Repeated Games Without Public Randomization: A Constructive Approach*

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Abstract

We study infinitely repeated games with perfect monitoring and without public randomization devices. Both symmetric and asymmetric discounting cases are considered; a new geometric construct called "self-accessibility" is proposed and used to unify the analyses of these two cases. In the case of symmetric discounting, our approach delivers a constructive version of the folk theorem of Fudenberg and Maskin (1991). If discounting is asymmetric, we show that any payoff that is in the interior of the smallest rectangular region that contains the stage game feasible set is realizable in the repeated game for identifiable sets of discount factor vectors. Next, we provide necessary and sufficient conditions for payoff vectors to be subgame-perfect equilibrium payoffs for *some* discount factor vector. Sets that are defined by these conditions are easily described; moreover, discount factor vectors and strategies that support a specific payoff vector can be explicitly constructed.

Keywords: Repeated Games, Public Randomization, Asymmetric Discounting

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In the words of Kronecker, the positive integers were created by God. ...When a man proves a positive integer to exist, he should show how to find it. If God has mathematics of his own that needs to be done, let him do it himself. - Errett Bishop[1]

1 Introduction

The pioneering work of Fudenberg and Maskin (1986), hereafter FM 1986, demonstrated that provided the players are patient enough, any FSIR (feasible and strictly individual rational) payoff vector for a game is supportable as the discounted average of payoffs arising from an SPNE (subgame perfect Nash equilibrium) of the corresponding repeated game. This result is often construed as the foundation of the argument behind the assertion that rational, self-interested agents can be induced to cooperate with one another as long as the future is reasonably important to them. To accord legitimacy to this argument, however, it is vital to ensure that it does not depend on fragile assumptions. Two such assumptions made in the original FM 1986 paper are of concern to us; the first is the availability of a public randomization device (hereafter PRD) and second is the assumption that all players have exactly the same discount factor.

While the arguments in the original FM 1986 paper were 'constructive', existing papers that have attempted to weaken one or both of these assumptions either are unable to provide explicit equilibrium strategies, or can not compute numerical bounds on discount factors for folk-theorem type results, or (in the case of asymmetric discounting) require very specific patterns of discount factor vectors, or are unable to describe in simple terms, the possible equilibrium payoff sets. Using a new geometric concept called *self-accessibility*, we are able to unify the study of 'PRD-less' games with both symmetric and asymmetric discounting. We strengthen results in the former case and offer new results in the latter case, while maintaining a constructive approach throughout.

Dispensing with PRDs is not a minor addendum because it is problematic to argue that all players have access to a correlating device fine enough to realize all FSIR payoffs. While, following Aumann and Maschler (1995), communication can be used to construct 'jointly controlled lotteries' that serve as correlating devices, in the industrial organization context, this might run foul of antitrust regulations that forbid communication among firms. Similarly, requiring exact equality of discount factors seems too restrictive; if we allow economic agents to have differential preferences over consumption bundles, why should we require them to hold identical time preferences?[2] Lehrer and Pauzner (1999), hereafter LP 1999, have argued that even when payoffs are monetary and players can borrow in an outside market to smooth their consumption streams, different agents, because of their

[1] From "A Constructivist Manifesto", quoted in Errett Bishop and Douglas Bridges, "Constructive Analysis", Springer, 1985, Chapter 1, Page 4.

[2] See for example Harrington (1989), Obara and Zincenko (2017) and Haag and Lagunoff (2007) who provide interesting economic models with differential discounting, the first two in the context of price-setting oligopolies, and the last in the context of collective effort-expending games.

differential financial standings, may be subject to different interest rates. Finally, it is odd to argue against the importance of asymmetric discounting in complete information games when the literature on incomplete information games is rife with such models.

A plethora of interesting and counterintuitive things happen as soon as one disallows PRDs, even when players are patient. For example, in the symmetric discounting case Yamamoto (2010) shows that for *large enough common discount factors*, the set of SPNE payoffs can be non-convex and non-monotonic with respect to the common discount factor. Salonen and Vartiainen (2008) show that for *large enough but unequal discount factors* the feasible payoff set of the repeated game can be totally disconnected,[3] and the Pareto frontier function can be everywhere discontinuous! For a class of two-player *finite* games, Olszewski (1998) shows that the undiscounted folk theorem does not hold.

