf{1}_{\{t > \lfloor T \lambda_0 \rfloor\}} + e_t$, $Z_t = 0.3Z_{t-1} + u_t - 0.1u_{t-1}$, $u_t \sim \text{i.i.d. } \mathcal{N}(0, 1)$, $e_t \sim \text{i.i.d. } \mathcal{N}(0, 1)$, $\{u_t\}$ independent from $\{e_t\}$, $T = 100$ with break magnitude $\delta^0 = 0.3$ and true break point $\lambda_0 = 0.25$ and 0.5 (the left and right panel, respectively). The blue solid (resp., green broken) line is the density of the infeasible (reps., feasible) continuous record asymptotic distribution of CP, the black broken line is the density of the asymptotic distribution from Bai (1997) and the red broken line break is the density of the finite-sample distribution.

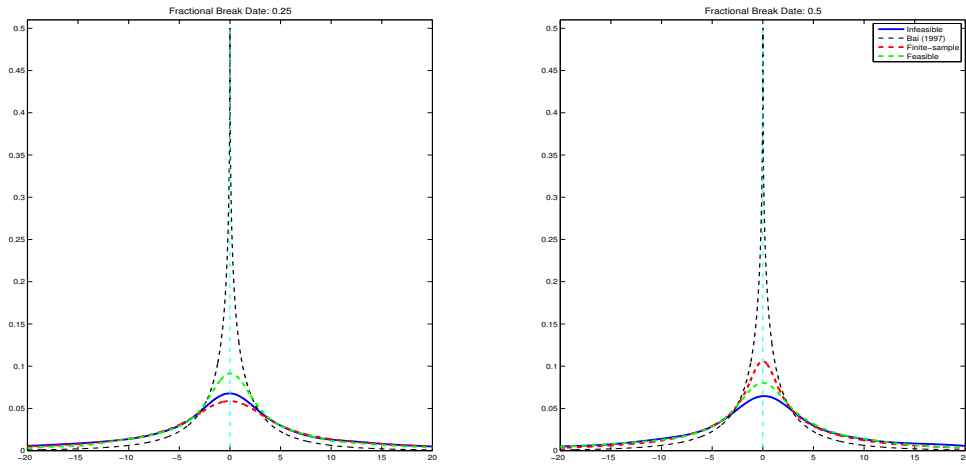


Figure 2: The comments in Figure 1 apply but with a break magnitude $\delta^0 = 1.5$.

Table 1: Small-sample accuracy of the estimates of the break point T_b^0 for model M1

		MAE	Std	RMSE	$Q_{0.25}$	$Q_{0.75}$	MAE	Std	RMSE	$Q_{0.25}$	$Q_{0.75}$
		$\lambda_0 = 0.3$					$\lambda_0 = 0.5$				
$\delta^0 = 0.3$	OLS	25.04	30.23	32.50	18	72	25.58	29.15	29.38	29	82
	GL-CR	14.20	10.60	19.06	29	57	4.94	9.82	9.97	46	54
	GL-CR-Iter	17.32	22.64	22.64	29	59	8.16	9.66	9.69	41	59
	GL-Uni	19.17	20.68	25.32	28	62	16.30	19.67	19.66	35	65
$\delta^0 = 0.4$	OLS	21.65	25.82	28.48	25	62	19.85	24.84	24.85	35	70
	GL-CR	13.31	11.63	17.83	28	50	4.77	9.75	9.86	46	52
	GL-CR-Iter	14.93	10.02	17.97	30	56	7.41	9.13	9.14	43	57
	GL-Uni	16.22	19.94	22.89	27	55	14.46	18.18	18.17	37	63
$\delta^0 = 0.6$	OLS	12.95	19.92	20.68	26	38	11.69	17.10	17.10	44	56
	GL-CR	12.25	12.51	15.56	29	49	4.46	8.82	8.64	49	51
	GL-CR-Iter	11.99	8.56	14.72	30	44	5.23	7.13	7.14	46	54
	GL-Uni	11.24	16.19	17.29	26	40	9.60	13.75	13.36	44	56
$\delta^0 = 1$	OLS	9.72	14.91	14.90	27	40	4.89	8.25	8.24	48	52
	GL-CR	5.06	8.17	8.21	27	32	2.81	7.17	7.21	49	51
	GL-CR-Iter	8.41	4.80	9.67	29	39	3.81	6.31	6.34	48	52
	GL-Uni	4.69	8.35	8.36	27	32	4.51	7.47	7.47	48	52

The model is $y_t = \varrho^0 + Z_t \delta_1^0 + Z_t \delta^0 \mathbf{1}_{\{t > \lfloor T \lambda_0 \rfloor\}} + e_t$, $Z_t = 0.3Z_{t-1} + u_t$, $u_t \sim i.i.d. \mathcal{N}(0, 1)$, $e_t \sim i.i.d. \mathcal{N}(0, 1.21)$, $T = 100$. The columns refer to Mean Absolute Error (MAE), standard deviation (Std), Root Mean Squared Error (RMSE) and the 25% and 75% empirical quantiles. OLS is the least-squares estimator; GL-CR is the GL estimator under a least-absolute loss function with the continuous record prior; GL-CR-Iter is the median of the density of the continuous record distribution which uses the GL estimator in place of \widehat{N}_b^{LS} ; GL-Uni is the GL estimator under a least-absolute loss function with a uniform prior.

Table 2: Small-sample accuracy of the estimates of the break point T_b^0 for model M2

		MAE	Std	RMSE	$Q_{0.25}$	$Q_{0.75}$	MAE	Std	RMSE	$Q_{0.25}$	$Q_{0.75}$
		$\lambda_0 = 0.3$					$\lambda_0 = 0.5$				
$\delta^0 = 0.3$	OLS	26.84	28.12	33.00	21	76	23.02	26.86	26.76	25	75
	GL-CR	12.79	13.13	18.46	29	57	11.84	13.17	13.12	35	65
	GL-CR-Iter	14.47	10.29	20.21	28	58	8.76	10.01	10.24	41	59
	GL-Uni	21.78	21.73	27.71	28	66	17.84	20.90	20.98	32	68
$\delta^0 = 0.4$	OLS	23.62	26.99	30.23	21	70	21.23	25.43	25.44	25	75
	GL-CR	16.36	13.86	21.49	29	61	11.56	11.97	12.25	36	64
	GL-CR-Iter	17.19	10.81	20.35	28	57	8.30	9.95	10.01	43	57
	GL-Uni	20.18	21.25	26.30	28	64	16.53	19.97	19.98	34	64
$\delta^0 = 0.6$	OLS	19.80	24.62	26.25	21	57	17.34	22.39	22.34	37	65
	GL-CR	12.84	13.66	18.23	30	56	9.96	11.93	11.99	38	58
	GL-CR-Iter	14.85	11.52	17.56	29	52	7.26	9.20	9.22	44	55
	GL-Uni	16.04	20.05	22.77	26	56	13.85	17.81	17.94	38	60
$\delta^0 = 1$	OLS	11.69	18.43	19.26	27	40	9.38	14.40	14.40	46	54
	GL-CR	6.82	10.85	12.81	27	38	6.96	9.43	9.52	44	53
	GL-CR-Iter	10.67	7.54	13.02	30	39	4.44	6.71	6.85	47	53
	GL-Uni	9.44	14.60	15.15	27	37	8.17	12.34	12.34	45	54

The model is $y_t = \delta_1^0 + \delta^0 \mathbf{1}_{\{t > \lfloor T\lambda_0 \rfloor\}} + e_t$, $e_t = 0.6e_{t-1} + u_t$, $u_t \sim i.i.d. \mathcal{N}(0, 0.49)$, $T = 100$. The notes of Table 1 apply.

Table 3: Small-sample accuracy of the estimates of the break point T_b^0 for model M3

		MAE	Std	RMSE	$Q_{0.25}$	$Q_{0.75}$	MAE	Std	RMSE	$Q_{0.25}$	$Q_{0.75}$
		$\lambda_0 = 0.3$					$\lambda_0 = 0.5$				
$\delta^0 = 0.3$	OLS	22.33	26.93	29.88	25	43	17.23	22.54	22.48	36	64
	GL-CR	12.79	13.32	18.41	29	57	10.18	12.11	12.16	38	62
	GL-CR-Iter	13.92	15.42	19.05	29	56	10.85	13.62	13.69	39	61
	GL-Uni	15.92	16.81	21.40	30	55	11.28	14.54	14.54	41	59
$\delta^0 = 0.4$	OLS	22.30	26.62	29.55	22	61	13.98	19.70	19.68	42	58
	GL-CR	10.08	12.73	16.23	29	46	9.08	11.41	11.41	42	58
	GL-CR-Iter	11.08	14.13	16.12	29	46	9.10	12.25	12.28	43	57
	GL-Uni	12.87	15.64	18.53	29	49	9.62	12.98	12.97	43	57
$\delta^0 = 0.6$	OLS	8.56	15.91	16.15	27	33	8.04	13.24	13.22	46	54
	GL-CR	4.68	8.99	10.13	29	35	5.62	8.15	8.15	47	53
	GL-CR-Iter	7.06	10.90	11.35	29	35	6.17	9.36	9.36	46	53
	GL-Uni	7.77	11.85	12.79	28	38	6.40	9.63	9.62	46	54
$\delta^0 = 1$	OLS	2.55	5.39	5.45	29	31	2.42	4.52	4.52	49	51
	GL-CR	1.24	3.26	3.48	29	31	2.31	4.11	4.11	49	51
	GL-CR-Iter	2.06	5.30	5.33	29	31	2.33	4.56	4.59	48	52
	GL-Uni	3.07	6.18	6.21	29	31	2.65	4.79	4.81	48	51

The model is $y_t = 1.4\rho^0 \delta^0 \mathbf{1}_{\{t > \lfloor T\lambda_0 \rfloor\}} + \rho^0 y_{t-1} + e_t$, $e_t \sim i.i.d. \mathcal{N}(0, 0.5)$, $\rho^0 = 0.6$, $T = 100$. The notes of Table 1 apply.

Table 4: Small-sample coverage rates and lengths of the confidence sets for model M1

		$\delta^0 = 0.4$		$\delta^0 = 0.8$		$\delta^0 = 1.2$		$\delta^0 = 1.6$	
		Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.
$\lambda_0 = 0.5$	OLS-CR	0.959	80.98	0.958	62.61	0.958	33.03	0.937	15.60
	Bai (1997)	0.803	60.84	0.837	32.03	0.874	15.84	0.875	9.34
	$\widehat{U}_T(T_m).neq$	0.945	80.55	0.945	53.41	0.945	31.68	0.945	22.05
	ILR	0.962	77.99	0.970	39.64	0.977	18.00	0.985	10.41
	GL-CR	0.924	61.86	0.919	53.51	0.904	25.99	0.877	18.82
	GL-CR-Iter	0.957	81.87	0.958	64.07	0.957	33.04	0.938	15.64
	sup-W	0.300		0.865		0.996		0.997	
$\lambda_0 = 0.35$	OLS-CR	0.960	79.26	0.920	61.18	0.956	32.07	0.938	15.58
	Bai (1997)	0.828	59.47	0.833	43.92	0.871	15.96	0.877	9.39
	$\widehat{U}_T(T_m).neq$	0.949	81.58	0.940	68.51	0.949	33.23	0.949	22.53
	ILR	0.963	79.92	0.972	44.71	0.978	19.81	0.980	11.01
	GL-CR	0.929	61.18	0.918	50.61	0.900	32.84	0.871	17.96
	GL-CR-Iter	0.965	80.58	0.927	63.34	0.957	32.64	0.939	15.65
	sup-W	0.276		0.592		0.992		1.000	
$\lambda_0 = 0.2$	OLS-CR	0.954	78.14	0.967	54.93	0.972	29.74	0.951	15.73
	Bai (1997)	0.827	59.48	0.884	33.24	0.900	17.11	0.896	9.84
	$\widehat{U}_T(T_m).neq$	0.951	84.04	0.951	65.09	0.951	57.96	0.951	28.16
	ILR	0.958	83.06	0.968	53.62	0.978	25.73	0.986	12.69
	GL-CR	0.914	60.21	0.939	47.40	0.931	28.07	0.910	14.47
	GL-CR-Iter	0.965	79.51	0.972	57.07	0.974	30.64	0.957	15.92
	sup-W	0.161		0.576		0.901		0.987	

The model is $y_t = \varrho^0 + Z_t \delta_1^0 + Z_t \delta^0 \mathbf{1}_{\{t > \lfloor T \lambda_0 \rfloor\}} + e_t$, $Z_t = 0.3Z_{t-1} + u_t$, $u_t \sim i.i.d. \mathcal{N}(0, 1)$, $e_t \sim i.i.d. \mathcal{N}(0, 1.21)$, $T = 100$. Cov. and Lgth. refer to the coverage probability and the average length of the confidence set (i.e., the average number of dates in the confidence set). sup-W refers to the rejection probability of the sup-Wald test using a 5% asymptotic critical value. The number of simulations is 5,000.

Table 5: Small-sample coverage rates and lengths of the confidence sets for model M2

		$\delta^0 = 0.4$		$\delta^0 = 0.8$		$\delta^0 = 1.2$		$\delta^0 = 1.6$	
		Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.
$\lambda_0 = 0.5$	OLS-CR	0.910	67.57	0.911	68.87	0.925	56.25	0.945	42.30
	Bai (1997)	0.808	67.57	0.811	50.22	0.843	32.67	0.894	20.74
	$\widehat{U}_T(T_m).neq$	0.981	91.66	0.973	87.38	0.973	82.24	0.973	79.19
	ILR	0.944	83.70	0.946	67.01	0.964	45.75	0.980	28.30
	GL-CR	0.885	60.05	0.884	52.63	0.898	43.41	0.926	32.61
	GL-CR-Iter	0.911	76.72	0.911	69.06	0.923	56.31	0.944	42.20
	sup-W	0.333		0.565		0.878		0.926	
$\lambda_0 = 0.35$	OLS-CR	0.927	75.58	0.910	66.20	0.921	52.89	0.944	39.15
	Bai (1997)	0.838	66.86	0.821	49.34	0.857	32.43	0.893	20.77
	$\widehat{U}_T(T_m).neq$	0.976	91.72	0.976	87.25	0.976	81.76	0.976	77.62
	ILR	0.941	83.24	0.945	68.31	0.959	47.58	0.975	29.32
	GL-CR	0.898	57.32	0.888	50.29	0.902	43.08	0.924	29.06
	GL-CR-Iter	0.930	75.87	0.913	66.13	0.921	52.66	0.944	38.71
	sup-W	0.489		0.625		0.868		1.000	
$\lambda_0 = 0.2$	OLS-CR	0.910	75.24	0.917	64.17	0.931	48.76	0.953	34.26
	Bai (1997)	0.808	67.03	0.852	50.40	0.897	33.62	0.937	21.76
	$\widehat{U}_T(T_m).neq$	0.981	92.17	0.981	89.14	0.981	84.72	0.981	80.39
	ILR	0.938	85.37	0.951	74.32	0.963	57.03	0.977	36.24
	GL-CR	0.912	56.87	0.909	48.68	0.920	38.47	0.932	23.91
	GL-CR-Iter	0.894	75.15	0.923	64.14	0.934	48.75	0.953	34.06
	sup-W	0.331		0.565		0.795		0.931	

The model is $y_t = \delta_1^0 + \delta^0 \mathbf{1}_{\{t > [T\lambda_0]\}} + e_t$, $e_t = 0.6e_{t-1} + u_t$, $u_t \sim i.i.d. \mathcal{N}(0, 0.49)$, $T = 100$. The notes of Table 4 apply.

Table 6: Small-sample coverage rates and lengths of the confidence sets for model M3

		$\delta^0 = 0.4$		$\delta^0 = 0.8$		$\delta^0 = 1.2$		$\delta^0 = 1.6$	
		Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.
$\lambda_0 = 0.5$	OLS-CR	0.895	67.38	0.923	39.66	0.952	19.23	0.972	10.30
	Bai (1997)	0.759	43.84	0.851	17.50	0.904	8.77	0.935	5.49
	$\widehat{U}_T(T_m).neq$	0.948	81.26	0.947	71.72	0.949	80.79	0.952	90.23
	ILR	0.934	77.87	0.951	50.62	0.961	31.01	0.975	22.80
	GL-CR	0.871	66.58	0.876	36.80	0.893	15.76	0.914	7.50
	GL-CR-Iter	0.896	71.50	0.916	42.49	0.948	20.23	0.967	10.30
	sup-W	0.300		0.988		0.996		1.000	
$\lambda_0 = 0.35$	OLS-CR	0.902	66.58	0.938	39.84	0.953	19.71	0.973	10.47
	Bai (1997)	0.766	43.21	0.855	17.61	0.901	8.82	0.935	5.52
	$\widehat{U}_T(T_m).neq$	0.950	82.13	0.950	72.22	0.952	78.81	0.953	87.02
	ILR	0.936	78.13	0.944	47.77	0.959	30.81	0.976	22.16
	GL-CR	0.885	66.06	0.887	35.98	0.893	15.79	0.917	7.59
	GL-CR-Iter	0.908	71.08	0.937	42.99	0.951	20.71	0.967	10.55
	sup-W	0.595		0.973		1.000		1.000	
$\lambda_0 = 0.2$	OLS-CR	0.910	65.50	0.950	38.42	0.956	18.13	0.969	10.41
	Bai (1997)	0.802	44.43	0.884	18.47	0.952	8.68	0.932	5.32
	$\widehat{U}_T(T_m).neq$	0.948	89.91	0.949	77.46	0.950	79.01	0.932	88.41
	ILR	0.932	79.97	0.941	49.72	0.965	33.97	0.951	21.62
	GL-CR	0.897	57.65	0.903	34.51	0.889	15.21	0.912	7.43
	GL-CR-Iter	0.903	65.07	0.923	37.65	0.939	17.46	0.959	10.39
	sup-W	0.443		0.914		0.999		1.000	

The model is $y_t = 1.4\varrho^0\delta^0\mathbf{1}_{\{t > [T\lambda_0]\}} + \varrho^0 y_{t-1} + e_t$, $e_t \sim i.i.d. \mathcal{N}(0, 0.5)$, $\varrho^0 = 0.6$, $T = 100$. The notes of Table 4 apply.

Supplement to Continuous Record Laplace-based Inference about the Break Date in Structural Change Models

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Abstract

This supplemental material includes the Mathematical Appendix to [Casini and Perron \(2020b\)](#), which includes all proofs of the results in the paper.

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A Mathematical Appendix

The supplement is structured as follows. In Section A.1 we present some preliminary lemmas. The proof of the results of Section 4 can be found in Section A.2 while those about inference are in Section A.3.

A.1 Additional Notation and Preliminary Results

We denote the (i, j) -th element of the outer product matrix $A'A$ by $(A'A)_{i,j}$ and the $i \times j$ upper-left (resp., lower-right) sub-block of $A'A$ by $[A'A]_{\{i \times j, \cdot\}}$ (resp., $[A]_{\{\cdot, i \times j\}}$). For a random variable ξ and a number $r \geq 1$, we write $\|\xi\|_r = (\mathbb{E} \|\xi\|^r)^{1/r}$. C is used as a generic constant that may vary from line to line; we may sometime write C_r to emphasize the dependence of C on some number r . For two scalars a and b the symbol $a \wedge b$ means the infimum of $\{a, b\}$. The symbol “ $\xrightarrow{\text{u.c.P.}}$ ” signifies uniform locally in time convergence under the Skorokhod topology, which implies convergence in probability.

It is typical in the high frequency statistics literature to use a localization argument [cf. Section I.1.d in Jacod and Shiryaev (2003)] which allows us to replace Assumption 2.1-2.2 by the following stronger assumption which basically turns all the local conditions into global ones.

Assumption A.1. *Under Assumption 2.1-2.6, $\{Y_t, D_t, Z_t\}_{t \geq 0}$ takes value in some compact set, $\{\sigma_{\cdot,t}, \sigma_{e,t}\}_{t \geq 0}$ is bounded càdlàg and $\{\mu_{\cdot,t}\}$ are bounded càdlàg or càglàd.*

Next, we collect a few lemmas. They follow from Lemma S.A.1-S.A.4 in CP (2020a).

Lemma A.1. *The following inequalities hold \mathbb{P} -a.s.:*

$$(Z'_0 M Z_0) - (Z'_0 M Z_2) (Z'_2 M Z_2)^{-1} (Z'_2 M Z_0) \geq H' (X'_\Delta X_\Delta) (X'_2 X_2)^{-1} (X'_0 X_0) H, \quad T_b < T_b^0 \quad (\text{A.1})$$

$$(Z'_0 M Z_0) - (Z'_0 M Z_2) (Z'_2 M Z_2)^{-1} (Z'_2 M Z_0) \geq H' (X'_\Delta X_\Delta) (X'X - X'_2 X_2)^{-1} (X'X - X'_0 X_0) H, \quad T_b \geq T_b^0 \quad (\text{A.2})$$

Lemma A.2. *Let X_t (\tilde{X}_t) be a q -dimensional (resp. p -dimensional) Itô continuous semimartingale defined on $[0, N]$. Let Σ_t denote the date t instantaneous covariation between X_t and \tilde{X}_t . Choose a fixed number $\epsilon > 0$ and ϖ satisfying $1/2 - \epsilon \geq \varpi \geq \epsilon > 0$. Further, let $B_T \triangleq \lfloor N/h - T^\varpi \rfloor$. Define the moving average of Σ_t as $\bar{\Sigma}_{kh} \triangleq (T^\varpi h)^{-1} \int_{kh}^{kh+T^\varpi h} \Sigma_s ds$, and let $\hat{\Sigma}_{kh} \triangleq (T^\varpi h)^{-1} \sum_{i=1}^{\lfloor T^\varpi \rfloor} \Delta_h X_{k+i} \Delta_h \tilde{X}'_{k+i}$. Then, $\sup_{1 \leq k \leq B_T} \|\hat{\Sigma}_{kh} - \bar{\Sigma}_{kh}\| = o_{\mathbb{P}}(1)$.*

Lemma A.3. *Under Assumption A.1 we have as $h \downarrow 0, T \rightarrow \infty$ with N fixed and for any $1 \leq i, j \leq p$, (i) $\left| (Z'_2 e)_{i,1} = \sum_{k=T_b+1}^T z_{kh}^{(i)} e_{kh} \right| \xrightarrow{\mathbb{P}} 0$; (ii) $\left| (Z'_0 e)_{i,1} = \sum_{k=T_b^0+1}^T z_{kh}^{(i)} e_{kh} \right| \xrightarrow{\mathbb{P}} 0$; (iii) $\left| (Z'_2 Z_2)_{i,j} - \int_{(T_b+1)h}^N \Sigma_{ZZ,s}^{(i,j)} ds \right| \xrightarrow{\mathbb{P}} 0$ where $(Z'_2 Z_2)_{i,j} = \sum_{k=T_b+1}^T z_{kh}^{(i)} z_{kh}^{(j)}$; (iv) $\left| (Z'_0 Z_0)_{i,j} - \int_{(T_b^0+1)h}^N \Sigma_{ZZ,s}^{(i,j)} ds \right| \xrightarrow{\mathbb{P}} 0$ with $(Z'_0 Z_0)_{i,j} = \sum_{k=T_b^0+1}^T z_{kh}^{(i)} z_{kh}^{(j)}$.*

For the following estimates involving X , we have, for any $1 \leq r \leq p$ and $1 \leq l \leq q+p$, (v) $\left| (Xe)_{l,1} \right| \xrightarrow{\mathbb{P}} 0$ where $(Xe)_{l,1} = \sum_{k=1}^T x_{kh}^{(l)} e_{kh}$; (vi) $\left| (Z'_2 X)_{r,l} - \int_{(T_b+1)h}^N \Sigma_{ZX,s}^{(r,l)} ds \right| \xrightarrow{\mathbb{P}} 0$ where $(Z'_2 X)_{r,l} = \sum_{k=T_b+1}^T z_{kh}^{(r)} x_{kh}^{(l)}$; (vii) $\left| (Z'_0 X)_{r,l} - \int_{(T_b^0+1)h}^N \Sigma_{ZX,s}^{(r,l)} ds \right| \xrightarrow{\mathbb{P}} 0$ where $(Z'_0 X)_{r,l} = \sum_{k=T_b^0+1}^T z_{kh}^{(r)} x_{kh}^{(l)}$. Further, for $1 \leq u, d \leq q+p$, (viii) $\left| (X'X)_{u,d} - \int_0^N \Sigma_{XX,s}^{(u,d)} ds \right| \xrightarrow{\mathbb{P}} 0$ where $(X'X)_{u,d} = \sum_{k=1}^T x_{kh}^{(u)} x_{kh}^{(d)}$.