To establish whether a given point in the payoff space is an equilibrium payoff vector when PRDs are unavailable, the first question that must be answered is: Is this payoff feasible in the repeated game using uncorrelated (but possibly mixed) actions? If this is answered in the affirmative, the second question is: Is this an equilibrium (SPNE) payoff? To fix terminology, we shall henceforth call these questions those of *realizability* and *supportability* respectively.

In the context of *symmetric discounting*, Fudenberg and Maskin (1991), hereafter FM 1991, was the first paper to prove a folk theorem without public randomization. They start by addressing realizability and provide a lower bound for discount factors such that any point in the feasible set is representable as the discounted average of an infinite sequence of its vertices. This result goes by the name 'Sorin's lemma' as Sylvain Sorin (1986) had first analyzed a similar representation. Going from realizability to supportability requires work; the sequence of pure actions given by Sorin's lemma may not be an equilbrium path. As the standard text of Mailath and Samuelson (2006) notes, "The difficulty is that some of the continuation values generated by these sequences may fail to be even weakly individual rational." In trying to address this issue FM 1991 relies on complex as well as non-constructive arguments to build on top of Sorin's result which in the end, delivers strategies to support a specific payoff vector (like FM 1986), but is unable to yield a computable discount factor bound (unlike FM 1986).[4] Our key contribution is to use the the recursion-based notion of self-accessibility to address both realizability and individual rationality *simultaneously.* A set of payoffs is self-accessible for a discount factor vector if, for any point in the set, there is a pure action that can be played such that the induced continuation payoff also lies in the set. In the case of symmetric discounting we show that closed balls of small enough radii strictly inside the FSIR set are self-accessible for discount factors above a bound that can be explicitly computed using a simple non-linear program. This insight enables us to strengthen and offer a simpler yet completely

[3]A set is totally disconnected if the only connected subsets it has are either empty or singleton.

[4]As an example of a non-constructive argument, consider covering a compact set S with an infinite collection of open balls and then choosing some characteristic of the finite subcover of S that is guaranteed because of compactness. But then, how does one figure out *which* finite subcover will do the job?

constructive argument of the FM 1991 folk theorem.[5]

Why might knowing a bound be important? From a practical standpoint, the whole argument of repeated interaction being a prime motivator behind cooperation among otherwise selfish individuals is much more plausible when the required discount factor bound is, say, .7 rather than .9999. In fact, if the bound works out to be .2, one has described what can be achieved for fairly myopic players as well. From a theoretical standpoint, even when we know that the set of equilibrium payoffs approaches the FSIR set as the common discount factor increases, it is of some interest to know if the approach is fast enough.[6]

The first paper to systematically study the asymmetric discounting case was LP 1999. They noted that in this setup, unlike in the symmetric discounting case, as the players become increasingly patient, a) realizable payoffs may lie outside the stage game feasible set and b) the limiting set of supportable payoffs could be very different from the FSIR set. However, they only analyze 2-player games, require fixed ratios of log discount factors, and their proof crucially relies on existence of PRDs. They conjecture that it might be possible to remove this last restriction using techniques similar to FM 1991; however, we show by a simple counterexample that the building block of the FM approach, Sorin's Lemma does not hold in this situation. Under unequal discounting, Chen and Takahashi (2012) show that a sequence of action profiles that delivers a given payoff can be supported in equilibrium if all its continuation payoffs are uniformly bounded away from the minmax values. However, without a PRD only an 'approximate' folk theorem is proved. The question of realizability is not tackled either.[7]

Repeated games with imperfect public monitoring are studied using 'self-generation', a technique originally advanced in Abreu, Pearce, and Stacchetti (1990), and subsequently extended by Fudenberg and Levine (1994) and Fudenberg, Levine, and Maskin (1994). Using this approach, Sugaya (2015) extends LP 1999 to prove a very comprehensive folk theorem that applies to any finite number of players, perfect and imperfect public monitoring and possibly asymmetric discounting while dispensing with PRDs.[8] He too, like Lehrer and Pauzner works with restrictive discount factor vector sequences, specifically assuming that pairwise ratios of *discount rates*[9] are either constants or are converging to constants. His main result shows that the limiting sets of payoff vectors that are individually rational each period and that of equilibrium payoffs are identical.

Although results based on self-generation apply to perfect monitoring, the following points about that approach are worth noting and will be contrasted with our approach.