Denote

$$X_\Delta \triangleq X_2 - X_0 = \left(0, \dots, 0, x_{(T_b+1)h}, \dots, x_{T_b^0 h}, 0, \dots, \right)', \quad \text{for } N_b < N_b^0$$

$$X_\Delta \triangleq -(X_2 - X_0) = \left(0, \dots, 0, x_{(T_b^0+1)h}, \dots, x_{T_b h}, 0, \dots, \right)', \quad \text{for } N_b > N_b^0$$

whereas $X_\Delta \triangleq 0$ for $N_b = N_b^0$. Observe that $X_2 = X_0 + X_\Delta \text{sign}(N_b^0 - N_b)$. When the sign is immaterial, we simply write $X_2 = X_0 + X_\Delta$.

Lemma A.4. *Under Assumption A.1, we have as $h \downarrow 0, T \rightarrow \infty$ with N fixed, $|N_b^0 - N_b| > \gamma > 0$ and for any $1 \leq i, j \leq p$,*

$$(i) \text{ with } (Z'_\Delta Z_\Delta)_{i,j} = \sum_{k=T_b^0+1}^{T_b} z_{kh}^{(i)} z_{kh}^{(j)} \text{ then } \begin{cases} |(Z'_\Delta Z_\Delta)_{i,j} - \int_{(T_b+1)h}^{T_b^0 h} \Sigma_{ZZ,s}^{(i,j)} ds| \xrightarrow{\mathbb{P}} 0, & \text{if } T_b < T_b^0 \\ |(Z'_\Delta Z_\Delta)_{i,j} - \int_{T_b^0 h}^{(T_b+1)h} \Sigma_{ZZ,s}^{(i,j)} ds| \xrightarrow{\mathbb{P}} 0, & \text{if } T_b > T_b^0 \end{cases}$$

and for $1 \leq r \leq p+q$

$$(ii) \text{ with } (Z'_\Delta X_\Delta)_{i,r} = \sum_{k=T_b^0+1}^{T_b} z_{kh}^{(i)} x_{kh}^{(r)} \text{ then } \begin{cases} |(Z'_\Delta X_\Delta)_{i,r} - \int_{(T_b+1)h}^{T_b^0 h} \Sigma_{ZX,s}^{(i,r)} ds| \xrightarrow{P} 0, & \text{if } T_b < T_b^0 \\ |(Z'_\Delta X_\Delta)_{i,r} - \int_{T_b^0 h}^{(T_b+1)h} \Sigma_{ZX,s}^{(i,r)} ds| \xrightarrow{P} 0, & \text{if } T_b > T_b^0 \end{cases}$$

Moreover, each quantity on the right hand side above are bounded in probability.

We will also use the following Central Limit Theorem [see Theorem 5.4.2. in [Jacod and Protter \(2012\)](#)]. We choose a progressively measurable ‘‘square-root’’ process Υ_Z of the \mathbb{M}_q^+ -valued process $\widehat{\Sigma}_{Z,s}$ whose elements are given by $\widehat{\Sigma}_{Z,s}^{(ij,kl)} = \Sigma_{Z,s}^{(ik)} \Sigma_{Z,s}^{(jl)}$. Note that by the symmetry of $\Sigma_{Z,s}$, the matrix with entries $(\Upsilon_{Z,s}^{(ij,kl)} + \Upsilon_{Z,s}^{(ji,kl)})/\sqrt{2}$ is a square-root of the matrix with entries $\widehat{\Sigma}_{Z,s}^{(ij,kl)} + \widehat{\Sigma}_{Z,s}^{(il,jk)}$. Let W_s^* a q^2 -dimensional standard Wiener process defined on an extension of $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Then the process \mathcal{U}_t with components $\mathcal{U}_t^{(rj)} = 2^{-1/2} \sum_{k,l=1}^q \int_0^t (\Upsilon_{Z,s}^{(rj,kl)} + \Upsilon_{Z,s}^{(jr,kl)}) dW_s^{*(kl)}$ is, conditionally on \mathcal{F} , a continuous Gaussian process with independent increments and (conditional) covariance $\widehat{\mathbb{E}}(\mathcal{U}^{(rj)}(v) \mathcal{U}^{(kl)}(v) | \mathcal{F}) = \int_{N_b^0+v}^{N_b^0} (\Sigma_{Z,s}^{(rk)} \Sigma_{Z,s}^{(jl)} + \Sigma_{Z,s}^{(rl)} \Sigma_{Z,s}^{(jk)}) ds$, with $v < 0$. The Central Limit Theorem we shall use is the following [cf. Lemma S.A.5 in [CP \(2020a\)](#)].

Lemma A.5. *Let Z be a continuous Itô semimartingale satisfying Assumption A.1. Then, $h^{-1/2}(Z'_2 Z_2 - ([Z, Z]_{Th} - [Z, Z]_{(T_b+1)h})) \xrightarrow{\mathcal{L}}^s \mathcal{U}$.*

Recall $Q_h(\widehat{\delta}(N_b), N_b) = \widehat{\delta}(N_b)'(Z'_2 M Z_2) \widehat{\delta}(N_b)$. We decompose $Q_h(\widehat{\delta}(N_b), N_b) - Q_h(\widehat{\delta}(N_b^0), N_b^0)$ into a ‘‘deterministic’’ and a ‘‘stochastic’’ component. By definition,

$$\widehat{\delta}(N_b) = (Z'_2 M Z_2)^{-1} (Z'_2 M Y) = (Z'_2 M Z_2)^{-1} (Z'_2 M Z_0) \delta^0 + (Z'_2 M Z_2)^{-1} Z_2 M e,$$

and $\widehat{\delta}(N_b^0) = (Z'_0 M Z_0)^{-1} (Z'_0 M Y) = \delta^0 + (Z'_0 M Z_0)^{-1} (Z'_0 M e)$. Therefore,

$$\begin{aligned} Q_h(\widehat{\delta}(N_b), N_b) - Q_h(\widehat{\delta}(N_b^0), N_b^0) &= \widehat{\delta}(N_b)' (Z'_2 M Z_2) \widehat{\delta}(N_b) - \widehat{\delta}(N_b^0)' (Z'_0 M Z_0) \widehat{\delta}(N_b^0) \\ &\triangleq D_h(\delta^0, N_b) + S_{e,h}(\delta^0, N_b), \end{aligned} \quad (\text{A.3})$$

where

$$D_h(\delta^0, N_b) \triangleq (\delta^0)' \left\{ (Z'_0 M Z_2) (Z'_2 M Z_2)^{-1} (Z'_2 M Z_0) - Z'_0 M Z_0 \right\} \delta^0 \quad (\text{A.4})$$

is the deterministic part and

$$S_{e,h}(\delta^0, N_b) \triangleq 2 (\delta^0)' (Z'_0 M Z_2) (Z'_2 M Z_2)^{-1} Z_2 M e - 2 (\delta^0)' (Z'_0 M e) \quad (\text{A.5})$$

$$+ e' M Z_2 (Z'_2 M Z_2)^{-1} Z_2 M e - e' M Z_0 (Z'_0 M Z_0)^{-1} Z'_0 M e \quad (\text{A.6})$$

is the stochastic part. Next, let $Z_\Delta = X_\Delta H$, and define

$$\overline{D}_h(\delta^0, N_b) \triangleq -D_h(\delta^0, N_b) / |T_b - T_b^0|. \quad (\text{A.7})$$

We arbitrarily define $\bar{D}_h(\delta^0, N_b) = (\delta^0)' \delta^0$ when $N_b = N_b^0$ since both the numerator and denominator of $\bar{D}_h(\delta^0, N_b)$ are zero. Observe that $\bar{D}_h(\delta^0, N_b)$ is non-negative because the matrix inside the braces in $D_h(\delta^0, N_b)$ is negative semidefinite. Equation (A.3) can be rewritten as

$$Q_h(\delta(N_b), N_b) - Q_h(\delta(N_b^0), N_b^0) = -|T_b - T_b^0| \bar{D}_h(N_b, \delta^0) + S_{e,h}(N_b, \delta^0) \quad \text{for all } T_b. \quad (\text{A.8})$$

Lemma A.6. *Under Assumption 2.2-2.6 and 3.1, uniformly in N_b ,*

$$Q_h(\delta(N_b), N_b) - Q_h(\delta(N_b^0), N_b^0) = -\delta_h(Z'_\Delta Z_\Delta) \delta_h + 2\delta'_h(Z'_\Delta \tilde{e}) \text{sgn}(T_b^0 - T_b) + o_{\mathbb{P}}(h^{3/2-\kappa}). \quad (\text{A.9})$$

Proof. It follows from Lemma 4.1 in CP (2020a). \square

Lemma A.7. *For any $\epsilon > 0$ there exists a $C > 0$ such that*

$$\liminf_{h \downarrow 0} \mathbb{P} \left[\sup_{K \leq |u| \leq \eta T^{1-\kappa}} Q_h(N_b^0 + u/r_h) - Q_h(N_b^0) < -C \right] \geq 1 - \epsilon,$$

for some large K and small $\eta > 0$.

Proof. Note that on $\{K \leq |u| \leq \eta T^{1-\kappa}\}$ we have $KT^\kappa \leq |T_b - T_b^0| < \eta T$. Suppose $N_b < N_b^0$. Let $\mathbf{T}_{K,\eta} = \{T_b : N\eta > |N_b - N_b^0| > K(T^{1-\kappa})^{-1}\}$. It is enough to show that

$$\mathbb{P} \left(\sup_{T_b \in \mathbf{T}_{K,\eta}} Q_h(N_b^0 + u/r_h) - Q_h(N_b^0) \geq 0 \right) < \epsilon,$$

or using (A.8), $\mathbb{P} \left(\sup_{T_b \in \mathbf{T}_{K,\eta}} h^{-3/2} S_{e,h}(\delta_h, N_b) / |T_b - T_b^0| \geq \inf_{T_b \in \mathbf{T}_{K,\eta}} h^{-3/2} \bar{D}_h(\delta_h, N_b) \right) < \epsilon$. The difficulty is to control the estimates that depend on the difference $|N_b - N_b^0|$. By Lemma A.1,

$$\inf_{T_b \in \mathbf{T}_{K,\eta}} \bar{D}_h(\delta_h, N_b) \geq \inf_{T_b \in \mathbf{T}_{K,\eta}} \delta'_h H' \left(X'_\Delta X_\Delta / |T_b - T_b^0| \right) (X'_2 X_2)^{-1} (X'_0 X_0) H \delta_h,$$

and, since $T\eta > |T_b - T_b^0| \geq KT^\kappa/N$, we need to study the behavior of $X'_\Delta X_\Delta = \sum_{k=T_b+1}^{T_b^0} x_{kh} x'_{kh}$. We shall apply asymptotic results for the local approximation of the covariation between processes. By Theorem 9.3.2 part (i) in Jacod and Protter (2012),

$$\left(h \left(T_b^0 - T_b \right) \right)^{-1} \sum_{k=T_b+1}^{T_b^0} x_{kh} x'_{kh} \xrightarrow{\mathbb{P}} \Sigma_{X_X, N_b^0}, \quad (\text{A.10})$$

since $|N_b - N_b^0|$ shrinks at a rate no faster than $h^{1-\kappa}$ and $(1/h^{1-\kappa}) \rightarrow \infty$. Since $X'_0 X_0$ and $X'_2 X_2$ both involve at least a fixed positive fraction of the data, we can apply a simple law of large numbers for approximate covariation processes [cf. Lemma A.3] to show that $X'_0 X_0$ and $X'_2 X_2$ are each $O_{\mathbb{P}}(1)$. By Lemma A.2, the approximation in (A.10) is uniform, and thus

$$\begin{aligned} & h^{-1/2} \inf_{T_b \in \mathbf{T}_{K,\eta}} \delta'_h H' \left(X'_\Delta X_\Delta / h |T_b - T_b^0| \right) (X'_2 X_2)^{-1} (X'_0 X_0) H \delta_h \\ &= \inf_{T_b \in \mathbf{T}_{K,\eta}} \left(\delta^0 \right)' H' \left(X'_\Delta X_\Delta / h |T_b - T_b^0| \right) (X'_2 X_2)^{-1} (X'_0 X_0) H \delta^0 \end{aligned}$$

is bounded away from zero, in view of Assumption 2.2-(iii). It remains to show that

$$\mathbb{P} \left(\sup_{T_b \in \mathbf{T}_{K,\eta}} h^{-3/2} S_{e,h}(\delta_h, N_b) / (T_b^0 - T_b) \geq C_2 \right) < \epsilon, \quad (\text{A.11})$$

for any $C_2 > 0$. Consider the terms of $S_{e,h}(\delta_h, N_b)$ in (A.5). Using $Z_2 = Z_0 \pm Z_\Delta$, the first term can be expanded as

$$\begin{aligned} \delta'_h (Z'_0 M Z_2) (Z'_2 M Z_2)^{-1} Z_2 M e &= \delta'_h ((Z'_2 \pm Z'_\Delta) M Z_2) (Z'_2 M Z_2)^{-1} Z_2 M e \\ &= \delta'_h Z'_0 M e \pm \delta'_h Z'_\Delta M e \pm \delta'_h (Z'_\Delta M Z_2) (Z'_2 M Z_2)^{-1} Z_2 M e. \end{aligned} \quad (\text{A.12})$$

Given Assumption 2.2-(iii), we apply Lemma A.3 to the estimates that do not involve $|N_b - N_b^0|$. Let us focus on the third term,

$$Z'_\Delta M Z_2 / h (T_b^0 - T_b) = Z'_\Delta Z_2 / h (T_b^0 - T_b) - (Z'_\Delta X_\Delta / (h (T_b^0 - T_b))) (X' X)^{-1} X' Z_2. \quad (\text{A.13})$$

Consider $Z'_\Delta Z_\Delta$ (the argument for $Z'_\Delta X_\Delta$ is analogous). By Lemma A.2, $Z'_\Delta Z_\Delta / h (T_b^0 - T_b)$ uniformly approximates the moving average of $\Sigma_{ZZ,t}$ over the window $(N_b^0 - T^\kappa h, N_b^0]$ for large K and small η . Hence, as $h \downarrow 0$, by using the same argument as in (A.10), the first term in (A.13) is

$$Z'_\Delta Z_\Delta / h (T_b^0 - T_b) = CO_{\mathbb{P}}(1), \quad (\text{A.14})$$

for some $C > 0$, uniformly in T_b . Using Lemma A.3, we deduce that the second term is thus also $O_{\mathbb{P}}(1)$ uniformly. Combining (A.12)-(A.14), we have

$$\begin{aligned} h^{-1/2} (\delta_h)' (Z'_\Delta M Z_2 / h (T_b^0 - T_b)) (Z'_2 M Z_2)^{-1} Z_2 M e & \quad (\text{A.15}) \\ \leq h^{-1/4} \frac{Z'_\Delta M Z_2}{h (T_b^0 - T_b)} O_{\mathbb{P}}(1) O_{\mathbb{P}}(h^{(5/4-\kappa) \wedge 1/2}) &= O_{\mathbb{P}}(h^{1/4}), \end{aligned}$$

where $(Z'_2 M Z_2)^{-1} = O_{\mathbb{P}}(1)$ and the term $O_{\mathbb{P}}(h^{(5/4-\kappa) \wedge 1/2})$ term follows from equation (S.33) in CP (2020a). So the right-and side of (A.15) is less than $\varepsilon/4$ in probability for large T . The second term of (A.12) can be dealt with as in equation (S.41) in CP (2020a). We deduce that the second term of (A.12) is such that

$$\begin{aligned} K^{-1} h^{-(1-\kappa)-1/2} \delta'_h Z'_\Delta M e &= \frac{h^{-1/2}}{K h^{1-\kappa}} \delta'_h \sum_{k=T_b+1}^{T_b^0} z_{kh} e_{kh} - \frac{h^{-1/2}}{h^{1-\kappa}} \delta'_h \left(\sum_{k=T_b+1}^{T_b^0} z_{kh} x'_{kh} \right) (X' X)^{-1} (X' e) \\ &\leq \frac{h^{-1/2}}{K h^{1-\kappa}} \delta'_h \sum_{k=T_b+1}^{T_b^0} z_{kh} e_{kh} - C \frac{1}{K} \frac{h^{-1/4}}{h^{1-\kappa}} (\delta^0) \left(\sum_{k=T_b+1}^{T_b^0} z_{kh} x'_{kh} \right) (X' X)^{-1} (X' e) \\ &\leq \frac{h^{-1/2}}{K h^{1-\kappa}} \delta'_h \sum_{k=T_b+1}^{T_b^0} z_{kh} e_{kh} - h^{-1/4} O_{\mathbb{P}}(1) O_{\mathbb{P}}(h^{1/2}). \end{aligned} \quad (\text{A.16})$$

Thus, using (A.12), (A.5) can be written as

$$\begin{aligned} \frac{h^{-1/2}}{K h^{1-\kappa}} 2\delta'_h Z'_0 M e \pm \frac{h^{-1/2}}{K h^{1-\kappa}} 2\delta'_h Z'_\Delta M e \pm \frac{h^{-1/2}}{K h^{1-\kappa}} 2\delta'_h (Z'_\Delta M Z_2) (Z'_2 M Z_2)^{-1} Z_2 M e &- 2 \frac{h^{-1/2}}{K h^{1-\kappa}} \delta'_h (Z'_0 M e) \\ \leq \frac{h^{-1/2}}{K h^{1-\kappa}} (\delta^0)' \sum_{k=T_b+1}^{T_b^0} z_{kh} \tilde{e}_{kh} \pm O_{\mathbb{P}}(h^{1/4}), & \end{aligned}$$

in view of (A.15) and (A.16). It remains to consider (A.6). We can use the results in CP (2020a). In particular, following the steps from their equation (S.42)-(S.43) to their displayed equation before (S.44), yields that (A.6) is stochastically small uniformly in $T_b \in \mathbf{T}_{K,\eta}$ as $h \downarrow 0$. Combining all the results the

relationship in (A.11) holds, concludes the proof. \square

A.2 Proofs of the Results of Section 4

The expression for G_h , \bar{Q}_h and $Q_{0,h}$ are given by, respectively, $G_h(N_b) = S_{e,h}(\delta^0, N_b)$, $\bar{Q}_h(N_b) = D_h(N_b) + S_{e,h}(N_b)$ and $Q_{0,h}(N_b) = D_h(N_b)$.

A.2.1 Proof of Lemma 4.1

Proof. We show that for any $\eta > 0$, there exists a $C_2 > 0$ such that

$$Q_h(\delta_h, N_b) - Q_h(\delta_h, N_b^0) < -C_2, \quad (\text{A.17})$$

for every N_b that satisfies $|N_b - N_b^0| \geq \eta$. Recall the decomposition in (A.8), then (A.17) can be proved by showing that for any $\epsilon_2 > 0$,

$$\mathbb{P} \left(\sup_{|N_b - N_b^0| \geq \eta} S_{e,h}(\delta_h, N_b) \geq \inf_{|N_b - N_b^0| \geq \eta} |T_b - T_b^0| \bar{D}_h(\delta_h, N_b) \right) < \epsilon_2. \quad (\text{A.18})$$

Suppose that $N_b < N_b^0$. The case with $N_b \geq N_b^0$ can be proved similarly. By definition, note that $|N_b - N_b^0| \geq \eta$ is equivalent to $|T_b - T_b^0| \geq T\eta$. Then,

$$\begin{aligned} & \mathbb{P} \left(\sup_{|N_b - N_b^0| \geq \eta} h^{-1/2} |S_{e,h}(\delta_h, N_b)| \geq \inf_{|N_b - N_b^0| \geq \eta} h^{-1/2} |T_b - T_b^0| \bar{D}_h(\delta_h, N_b) \right) \\ & \leq \mathbb{P} \left(\bar{D}_{\eta,h}^{-1} \sup_{|T_b - T_b^0| \geq T\eta} h^{-1/2} |S_{e,h}(\delta_h, N_b)| \geq \eta \right), \end{aligned}$$

where $D_{\eta,h} = T \inf_{|T_b - T_b^0| \geq T\eta} h^{-1/2} \bar{D}_h(N_b)$. Lemma A.8 below shows that $D_{\eta,h}$ is positive and bounded away from zero. Thus, it is sufficient to show that $\sup_{|T_b - T_b^0| \geq T\eta} h^{-1/2} |S_{e,h}(\delta_h, N_b)| = o_{\mathbb{P}}(1)$. The latter result was shown to hold by CP (2020a) [cf. (S.31)]. Thus,

$$\mathbb{P} \left(\bar{D}_{\eta,h}^{-1} \sup_{|T_b - T_b^0| \geq T\eta} h^{-1/2} |S_{e,h}(\delta_h, N_b)| \geq \eta \right) < \epsilon_2,$$

which implies (A.18) and concludes the proof. \square

Lemma A.8. *For any $B > 0$, let $D_{B,h} = \inf_{|T_b - T_b^0| > TB} Th^{-1/2} \bar{D}_h(\delta_h, N_b)$. There exists an $A > 0$ such that for every $\epsilon > 0$, there exists a $B < \infty$ such that $P(D_{B,h} \geq A) \leq 1 - \epsilon$. That is, $D_{B,h}$ is positive and bounded away from zero with high probability.*

Proof. It follows from Lemma S.A.8 in CP (2020a). \square

A.2.2 Proof of Theorem 4.1

We shall first prove a number of preliminary lemmas. We need the following additional notation. Recall that $l \in \mathbf{L}$ and thus there exist some real number B sufficiently large and some a sufficiently small such

that

$$\inf_{|u|>B} l(u) - \sup_{|u|\leq B^a} l(u) \geq 0. \quad (\text{A.19})$$

Let $\zeta_h(u) = \exp(\gamma_h \tilde{G}_h(u) - \Lambda^0(u))$, $\Gamma_M = \{u \in \mathbb{R} : M \leq |u| < M+1\} \cap \Gamma_h$ and set

$$J_{1,M} \triangleq \int_{\Gamma_M} \zeta_h(u) \pi_h(u) du, \quad J_2 \triangleq \int_{\Gamma_h} \zeta_h(u) \pi_h(u) du. \quad (\text{A.20})$$

The proof involves showing the weak convergence toward the Gaussian process $\mathcal{V}(u) = \mathcal{W}(u) - \Lambda^0(u)$ where

$$\mathcal{W}(u) = \begin{cases} 2(\delta^0)' \mathcal{W}_1(u), & u \leq 0 \\ 2(\delta^0)' \mathcal{W}_2(u), & u > 0 \end{cases}, \quad \Lambda^0(u) = \begin{cases} |u|(\delta^0)' \Sigma_{Z,1} \delta^0, & u \leq 0 \\ u(\delta^0)' \Sigma_{Z,2} \delta^0, & u > 0 \end{cases}.$$

Finally, we introduce the following class of functions. The function $f_h : \mathbb{R} \rightarrow \mathbb{R}$ is said to belong to the family \mathbf{F} if it possesses the following properties: (1) For a fixed h , $f_h(x) \uparrow \infty$ monotonically as $x \uparrow \infty$ and $x \in [0, \infty)$; (2) For any $b < \infty$, $x^b \exp(-f_h(x)) \rightarrow 0$ as $h \downarrow 0$, and $x \rightarrow \infty$.

For $v \leq 0$ define $Q_{0,h}^*(v) \triangleq -(\delta^0)' (\sum_{k=T_b^0+[v/h]}^{T_b^0} z_{kh} z'_{kh}) \delta^0$, and $G_h^*(v) \triangleq 2(\delta^0)' \sum_{k=T_b^0+[v/h]}^{T_b^0} z_{kh} \tilde{e}_{kh}$, where $\tilde{e}_{kh} \sim \mathcal{N}(0, \sigma_{e,(k-1)}^2 h)$. Define analogously $Q_{0,h}^*(v)$ and $G_h^*(v)$ for $v > 0$, so that $\tilde{Q}_h^*(v) = \tilde{Q}_{0,h}^*(v) + \tilde{G}_h^*(v)$. The following lemma follows from Proposition 3.2.