[5] Although the strategies we use are very similar to the ones used in FM 1991, one of our innovation is to design a punishment phase that does not increase without bound as players become arbitrarily patient.

[6] A recent paper, Hörner and Takahashi (2016) addresses this question systematically for the first time.

[7] Chen and Fujishige (2013) shows that the set of realizable payoffs in a *finitely* repeated game with unequal discounting is monotonically increasing in the length of the horizon.

[8] See also Hörner and Olszewski (2005) who prove a folk theorem for almost-perfect private monitoring without using a PRD, though their results do not encompass unequal discounting.

[9] If δ_i is the i-th player's discount factor, his discount rate is $\frac{1}{\delta_i} - 1$.

- Self-generation proofs do not offer a clear-cut way of calculating a bound that discount factors must obey for supporting a target payoff vector.
- The actual computation of actions to be played requires solving an infinite sequence of nonlinear programs for the asymmetric discounting case, a computationally onerous task.
- Self-generation considers realizability and supportability together. The issue of realizability which is interesting in its own right is not addressed separately.
- When the number of players is three and higher, this approach does not yield a simple condition for verifying if a given point is an equilibrium point for some discount factor vector.

As it turns out, we can use the notion of self-accessibility to fill many of these voids. To support points inside the interior of the FSIR set, we can find a common bound for each player's discount factor - the same one we can compute with symmetric discounting! We show that for games with full-dimensional feasible sets, if one is allowed to choose discount rates, for large enough (computable) discount factors, one can make any payoff vector realizable as long as its i-th coordinate is strictly between the maximum and the minimum of player i's stage game payoffs. Next, we offer necessary and sufficient conditions for payoffs to belong to the limiting equilibrium payoff set[10]. These conditions, respectively referred to as the *weak and strict diagonal conditions*, characterize sets that are quite easy to compute and visualize. We show that with a symmetric stage game, if we wish to give a player (close to) the maximum possible payoff (of the stage game), we have to make him the *least* patient among all players. We offer specific strategies and sets of discount factors that realize or support a given payoff vector satisfying the sufficiency conditions of realizability and supportability respectively. As such, we go well beyond proving folk theorems: our constructive approach empower game theorists with prescriptive powers above and beyond explicative ones.[11]

The paper is structured as follows. In the next section, we formally introduce the model and notation, define self-accessibility and briefly explain its significance. In section 3, a numerical example is discussed to illustrate the constructive nature of our arguments and present the flavor of some of our findings. Section 4 analyzes self-accessibility when discounting is symmetric and presents a constructive extension of FM 1991. Section 5 discusses self-accessibility in the asymmetric discounting case and explores the supportability of payoffs within the interior of the FSIR set. Section 6 addresses the issue of realizability in the asymmetric discounting case outside the feasible set. Section 7 defines the weak and strict diagonal conditions and shows how they relate to supportability of payoff vectors outside the FSIR set. Section 8 concludes. All proofs are collected in an appendix.

[10] By this we mean the set of payoffs which are SPNE payoffs for some discount factor vector.

[11] In fairness, we readily admit that for the n-player case we do *not* solve the problem of characterizing the set of supportable payoffs for a specific discount factor vector (or even the corresponding limiting set for fixed discount rate ratios). However, we do throw some light on it since, our necessary condition tells us what *cannot* be part of that set.

2 Preliminaries

2.1 Notation and The Model

We consider a standard infinitely repeated game of perfect monitoring with possibly unequal discounting. At each $t \in \{0, 1, 2, \ldots\}$ the (finite) stage-game $G = \langle I; (A_i)_i; (g_i)_i \rangle$ is played, where I is the set of players $\{1, \ldots, n\}$, A_i is player i's finite set of actions, $A := \times_j A_i$ is the set of all pure action profiles, and $g_i : A \to \mathbb{R}$ is player i's payoff function. A mixed action of i is $\alpha_i \in \triangle A_i$, where for any set S, $\triangle S$ denotes the set of all probability distributions on the set S. Let $\boldsymbol{a}^{(t)} \in A$ be the (realized) action profile played at time t.[12] When player i discounts future payoffs using the discount factor δ_i, player i's average discounted utility defined over infinite sequences of pure actions in A is

$$u_i(\left\{\boldsymbol{a}^{(t)}\right\}_{t=0}^{\infty}) := (1 - \delta_i) \sum_{t=0}^{\infty} {\delta_i}^t g_i \left(\boldsymbol{a}^{(t)}\right).$$