Lemma A.9. *Let $v \in \Gamma^*$. If $v \leq 0$, then $Q_{0,h}^*(v) \xrightarrow{\text{u.c.p.}} [Z, Z]_1$ where $[Z, Z]_1 \triangleq [Z, Z]_h [N_b^0/h] - [Z, Z]_{h[t_v/h]}$, $t_v \triangleq N_b^0 + v$. If $v > 0$, then $Q_{0,h}^*(v) \xrightarrow{\text{u.c.p.}} [Z, Z]_2$ where $[Z, Z]_2 \triangleq [Z, Z]_{h[t_v/h]} - [Z, Z]_{h[N_b^0/h]}$, $t_v \triangleq N_b^0 + v$.*

Lemma A.10. *We have $h^{-1/2} G_h^*(v) \Rightarrow \mathcal{W}(v)$ in $\mathbb{D}_b(\mathbf{C})$, where $\mathbf{C} \subset \mathbb{R}$ is a compact set and $\mathcal{W}(v) = ((\delta^0)' \Omega_1 \delta^0)^{1/2} W_1(-v)$ if $v < 0$ and $\mathcal{W}(v) = ((\delta^0)' \Omega_2 \delta^0)^{1/2} W_2(v)$ if $v > 0$.*

Proof. This result also follows from Proposition 3.2 adapted to the case of stationary regimes. Hence, $h^{-1/2} G_h^*(v) \Rightarrow \mathcal{W}(v)$ in $\mathbb{D}_b(\mathbf{C})$. \square

Lemma A.11. *On $\{u \leq K\}$, $\gamma_h \tilde{Q}_h(u) \stackrel{d}{=} \tilde{Q}_h^*(v) \Rightarrow \mathcal{V}(v)$, where “ $\stackrel{d}{=}$ ” signifies equivalence in distribution and $v \in \Gamma^* = (-\vartheta N_b^0, \vartheta(N - N_b^0))$. On the other hand, $\exp(\mathcal{V}(v)) \xrightarrow{\mathbb{P}} 0$ for all $v \notin \Gamma^*$.*

Proof. Assume $u \leq 0$ (i.e., $N_b \leq N_b^0$). We now employ a change of time scale as in CP (2020a). On the old time scale, given $\{|u| \leq K\}$, $N_b(u)$ varies on the time interval $[N_b^0 - Kh^{1-\kappa}, N_b^0 + Kh^{1-\kappa}]$ for some $K < \infty$. Lemma A.6 shows that the asymptotic behavior of $Q_T(T_b(u)) - Q_T(T_b^0)$ is determined by $-\delta'_h (Z'_\Delta Z_\Delta) \delta_h \pm 2\delta'_h (Z'_\Delta e)$. Next, observe that scaling the criterion function $Q_T(T_b(u)) - Q_T(T_b^0)$ by ψ_h^{-1} has the effect of changing the time scale $s \mapsto t \triangleq \psi_h^{-1} s$. That is, recall the processes Z_s and \tilde{e}_s as defined in (2.3) and (3.1), respectively, and let $Z_{\psi,s} \triangleq \psi_h^{-1/2} Z_s$, $W_{\psi,e,s} \triangleq \psi_h^{-1/2} W_{e,s}$. Then, for $s \in [N_b^0 - Kh^{1-\kappa}, N_b^0 + Kh^{1-\kappa}]$,

$$Z_{\psi,s} = \psi_h^{-1/2} \sigma_{Z,s} dW_{Z,s}, \quad W_{\psi,e,s} = \psi_h^{-1/2} \sigma_{e,s} dW_{e,s}. \quad (\text{A.21})$$

With $s \in [N_b^0 - Kh^{1-\kappa}, N_b^0 + Kh^{1-\kappa}]$ and $v = \psi_h^{-1} (N_b^0 - s)$, using the properties of $W_{\cdot,s}$ and the \mathcal{F}_s -measurability of $\sigma_{Z,s}$, $\sigma_{e,s}$, we have

$$Z_{\psi,t} = \sigma_{Z,t} dW_{Z,t}, \quad W_{\psi,e,t} = \sigma_{e,t} dW_{e,t}, \quad t \in [-\vartheta N_b^0, \vartheta(N - N_b^0)]. \quad (\text{A.22})$$

This results in a change of time scale which we apply to $Z'_\Delta Z_\Delta/\psi_h$ and $Z'_\Delta \tilde{e}/\psi_h$ so that

$$Z'_\Delta Z_\Delta/\psi_h = \sum_{k=T_b^0+\lfloor v/h \rfloor}^{T_b^0} z_{\psi,kh} z'_{\psi,kh}, \quad Z'_\Delta \tilde{e}/\psi_h = \sum_{k=T_b^0+\lfloor v/h \rfloor}^{T_b^0} z_{\psi,kh} \tilde{e}_{\psi,kh}, \quad (\text{A.23})$$

where $z_{\psi,kh} = z_{kh}/\sqrt{\psi_h}$ and $\tilde{e}_{\psi,kh} = \tilde{e}_{kh}/\sqrt{\psi_h}$ with $v \in \Gamma^*$. In view of Condition 1, $\gamma_h \asymp h^{-1/2}\psi_h^{-1}$. By Lemma A.6 and (A.23),

$$\left(Q_h(N_b) - Q_h(N_b^0) \right) / \psi_h = -\delta_h(Z'_\Delta Z_\Delta) \delta_h \pm 2\delta'_h(Z'_\Delta \tilde{e}) + o_{\mathbb{P}}(h^{1/2}).$$

or $\gamma_h \bar{Q}_h(N_b) = -(\delta^0)' Z'_\Delta Z_\Delta (\delta^0) \pm 2(\delta^0)' (h^{-1/2} Z'_\Delta \tilde{e}) + o_{\mathbb{P}}(1)$ where $e_{kh} \sim \text{i.n.d. } \mathcal{N}(0, \sigma_{h,k-1}^2 h)$, $\sigma_{h,k} = O(h^{-1/4})\sigma_{e,k}$ and \tilde{e}_{kh} is the normalized error [i.e., $\tilde{e}_{kh} \sim \text{i.n.d. } \mathcal{N}(0, \sigma_{e,k-1}^2 h)$] from (3.1). In summary, $\gamma_h \tilde{Q}_h(u) \stackrel{d}{=} \tilde{Q}_h^*(v)$, while by Lemma A.9-A.10, $\tilde{Q}_h^*(v) \Rightarrow \mathcal{V}(v)$ on Γ^* . For $|u| > K$ and K large enough, we have $\gamma_h \tilde{Q}_h(v) \downarrow -\infty$ in probability upon using the same arguments as in Lemma A.12-A.13 and thus we use the shorthand notation $\exp(\mathcal{V}(u)) \xrightarrow{\mathbb{P}} 0$ for all $v \notin \Gamma^*$. \square

Lemma A.12. *For any $l \in \mathbf{L}$ and any $\epsilon > 0$,*

$$\liminf_{h \downarrow 0} \mathbb{P} \left[\int_{K \leq |u| \leq \eta T^{1-k}} l(s-u) \exp(\gamma_h \tilde{Q}_h(u)) \pi(N_b^0 + u/r_h) < \epsilon \right] \geq 1 - \epsilon, \quad (\text{A.24})$$

for sufficiently large K and small $\eta > 0$.

Proof. Since $l \in \mathbf{L}$, we have $l(s-u) \leq 1 + |s-u|^m$ for $m > 0$. Thus, we shall find a sufficiently large K such that

$$\exp(\gamma_h \tilde{Q}_h(u)) \pi_h(N_b^0 + u/r_h) \xrightarrow{\mathbb{P}} 0, \quad K \leq |u| \leq \eta T^{1-k}, \quad (\text{A.25})$$

as $h \downarrow 0$. By Assumption 4.3, $\pi(\cdot)$ satisfies $\pi(N_b^0) < C_\pi$, for some real number $C_\pi < \infty$. Thus, $\pi(\cdot)$ will play no role in proving (A.25). By Lemma A.7, for large K there exists a $C > 0$ such that

$$\liminf_{h \downarrow 0} \mathbb{P} \left[\sup_{K \leq |u| \leq \eta T^{1-\kappa}} Q_h(\delta_h, N_b) - Q_h(\delta_h, N_b^0) < -C \right] = 1.$$

Consequently, $\gamma_h \tilde{Q}_h(u)$ diverges to $-\infty$ for values of u satisfying $K \leq |u| \leq \eta T^{1-\kappa}$. By the property of the exponential function, we have for some finite $C_1, C_2 > 0$, $\exp(\gamma_h \tilde{Q}_h(u)) = C_1 \exp(-\kappa_\gamma T^{3/2-\kappa} C_2) \xrightarrow{\mathbb{P}} 0$ for all $K \leq |u| \leq \eta T^{1-k}$. Combining all the arguments above, the latter result implies (A.24), which concludes the proof. \square

Lemma A.13. *For every $l \in \mathbf{L}$, and any $\epsilon, \eta > 0$, we have*

$$\liminf_{h \downarrow 0} \mathbb{P} \left[\int_{|u| \geq \eta T^{1-\kappa}} l(s-u) \exp(\gamma_h \tilde{Q}_h(u)) \pi_h(N_b^0 + u/r_h) < \epsilon \right] \geq 1 - \epsilon. \quad (\text{A.26})$$

Proof. Since $l \in \mathbf{L}$, we have $l(s-u) \leq 1 + |s-u|^m$ for $m > 0$. We shall show

$$\int_{|u| \geq \eta T^{1-\kappa}} (1 + |s-u|^m) \left| \exp(\gamma_h \tilde{Q}_h(u)) \pi_h(N_b^0 + u/r_h) \right| du = o_{\mathbb{P}}(1). \quad (\text{A.27})$$

The left-hand side above is no larger than

$$\begin{aligned} & C \int_{|u| \geq \eta T^{1-\kappa}} \left| s - T^{1-\kappa} (N_b - N_b^0) \right|^m \exp \left(\gamma_h \tilde{Q}_h(u) \right) \pi_h \left(N_b^0 + u/r_h \right) du \\ & \leq C_m \left(T^{1-\kappa} \right)^{m+1} \int_{|N_b - N_b^0| \geq \eta/2} \left| N_b - N_b^0 \right|^m \exp \left(\gamma_h \tilde{Q}_h(u) \right) \pi(N_b) dN_b, \end{aligned}$$

where $C_m < \infty$ may depend on m . By Lemma 4.1 there exists a $c > 0$ such that

$$\liminf_{h \downarrow 0} \mathbb{P} \left(\sup_{|N_b - N_b^0| \geq \eta/2} \exp \left(\gamma_h \tilde{Q}_h(u) \right) \leq \exp(-c\gamma_h) \right) = 1.$$

Then, the left-hand side of (A.27) is bounded by

$$C_m \left(T^{1-\kappa} \right)^{m+1} e^{-c\gamma_h} \int_{[0, N]} N_b^m \pi(N_b) dN_b = o_{\mathbb{P}}(1),$$

since $\exp(-c\gamma_h) \rightarrow 0$ at a rate faster than $(T^{1-\kappa})^{m+1} \rightarrow \infty$. This concludes the proof. \square

Lemma A.14. *Let $u_1, u_2 \in \mathbb{R}$ being of the same sign and satisfying $0 < |u_1| < |u_2| < K < \infty$. For any integer $r > 0$ and some constants c_r and C_r which depend on r only, we have*

$$\mathbb{E} \left[\left(\zeta_h^{1/2r}(u_2) - \zeta_h^{1/2r}(u_1) \right)^{2r} \right] \leq c_r \left| \left(\delta^0 \right)' (|u_2 - u_1| \Sigma_i) \delta^0 \right|^r \leq C_r |u_2 - u_1|^r,$$

where $i = 1$ if $u_1 < 0$ and $i = 2$ if $u_1 > 0$.