Under perfect monitoring the public history at the end of period t is $h^t = (\boldsymbol{a}^{(0)}, \ldots, \boldsymbol{a}^{(t)}) \in A^{t+1}$. A pure strategy of i is a (sequence of) maps $s_i(t+1) : H^t \to A_i$ (for $t = -1, 0, 1, \ldots$) where H^t denotes the set of histories at the end of period t (with the convention that h^{-1} is the empty set). Mixed stategies are analogous, except that they map to the corresponding mixed actions $\triangle A_i$. This formulation implies that strategies cannot be conditioned on anything other than the history of actions actually played; in particular, there is no publicly observable random variable on whose realized value actions may be conditioned, and mixed actions are not observable; only their realizations are.

This describes the repeated game $G^{\infty}(\boldsymbol{\delta})$, where the vector $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n)$ is referred to as the discount factor vector. In the special case where each player discounts the future at the same rate δ, we denote the game (by a slight abuse of notation) as $G^{\infty}(\delta)$. We let $\mathcal{F}(\boldsymbol{\delta})$ denote the set of feasible payoff vectors and $\mathcal{V}(\boldsymbol{\delta})$ to denote the set of subgame perfect equilibrium payoff vectors in the repeated game.

Let $C = g(A)$. Player i's minmax value is $w_i := \min_{\boldsymbol{\alpha}_{-i} \in \times_{j \neq i}(\Delta A_j)} \max_{a_i \in A_i} \; g_i(a_i, \boldsymbol{\alpha}_{-i})$, Let $\boldsymbol{m}^i \in \times_{j=1}^{n}(\Delta A_j)$ be the profile that minmaxes i, with player i playing a best response. Whenever it is convenient to do so, we will assume without loss of generality that $w_i = 0$ for all i. The feasible set is $F := co(C)$, the feasible and weakly individually rational (FWIR) set is $F^+ := \{\boldsymbol{x} \in F \,|\, x_i \geqslant w_i\}$ and the feasible strictly individually rational (FSIR) set is $F^* := \{\boldsymbol{x} \in F \,|\, x_i > w_i\}$. The lower boundary of F is $\underline{\partial} F := \{\boldsymbol{x} \in F : \nexists \boldsymbol{y} \in F \text{ such that } \boldsymbol{y} << \boldsymbol{x}\}$. We let $M = \max_i \{|g_i(\boldsymbol{a})| : \boldsymbol{a} \in A\}$; when F^* is full-dimensional, this is strictly positive.

For any set $S \subset \mathbb{R}^n$, and $M \subset \{1, \ldots, n\}$, $Proj_M(S)$ denotes the projection of S along

[12] In what follows vectors are boldfaced while scalars and sets are not. Sequence indices are denoted by superscripts and sometimes they are enclosed in parentheses to distinguish them from exponents or from another sequence denoted by the same letter; for example, $\boldsymbol{c}^l$ denotes the l-th vertex of a polytope C, while $\{\boldsymbol{c}^{(t)}\}$ denotes an infinite sequence of vertices each element of which is a $\boldsymbol{c}^l$ for some l. Coordinates of vectors are denoted by subscripts.

the coordinates in M. In particular when $M = \{1, 2, \ldots, n-1\}$, we use the shorthand $Proj_{-n}(S)$ to denote the projection on the first $n-1$ coordinates. For a finite set C of pure action payoffs in an n-player game, we define the corresponding 'feasible set' for Players $1, \ldots, l$ $(l \leqslant n)$ as $F(1, \ldots, l) := co(Proj_{\{1,\ldots,l\}}(C)) = Proj_{\{1,\ldots,l\}}F$.