Proof. The proof is given only for the case $u_2 > u_1 > 0$. We follow Lemma III.5.2 in Ibragimov and Has'minskiĭ (1981). Let $\mathcal{V}(u_i) = \exp(\mathcal{V}(u_i))$ for $i = 1, 2$ where $\mathcal{V}(u_i) = \mathcal{W}(u_i) - \Lambda^0(u_i)$ and $\mathcal{V}_{u_1}(u_2) \triangleq \exp(\mathcal{V}(u_2) - \mathcal{V}(u_1))$. We have

$$\mathbb{E} \left[\left(\mathcal{V}^{1/2r}(u_2) - \mathcal{V}^{1/2r}(u_1) \right)^{2r} \right] = \sum_{j=0}^{2r} \binom{2r}{j} (-1)^j \mathbb{E}_{u_1} \left[\mathcal{V}_{u_1}^{j/2r}(u_2) \right].$$

For any given $u \in \mathbb{R}$, $\mathcal{V}(u)$ is the exponential of a Gaussian random variable, and thus

$$\mathbb{E}_{u_1} \left[\mathcal{V}_{u_1}^{j/2r}(u_2) \right] = \exp \left(\frac{1}{2} \left(\frac{j}{2r} \right)^2 4 \left(\delta^0 \right)' (|u_2 - u_1| \Sigma_2) \delta^0 - \frac{j}{2r} \left| \Lambda^0(u_2) - \Lambda^0(u_1) \right| \right). \quad (\text{A.28})$$

Letting $d \triangleq \exp \left(j(2r)^{-1} 2 \left(\delta^0 \right)' (|u_2 - u_1| \Sigma_2) \delta^0 - \left| \Lambda^0(u_2) - \Lambda^0(u_1) \right| \right)$, we have

$$\mathbb{E} \left[\left(\mathcal{V}^{1/2r}(u_2) - \mathcal{V}^{1/2r}(u_1) \right)^{2m} \right] = \sum_{j=0}^{2r} \binom{2r}{j} (-1)^j d^{j/2r}.$$

We need to study three different cases. Let $\varpi \triangleq 2 \left(\delta^0 \right)' (|u_2 - u_1| \Sigma_2) \delta^0 - \left| \Lambda^0(u_2) - \Lambda^0(u_1) \right|$.

(1) $\varpi < 0$. Note that

$$\begin{aligned} d &= \exp \left(\frac{j}{r} \left(\delta^0 \right)' (|u_2 - u_1| \Sigma_2) \delta^0 - 2 \left| \left(\delta^0 \right)' (|u_2 - u_1| \Sigma_2) \delta^0 \right| + \varpi \right) \\ &= \exp \left(-\frac{2r-j}{r} \left(\delta^0 \right)' (|u_2 - u_1| \Sigma_2) \delta^0 \right) e^{\varpi}, \end{aligned}$$

which then yields

$$\mathbb{E} \left[\left(\mathcal{V}^{1/2r} (u_2) - \mathcal{V}^{1/2r} (u_1) \right)^{2r} \right] \leq p_r (a), \quad (\text{A.29})$$

where $p_r (a) \triangleq \sum_{j=0}^{2r} \binom{2r}{j} (-1)^j a^{(2r-j)}$ and $a = \exp \left(-r^{-1} (\delta^0)' (|u_2 - u_1| \Sigma_2) \delta^0 \right)$.

(2) $2 (\delta^0)' (|u_2 - u_1| \Sigma_2) \delta^0 = |\Lambda^0 (u_2) - \Lambda^0 (u_1)|$. This case is the same as the previous one but with $a = \exp \left(-r^{-1} (\delta^0)' (|u_2 - u_1| \Sigma_2) \delta^0 \right)$.

(3) $\varpi > 0$. Upon simple manipulations, $d = \exp \left(- (2r - j) r^{-1} (\delta^0)' (|u_2 - u_1| \Sigma_2) \delta^0 \right) e^\varpi$. Then

$$\mathbb{E} \left[\left(\mathcal{V}^{1/2r} (u_2) - \mathcal{V}^{1/2r} (u_1) \right)^{2r} \right] \leq e^{\varpi/2r} \sum_{j=0}^{2r} \binom{2r}{j} (-1)^j a^{(2r-j)},$$

with $a = \exp \left(-r^{-1} (\delta^0)' (|u_2 - u_1| \Sigma_2) \delta^0 \right)$.

We focus on the first case only. It is not difficult to see that the same proof is valid for the other cases. It is enough to show that at the point $a = 1$, the polynomial $p_r (a)$ admits a root of multiplicity r . This follows from the equalities $p_r (1) = p_r^{(1)} (1) = \dots = p_r^{(r-1)} (1) = 0$. Then, note that $p_r^{(i)} (a)$ is a linear combination of summations \mathcal{S}_k ($k = 0, 1, \dots, 2i$) given by $\mathcal{S}_k = \sum_{j=0}^{2r} \binom{2r}{j} j^k$. Thus, all one needs to verify is that that $\mathcal{S}_k = 0$ for $k = 0, 1, \dots, 2r - 2$. \mathcal{S}_k can be found by applying the k -fold of the operator $a (d/da)$ to the function $(1 - a^2)^{2r}$ and evaluating it at $a = 1$. Consequently, $\mathcal{S}_k = 0$ for $k = 0, 1, \dots, 2r - 1$. Using this into equation (A.29),

$$\mathbb{E} \left[\left(\mathcal{V}^{1/2r} (u_2) - \mathcal{V}^{1/2r} (u_1) \right)^{2r} \right] \leq (1 - a)^r \tilde{p}_r (a) \leq \left(r^{-1} (\delta^0)' (|u_2 - u_1| \Sigma_2) \delta^0 \right)^r \tilde{p}_r (a), \quad (\text{A.30})$$

where $\tilde{p}_r (a)$ is a polynomial of degree $r^2 - r$, and the last inequality follows from $1 - e^{-c} \leq c$, for $c > 0$. Now, write $\bar{\zeta}_h^{1/2r} (u_2, u_1) = \zeta_h^{1/2r} (u_2) - \zeta_h^{1/2r} (u_1)$. By Lemma A.10, the continuous mapping theorem and (A.30), we have $\lim_{h \rightarrow 0} \mathbb{E} [(\bar{\zeta}_h^{1/2r} (u_2, u_1))^{2r}] \leq (1 - a)^r \tilde{p}_r (a)$. Since $j \leq 2r$, we set $c_r = \max_{0 \leq a \leq 1} \tilde{p}_r (a) / r^r < \infty$ and the claim of the lemma follows since $\|\delta^0\|, \Sigma_i < \infty$. \square

Lemma A.15. *Let $0 < C < \infty$ and $u_1, u_2 \in \mathbb{R}$ being of the same sign satisfying $0 < |u_1| < |u_2| < K < \infty$. Then, for all h sufficiently small, we have*

$$\mathbb{E} \left[\left(\zeta_h^{1/4} (u_2) - \zeta_h^{1/4} (u_1) \right)^4 \right] \leq C_2 |u_2 - u_1|^2. \quad (\text{A.31})$$

Furthermore, for some constant C_1 as in Lemma A.14 we have

$$\mathbb{P} [\zeta_h (u) > \exp (-3C_1 |u| / 2)] \leq \exp (-C_1 |u| / 4). \quad (\text{A.32})$$

Proof. Use Lemma A.14 with $r = 2$ to verify (A.31). For (A.32), assume $u > 0$. By Markov's inequality and Lemma A.14,

$$\begin{aligned} \mathbb{P} [\zeta_h (u) > \exp (-3C_1 |u| / 2)] &\leq \exp (3C_1 |u| / 4) \mathbb{E} \left[\zeta_h^{1/2} (u) \right] \\ &\leq \exp \left(\frac{3C_1 |u|}{4} - (\delta^0)' (|u| \Sigma_2) \delta^0 \right) \leq \exp \left(-\frac{C_1 |u|}{4} \right). \square \end{aligned}$$

Lemma A.16. *Under the conditions of Lemma A.15, for any $\varpi > 0$ there exists a finite real number c_ϖ and a \bar{h} such that for all $h < \bar{h}$, $\mathbb{P} [\sup_{|u| > M} \zeta_h (u) > M^{-\varpi}] \leq c_\varpi M^{-\varpi}$.*

Proof. It can be easily shown to follow from the previous lemma. \square

Lemma A.17. *For a small $\epsilon \leq \bar{\epsilon}$, where $\bar{\epsilon}$ may depend on $\pi(\cdot)$, there exists $0 < C < \infty$ such that*

$$\mathbb{P} \left[\int_0^\epsilon \zeta_h(u) \pi_h(u) du < \epsilon \pi^0 \right] < C \epsilon^{1/2}. \quad (\text{A.33})$$

Proof. Since $\mathbb{E}(\zeta_h(0)) = 1$ while $\mathbb{E}(\zeta_h(u)) \leq 1$ for sufficiently small $h > 0$,

$$\mathbb{E} |\zeta_h(u) - \zeta_h(0)| \leq \left(\mathbb{E} \left(\zeta_h^{1/2}(u) + \zeta_h^{1/2}(0) \right)^2 \mathbb{E} \left(\left(\zeta_h^{1/2}(u) - \zeta_h^{1/2}(0) \right)^2 \right) \right)^{1/2} \leq C |u|^{1/2}, \quad (\text{A.34})$$

where we used Lemma A.14 with $r = 1$. By Assumption 4.3, $|\pi_h(u) - \pi^0| \leq B(T^{1-\kappa})^{-1} |u|$, with $B > 0$. Given that $|u| < \bar{\epsilon}$, it holds that $\pi^0/2 < \pi_h(u)$. Thus, for a small $\bar{\epsilon} > 0$, $\int_0^\epsilon \zeta_h(u) \pi_h(u) du > 2^{-1} \pi^0 \int_0^\epsilon \zeta_h(u) du$. We can then use $\zeta_h(0) = 1$ to yield,

$$\mathbb{P} \left[\int_0^\epsilon \zeta_h(u) \pi_h(u) du < 2^{-1} \epsilon \right] \leq \mathbb{P} \left[\int_0^\epsilon (\zeta_h(u) - \zeta_h(0)) du < -2^{-1} \epsilon \right] \leq \mathbb{P} \left[\int_0^\epsilon |\zeta_h(u) - \zeta_h(0)| du > 2^{-1} \epsilon \right].$$

By Markov's inequality together with inequality (A.34), the right-hand side above is less than or equal to $2\epsilon^{-1} \int_0^\epsilon \mathbb{E} |\zeta_h(u) - \zeta_h(0)| du < 2C\epsilon^{1/2}$. \square

Lemma A.18. *For $f_h \in \mathbf{F}$, and M sufficiently large, there exist constants $c, C > 0$ such that*

$$\mathbb{P} [J_{1,M} > \exp(-cf_h(M))] \leq C (1 + M^C) \exp(-cf_h(M)). \quad (\text{A.35})$$

Proof. Since $u \in \Gamma_h$, and given the differentiability of $\pi(\cdot)$, we consider for simplicity the case of the uniform prior (i.e., $\pi_h(u) = 1$ for all u). We divide the open interval $\{u : M \leq |u| < M + 1\}$ into I disjoint segments and denote the i -th segment by Π_i . Choose a point $u_i \in \Pi_i$ and define $J_{1,M}^\Pi \triangleq \sum_{i \in I} \zeta_h(u_i) \text{Leb}(\Pi_i) = \sum_{i \in I} \int_{\Pi_i} \zeta_h(u) du$. Then,

$$\begin{aligned} \mathbb{P} \left[J_{1,M}^\Pi > 4^{-1} \exp(-cf_T(M)) \right] &\leq \mathbb{P} \left[\max_{i \in I} \zeta_h^{1/2}(u_i) (\text{Leb}(\Gamma_M))^{1/2} > (1/2) \exp(-f_h(M)/2) \right] \\ &\leq \sum_{i \in I} \mathbb{P} \left[\zeta_h^{1/2}(u_i) > (1/2) (\text{Leb}(\Gamma_M))^{-1/2} \exp(-f_h(M)/2) \right] \\ &\leq 2I (\text{Leb}(\Gamma_M))^{1/2} \exp(-f_h(M)/12), \end{aligned} \quad (\text{A.36})$$

where the last inequality follows from applying the second part of Lemma A.15 to each summand. For a sufficiently large M , $\exp(-f_h(M)/12) < 1/2$ and thus,

$$\mathbb{P} [J_{1,M} > \exp(-f_h(M)/2)] \leq \mathbb{P} \left[|J_{1,M} - J_{1,M}^\Pi| > 2^{-1} \exp(-f_h(M)/2) \right] + \mathbb{P} \left[J_{1,M}^\Pi > \exp(-f_h(M)) \right]. \quad (\text{A.37})$$

Let us focus on the first term:

$$\begin{aligned} \mathbb{E} \left[J_{1,M} - J_{1,M}^\Pi \right] &\leq \sum_{i \in I} \int_{\Pi_i} \mathbb{E} \left| \zeta_h^{1/2}(u) - \zeta_h^{1/2}(u_i) \right| du \\ &\leq \sum_{i \in I} \int_{\Pi_i} \left(\mathbb{E} \left| \zeta_h^{1/2}(u) + \zeta_h^{1/2}(u_i) \right| \mathbb{E} \left| \zeta_h^{1/2}(u) - \zeta_h^{1/2}(u_i) \right| \right)^{1/2} du \\ &\leq C (1 + M)^C \sum_{i \in I} \int_{\Pi_i} |u_i - u|^{1/2} du, \end{aligned}$$

where the last inequality uses Lemma A.14 and the fact that $u < M + 1$. Note that each summand on the right-hand side above is less than $C (MI^{-1})^{3/2}$. Thus, we can find numbers C_1 and C_2 such that

$$\mathbb{E} \left[J_{1,M} - J_{1,M}^{\Pi} \right] \leq C_1 \left(1 + M^{C_2} \right) I^{-1/2}. \quad (\text{A.38})$$

Using (A.36) and (A.38) into (A.37), we have

$$\mathbb{P} [J_{1,M} > \exp(-f_h(M)/2)] \leq C_1 \left(1 + M^{C_2} \right) I^{-1/2} + 2I (\text{Leb}(\Gamma_M))^{1/2} \exp(-f_h(M)/12).$$

The claim of the lemma follows choosing a I that satisfies $1 \leq I^{3/2} \exp(-f_h(M)/12) \leq 2$. \square

Lemma A.19. *For $f_h \in \mathbf{F}$, and M sufficiently large, there exist constants $c, C > 0$ such that*

$$\mathbb{E} [J_{1,M}/J_2] \leq C \left(1 + M^C \right) \exp(-cf_h(M)). \quad (\text{A.39})$$

Proof. Note that $J_{1,M}/J_2 \leq 1$ and thus, for an arbitrary $\epsilon > 0$,

$$\mathbb{E} [J_{1,M}/J_2] \leq \mathbb{P} [J_{1,M} > \exp(-cf_h(M)/2)] + 4\epsilon^{-1} \exp(-cf_h(M)) + \mathbb{P} \left[\int_{\Gamma_h} \zeta_h(u) du < \epsilon/4 \right]$$

where we used Lemma A.18 for the first term. In view of the relationship in (A.33), $\mathbb{P}[\int_{\Gamma_h} \zeta_h(u) du < \epsilon/4] \leq C\epsilon^{1/2}$. To conclude the proof, choose $\epsilon = \exp((-2c/3)f_h(M))$. \square

Proof of Theorem 4.1. Let $p_{1,h}(u) \triangleq \tilde{p}_h(u)/\bar{p}_h$, where $\tilde{p}_h(u) = \exp(\gamma_h(\tilde{G}_h(u) + \tilde{Q}_{0,h}(u)))$ and $\bar{p}_h \triangleq \int_{\Gamma_h} p_{1,h}(w) dw$. By definition, $\hat{N}_b^{\text{GL}} = N\hat{\lambda}_b^{\text{GL}}$ minimizes

$$\int_{\Gamma^0} l(r_h(s-u)) \exp\left(\gamma_h\left(\tilde{G}_h(u) + \tilde{Q}_{0,h}(u)\right)\right) \pi(u) du, \quad s \in \Gamma^0.$$

Changing variables and using simple manipulations, the expression above is equal to

$$r_h^{-1} \bar{p}_h \int_{\Gamma_h} l\left(r_h\left(s - N_b^0\right) - u\right) p_{1,h}(u) \pi_h(u) du, \quad (\text{A.40})$$

from which it follows that $\xi_{l,h} \triangleq \vartheta^{-1} T^{1-\kappa} (\hat{\lambda}_b^{\text{GL}} - \lambda_0)$ is the minimum of the function

$$\Psi_{l,h}(s) \triangleq \int_{\Gamma_h} l(s-u) \frac{\tilde{p}_h(u) \pi(N_b^0 + u/r_h)}{\int_{\Gamma_h} \tilde{p}_h(w) \pi(N_b^0 + w/r_h) dw} du. \quad (\text{A.41})$$

Lemma A.11 shows that, under Condition 1, the normalizing factor γ_h acts as a change of time scale $s \mapsto \psi_h^{-1}s$ since $r_h \asymp \psi_h^{-1}$, where $a_h \asymp b_h$ signifies $b_h/c \leq a_h \leq cb_h$ for some constant c . The change of time scale then implies that the sample criterion $\bar{Q}_h(N_b)$ is evaluated at $N_b = N_b^0 + \vartheta v$. It also suggests the following change of variable through the substitution $a = \vartheta\psi_h u$,

$$\Psi_{l,h}(s) = \int_{\Gamma^*} l\left((\vartheta\psi_h)^{-1}(\vartheta\psi_h s - a)\right) \frac{\exp\left(\tilde{G}_h^*(a) + \tilde{Q}_{0,h}^*(a)\right) \pi(N_b^0 + a)}{\int_{\Gamma^*} \exp\left(\tilde{G}_h^*(w) + \tilde{Q}_{0,h}^*(w)\right) \pi(N_b^0 + w) dw} da, \quad (\text{A.42})$$

where $\Gamma^* = (-\vartheta N_b^0, \vartheta(N - N_b^0))$ and the Quasi-prior is defined on the ‘‘fast time scale’’. This implies that $\Psi_{l,h}^*(s) \triangleq \Psi_{l,h}(\vartheta\psi_h s)$ is minimized by $\xi_{l,h} \triangleq Th(\hat{\lambda}_b^{\text{GL}} - \lambda_0)$. The next step involves showing the finite-dimensional convergence in distribution of the function $\Psi_{l,h}^*(s)$ to $\Psi_l^*(s)$ and the tightness of the sequence of functions $(\Psi_{l,h}^*(s))$ on the space $\mathbb{C}_b(\Gamma^*)$. This gives the weak convergence $\Psi_{l,h}^* \Rightarrow \Psi_l^*$ on

$\mathbb{C}_b(\Gamma^*)$. The final part of the proof requires showing that (a) any element of the set of minimizers of the sample criterion is stochastically bounded, and (b) the length of such set converges to zero as $h \downarrow 0$. Note that we can deduce certain tail properties of $\Psi_{l,h}^*$ from that of $\Psi_{l,h}$ on $\{u \in \mathbb{R}, |u| \leq K\}$ as $K \rightarrow \infty$. Thus, we shall sometimes study the behavior of $\Psi_{l,h}$ below. Given the boundedness of $\pi(\cdot)$ [cf. Assumption 4.3], Lemma A.18-A.19 imply that, for any $\nu > 0$, we can find a \bar{h} such that for all $h < \bar{h}$,

$$\mathbb{E} \left[\int_{\Gamma_M} \frac{\exp\left(\gamma_h\left(\tilde{G}_h(u) + \tilde{Q}_{0,h}(u)\right)\right)}{\int_{\Gamma_h} \exp\left(\gamma_h\left(\tilde{G}_h(u) + \tilde{Q}_{0,h}(u)\right)\right) dw} du \right] \leq \frac{c_\nu}{M^\nu}. \quad (\text{A.43})$$

Furthermore, by Lemma A.12-A.13, for $K < \infty$,

$$\Psi_{l,h}(s) = \int_{\{u \in \mathbb{R}, |u| \leq K\}} l(s-u) \frac{\tilde{p}_h(u)}{\int_{\{u \in \mathbb{R}, |u| \leq K\}} \tilde{p}_h(w) dw} du + o_{\mathbb{P}}(1).$$

The last two relationships extend to $\Psi_{l,h}^*(s)$ and in particular (A.43) shows the tail behavior of $\Psi_{l,h}(s)$. On the region $\{|u| \leq K\}$, the change of time scale $u \mapsto \psi_h u = v$ implies that we need to analyze the behavior of $\Psi_{l,h}^*(s)$. Thus, we show the convergence of the marginal distributions of the function $\Psi_{l,h}^*(s)$ to the marginals of $\Psi_l^*(s)$ defined in Definition 4.1. Choose a finite integer n , and arbitrary real numbers a_j ($j = 0, \dots, n$). Let $\zeta_h^*(v) = \tilde{G}_h^*(v) + \tilde{Q}_{0,h}^*(v)$. The estimate

$$\sum_{j=1}^n a_j \int_{|v| \leq K} l(s_j - v) \zeta_h^*(v) \pi(N_b^0 + v) dv + a_0 \int_{|v| \leq K} \zeta_h(v) \pi(N_b^0 + v) dv \quad (\text{A.44})$$

converges in distribution to the random variable

$$\sum_{j=1}^n a_j \int_{|v| \leq K} l(s_j - v) \exp(\mathcal{V}(v)) \pi(N_b^0 + v) dv + a_0 \int_{|v| \leq K} \exp(\mathcal{V}(v)) \pi(N_b^0 + v) dv,$$

where $\mathcal{V}(v) = \mathcal{W}(v) - \Lambda^0(v)$, since by Lemma A.14-A.15 we can apply Theorem I.A.22 in Ibragimov and Has'minskiĭ (1981). We can then use the Cramer-Wold Theorem [cf. Theorem 29.4 in Billingsley (1995)] to establish the convergence in distribution of the vector

$$\int_{|v| \leq K} l(s_i - v) \zeta_h^*(v) \pi(N_b^0 + v) dv, \dots, \int_{|v| \leq K} l(s_n - v) \zeta_h^*(v) \pi(N_b^0 + v) dv, \int_{|v| \leq K} \zeta_h^*(v) \pi(N_b^0 + v) dv,$$

to the marginal distributions of the vector

$$\int_{|v| \leq K} l(s_i - v) \exp(\mathcal{V}(v)) \pi(N_b^0 + v) dv, \dots, \int_{|v| \leq K} l(s_n - v) \exp(\mathcal{V}(v)) \pi(N_b^0 + v) dv, \\ \int_{|v| \leq K} \exp(\mathcal{V}(v)) \pi(N_b^0 + v) dv.$$

Note that the above integrals are equal to zero if $v \notin \Gamma^*$, because in the latter result there is an intermediate step involving the change of time scale. That is, $v \notin \Gamma^*$ corresponds to tail behavior as described in (A.43) and the equation right below it. Then, for any $K_1, K_2 < \infty$, the marginal distributions of

$$\frac{\int_{|v| \leq K_1} l(s-v) \exp\left(\tilde{G}_h^*(v) + \tilde{Q}_{0,h}^*(v)\right) \pi(N_b^0 + v) dv}{\int_{|w| \leq K_2} \exp\left(\tilde{G}_h^*(w) + \tilde{Q}_{0,h}^*(w)\right) \pi(N_b^0 + w) dw},$$

converge to the marginals of $\int_{|v| \leq K_1} l(s-v) \pi(N_b^0 + v) \exp(\mathcal{V}(v)) / \left(\int_{|w| \leq K_2} \exp(\mathcal{V}(w)) \pi(N_b^0 + w) dw \right) dv$. By the same reasoning, the marginal distributions of

$$\int_{M \leq |v| < M+1} \frac{\exp(\tilde{G}_h^*(v) + \tilde{Q}_{0,h}^*(v))}{\int_{|w| \leq K_2} \exp(\tilde{G}_h^*(w) + \tilde{Q}_{0,h}^*(w)) dw} dv,$$

converge to the distribution of $\int_{M \leq |v| < M+1} (\exp(\mathcal{V}(v)) \pi(N_b^0 + v)) / \int_{|w| \leq K_2} \exp(\mathcal{V}(w)) \pi(N_b^0 + w) dw dv$. In view of (A.43) which gives a bound on the tail behavior of the mean of the Quasi-posterior on the original time scale, we have

$$\Psi_{l,h}^*(s) = \int_{\Gamma^*} l(s-v) \frac{\exp(\mathcal{V}(v)) \pi(N_b^0 + v) dv}{\int_{\Gamma^*} \exp(\mathcal{V}(w)) \pi(N_b^0 + w) dw} + o_{\mathbb{P}}(1). \quad (\text{A.45})$$

This is sufficient for the convergence of the finite-dimensional distributions of $\Psi_{l,h}^*(s)$ to those of $\Psi_l^*(s)$. We now turn to the tightness of the sequence of functions $\{\Psi_{l,h}^*(s)\}$. We shall show that the probability distributions in $\mathbb{C}([-K, K])$ (i.e., the space of continuous functions on $[-K, K]$) generated by the contractions of the functions $\Psi_{l,h}^*(s)$ on $s \in [-K, K]$ are tight. For any $l \in \mathbf{L}$, we have the inequality $l(u) \leq 2^r(1 + |u|^2)^r$, for some r . Let

$$H_K(\varpi) \triangleq \int_{\mathbb{R}} \sup_{|s| \leq K, |y| \leq \varpi} |l(s+y-u) - l(s-u)| (1 + |u|^2)^{-r-1} du.$$

We show that $\lim_{\varpi \downarrow 0} H_K(\varpi) = 0$. Note that for any $c > 0$ we can choose a M such that

$$\int_{|u| > M} \sup_{|s| \leq K, |y| \leq \varpi} |l(s+y-u) - l(s-u)| (1 + |u|^2)^{-r-1} du < c.$$

We apply Lusin's theorem [cf. Section 3.3 of Royden and Fitzpatrick (2010)]. Let \bar{L} denote the upper bound of $l(\cdot)$ on the set $\{u \in \mathbb{R} : |u| \leq K + 2M\}$. By the measurability of $l(\cdot)$, we can find a continuous function $J(u)$ in the interval $\{u : |u| \leq K + 2M\}$ which equals $l(u)$ except on a set whose measure does not exceed $c(2\bar{L})^{-1}$. Denote the modulus of continuity of $J(\cdot)$ by $w_J(\varpi)$. Without loss of generality assume $|J(u)| \leq \bar{L}$ on $\{u : |u| \leq K + 2M\}$. Then,

$$\begin{aligned} & \int_{|u| > M} \sup_{|s| \leq K, |y| \leq \varpi} |l(s+y-u) - l(s-u)| (1 + |u|^2)^{-r-1} du \\ & \leq w_J(\varpi) \int_{\mathbb{R}} (1 + |u|^2)^{-r-1} du + 2\bar{L} \text{Leb}\{u \in \mathbb{R} : |u| \leq K + 2M, l \neq J\}, \end{aligned}$$

and $\bar{L} \leq Cw_J(\varpi) + c$ for some $C < \infty$. Accordingly, $H_K(\varpi) \leq Cw_J(\varpi) + 2c$, with the property that c can be chosen arbitrary small and $w_J(\varpi) \rightarrow 0$ as $\varpi \downarrow 0$ (holding for each fixed c) by definition. Definition 4.1 implies that we can find a number $C < \infty$ such that

$$\begin{aligned} & \mathbb{E} \left[\sup_{|s| \leq K, |y| \leq \varpi} |\Psi_{l,h}^*(s+y) - \Psi_{l,h}^*(s)| \right] \\ & \leq \int_{\mathbb{R}} \sup_{|s| \leq K, |y| \leq \varpi} |l(s+y-u) - l(s-u)| \mathbb{E} \left(\frac{\zeta_h(u) \pi_h(u)}{\int_{\Gamma_h} \zeta_h(w) \pi_h(w) dw} \right) du \leq CH_K(\varpi). \end{aligned}$$

The tightness of $\Psi_{l,h}^*(s)$ on $s \in \Gamma^*$ is established by using Markov's inequality together with the above bound. It remains to study the oscillations of the minimum points of the sample function $\Psi_{l,h}^*$. Consider

an open bounded interval \mathbf{A} satisfying $\mathbb{P}[\xi_l^0 \in b(\mathbf{A})] = 0$, where $b(\mathbf{A})$ denotes the boundary of the set \mathbf{A} . Define the functionals $H_{\mathbf{A}}(\Psi) = \inf_{s \in \mathbf{A}} \Psi_l^*(s)$ and $H_{\mathbf{A}^c}(\Psi) = \inf_{s \in \mathbf{A}^c} \Psi_l^*(s)$, where \mathbf{A}^c is the set complementary to \mathbf{A} . Let \mathbf{M}_h denote the set of absolute minimum points of the function $\Psi_{l,h}^*(s)$. By the definition of $\widehat{\lambda}_b^{\text{GL}}$ we have that $\limsup_{h \downarrow 0} \mathbb{E}[l(T_h(\widehat{\lambda}_b^{\text{GL}} - \lambda_0))] < \infty$. This implies that any minimum point $\xi \in \mathbf{M}_h$ of the sample criterion function is uniformly stochastically bounded, i.e.,

$$\lim_{K \rightarrow \infty} \mathbb{P}[\mathbf{M}_h \not\subseteq \{s : |s| \leq K\}] = 0. \quad (\text{A.46})$$

Next, note that

$$\begin{aligned} \mathbb{P}[\mathbf{M}_h \subset \mathbf{A}] &= \mathbb{P}[H_{\mathbf{A}}(\Psi) < H_{\mathbf{A}^c}(\Psi), \mathbf{M}_h \subset \{s : |s| \leq K\}] \\ &\geq \mathbb{P}[H_{\mathbf{A}}(\Psi) < H_{\mathbf{A}^c}(\Psi)] - \mathbb{P}[\mathbf{M}_h \not\subseteq \{s : |s| \leq K\}]. \end{aligned}$$

Furthermore, the relationships

$$\liminf_{h \downarrow 0} \mathbb{P}[\mathbf{M}_h \subset \mathbf{A}] \geq \mathbb{P}[H_{\mathbf{A}}(\Psi) < H_{\mathbf{A}^c}(\Psi)] - \sup_h \mathbb{P}[\mathbf{M}_h \not\subseteq \{s : |s| \leq K\}],$$

and $\limsup_{h \downarrow 0} \mathbb{P}[\mathbf{M}_h \subset \mathbf{A}] \leq \mathbb{P}[H_{\mathbf{A}}(\Psi) < H_{\mathbf{A}^c}(\Psi)]$ are valid. As for the minimum point of the population criterion $\Psi_l^*(\cdot)$, we have $\mathbb{P}[\xi_l^0 \in \mathbf{A}] \leq \mathbb{P}[H_{\mathbf{A}}(\Psi) < H_{\mathbf{A}^c}(\Psi)]$, and $\mathbb{P}[\xi_l^0 \in \mathbf{A}] + \mathbb{P}[|\xi_l^0| > K] \geq \mathbb{P}[H_{\mathbf{A}}(\Psi) \leq H_{\mathbf{A}^c}(\Psi)]$. The uniqueness assumption on the population criterion (cf. Assumption 4.2) and (A.46) leads to $\lim_{K \rightarrow \infty} \{\sup_{h \leq \bar{h}} \mathbb{P}[\mathbf{M}_h \not\subseteq \{s : |s| \leq K\}] + \mathbb{P}[|\xi_l^0| > K]\} = 0$, for a small $\bar{h} > 0$. Hence,

$$\lim_{h \downarrow 0} \mathbb{P}[\mathbf{M}_h \subset \mathbf{A}] = \mathbb{P}[\xi_l^0 \in \mathbf{A}]. \quad (\text{A.47})$$

In the last step of the proof, we show that the length of the set \mathbf{M}_h converges to zero in probability as $h \downarrow 0$. Let \mathbf{A}_d denote an interval in \mathbb{R} centered at the origin and of length $d < \infty$. In view of (A.47), $\lim_{d \rightarrow \infty} \sup_{h \downarrow 0} \mathbb{P}[\mathbf{M}_h \not\subseteq \mathbf{A}_d] = 0$. Fix any $\epsilon > 0$ and divide \mathbf{A}_d into admissible subintervals whose lengths do not exceed $\epsilon/2$. We have,

$$\mathbb{P}\left[\sup_{s_i, s_j \in \mathbf{M}_h} |s_i - s_j| > \epsilon\right] \leq \mathbb{P}[\mathbf{M}_h \not\subseteq \mathbf{A}_d] + (1 + 2d/\epsilon) \sup \mathbb{P}\left[H_{\mathbf{A}}(\Psi_{l,h}^*) = H_{\mathbf{A}^c}(\Psi_{l,h}^*)\right],$$

where $1 + 2d/\epsilon$ is an upper bound on the number of subintervals and the supremum is taken over all possible open bounded subintervals $\mathbf{A} \subset \mathbf{A}_d$. Given $\Psi_{l,h}^* \Rightarrow \Psi_l^*$, we have $\mathbb{P}[H_{\mathbf{A}}(\Psi_{l,h}^*) = H_{\mathbf{A}^c}(\Psi_{l,h}^*)] \rightarrow \mathbb{P}[H_{\mathbf{A}}(\Psi) = H_{\mathbf{A}^c}(\Psi)]$ as $h \downarrow 0$. Since $\mathbb{P}[H_{\mathbf{A}}(\Psi) = H_{\mathbf{A}^c}(\Psi)] = 0$ and d can be chosen large, we have $\mathbb{P}[\sup_{s_i, s_j \in \mathbf{M}_h} |s_i - s_j| > \epsilon] = o(1)$. Given that $\epsilon > 0$ can be chosen arbitrary small we have that the distribution of $\xi_{l,h} = T_h(\widehat{\lambda}_b^{\text{GL}} - \lambda_0)$ converges to the distribution of ξ_l^0 . \square

A.3 Proofs of Section 6

A.3.1 Proof of Theorem 6.1

The next tree lemmas correspond to Lemma A.29-A.31 from [Casini and Perron \(2020c\)](#), respectively.

Lemma A.20. *For $\varpi > 3/4$, we have $\lim_{h \downarrow 0} \limsup_{|s| \rightarrow \infty} \left| \widehat{\mathcal{W}}_h(s) \right| / |s|^\varpi = 0$ \mathbb{P} -a.s.*

Lemma A.21. *$\{\widehat{\mathcal{W}}_h\}$ converges weakly toward \mathcal{W} on compact subsets of \mathbb{D}_b .*

Lemma A.22. Fix $0 < a < \infty$. For $l \in \mathbf{L}$ and any positive sequence $\{a_h\}$ satisfying $a_h \xrightarrow{\mathbb{P}} a$,

$$\int_{\mathbb{R}} |l(r-s)| \exp(\widehat{\mathcal{W}}_h(s)) \exp(-a_h |s|) ds \xrightarrow{d} \int_{\mathbb{R}} |l(r-s)| \exp(\mathcal{W}(s)) \exp(-a |s|) ds.$$

Proof of Theorem 6.1. Let $\mathbf{C} \subset \mathbb{R}$ be compact. Suppose γ_h satisfies Condition 1. Then,

$$\widehat{\Psi}_{l,h}^*(s) = \int_{\widehat{\Gamma}^*} l(s-v) \frac{\exp(\widehat{\mathcal{V}}_h(v)) \pi(\widehat{N}_b^{\text{GL}} + v)}{\int_{\widehat{\Gamma}^*} \exp(\widehat{\mathcal{V}}_h(w)) \pi(\widehat{N}_b^{\text{GL}} + w) dw} dv + o_{\mathbb{P}}(1).$$

Using Lemma A.21, $\widehat{\mathcal{V}}_h \Rightarrow \mathcal{V}$ in $\mathbb{D}_b(\mathbf{C})$. From Assumption 6.1-(i), $\widehat{\Gamma}^*$ can be replaced by Γ^* for small enough h if we show that the integral as a function of $\widehat{N}_b^{\text{GL}}$ is continuous. Note that the integrand is Riemann integrable and thus the integral considered as a map is continuous. Since \mathcal{V} is \mathbb{P} -a.s. continuous, then the desired result follows by the continuity of the composition of continuous functions. Using the same steps as in the proof of Theorem 4.1 together with Lemma A.22 yields the result. \square

References

- Billingsley, P., 1995. Probability and measure. New York: John Wiley and Sons.
- Casini, A., Perron, P., 2019. Structural breaks in time series. Oxford Research Encyclopedia of Economics and Finance Oxford University Press.
- Casini, A., Perron, P., 2020a. Continuous record asymptotics for structural change models. arXiv preprint arXiv:1803.10881.
- Casini, A., Perron, P., 2020b. Continuous record Laplace-based inference in structural change models. arXiv preprint arXiv:1804.00232.
- Casini, A., Perron, P., 2020c. Generalized Laplace inference in multiple change-points models. arXiv preprint arXiv:1803.10871.
- Ibragimov, A., Has'minskiĭ, R.Z., 1981. Statistical estimation: asymptotic theory. Springer-Verlag New York.
- Jacod, J., Protter, P., 2012. Discretization of processes. Berlin Heidelberg: Springer.
- Jacod, J., Shiryaev, A.N., 2003. Limit theorems for stochastic processes. Berlin Heidelberg: Springer-Verlag.
- Royden, H.L., Fitzpatrick, P., 2010. Real analysis. Prentice Hall.