$\boldsymbol{\iota}$ is a vector of 1's while $\boldsymbol{e}^i$ is the i'th unit vector. For later use, we recall the definitions of a few geometric terms. The affine hull of a set $X \subset \mathbb{R}^n$ is

$$aff(X) := \left\{ \sum_{l=1}^{k} \lambda^l \boldsymbol{x}^l \;\middle|\; \boldsymbol{x}^l \in X, \sum_{l=1}^{k} \lambda^l = 1,\, k \in \mathbb{N} \right\}.$$

If $\lambda^l \geqslant 0$ above, we obtain the convex hull of X, denoted as $co(X)$. For $\boldsymbol{x} \in X$, the affine (closed) ball with center $\boldsymbol{x}$ and radius r is $B_X(\boldsymbol{x}, r) := \{\boldsymbol{y} \in aff(X) : d(\boldsymbol{y}, \boldsymbol{x}) \leqslant r\}$, while $B(\boldsymbol{x}, r)$ denotes the usual (closed) ball in $\mathbb{R}^n$. The relative interior of X is

$$relint(X) := \{\boldsymbol{x} : \exists\, r > 0 \text{ such that } B_X(\boldsymbol{x}, r) \subset X\}.$$

When $X = co(C)$, where $C = \{\boldsymbol{c}^1, \ldots, \boldsymbol{c}^L\}$, and every point in C is an extreme point of X, then each point in $relint(X)$ can be expressed as a convex combination of those points with strictly positive weights, i.e. $relint(X) = \left\{\sum_{l=1}^{L} \lambda^l \boldsymbol{c}^l \;\middle|\; \lambda^l > 0,\ \sum_{l=1}^{L} \lambda^l = 1\right\}$. The usual interior of a set S is denoted by $int(S)$. Finally, we introduce a new terminology: the *rectangular hull* of a bounded set in $\mathbb{R}^n$, denoted as $re(S)$ is the smallest closed rectangle that contains S. Formally,

$$re(S) := \bigcap_{R \in \mathcal{R}} R \quad \text{where } \mathcal{R} = \left\{ \prod_{i=1}^{n} [a_i, b_i] : a_i \leqslant b_i, \prod_{i=1}^{n} [a_i, b_i] \supset S \right\}.$$

2.2 Self-accessibility

We now define *self-accessibility*, for possibly unequal discounting and explain its usefulness. Although self-accessibility is an independent geometric notion, in the definition below the set C may be usefully thought of as the set of payoff vectors from pure action profiles of the stage game and δ_j may be thought of as the discount factor of player j.

Definition. Let $C \subset \mathbb{R}^n$ be a finite set. A set $S \subset co(C)$ is said to be **self-accessible** relative to C for a vector $\boldsymbol{\delta} = (\delta_1, \ldots \delta_n) \in [0, 1)^n$ if for any $\boldsymbol{x} \in S$ there exists $\boldsymbol{y} \in S$ and $\boldsymbol{c} \in C$ such that $x_j = (1 - \delta_j)c_j + \delta_j y_j$ for $j = 1, \ldots, n$.[13]

The definition is particularly intuitive for equal discounting. For some $\boldsymbol{x} \in S \subset co(C)$, suppose that we can find $\boldsymbol{c} \in C$, $\boldsymbol{y} \in S$ and $\delta \in [0, 1)$ such that $\boldsymbol{x} = (1 - \delta)\,\boldsymbol{c} + \delta \boldsymbol{y}$; this is a 'dynamic programming decomposition' of the target payoff $\boldsymbol{x}$, with the restrictions that the current payoff $\boldsymbol{c}$ is generated by a pure action profile and the continuation payoff $\boldsymbol{y}$ lies in the set S itself. If there is a *uniform* $\delta \in [0, 1)$ for which any point $\boldsymbol{x} \in S$ can be

[13]When there is no scope for confusion about C, we omit it. Note that if S is self-accessible relative to C for $\boldsymbol{\delta}$, then it is self-acessible relative to any superset of C for the same $\boldsymbol{\delta}$.

written as the $(1-\delta, \delta)$ convex combination of a pure-action payoff and a continuation payoff within the set itself, then S is self-accessible for $\boldsymbol{\delta} = \delta\boldsymbol{\iota}$.

Why is self-accessibility useful for dispensing with PRDs? Suppose C is the set of pure-action payoffs in the stage game, $S \subset co(C)$ is self-accessible for $\boldsymbol{\delta}$, and $\boldsymbol{x} \in S$. It follows that there exists $\boldsymbol{c}^{(0)} \in C$ such that $x_j = (1-\delta_j)c_j^{(0)} + \delta_j y_j^1$ for each j where $\boldsymbol{y}^1$ is also in S. Because of the latter, we can write $y_j^1 = (1-\delta_j)c_j^{(1)} + \delta_j y_j^2$ for each j for some $\boldsymbol{c}^1 \in C$ and $\boldsymbol{y}^2 \in S$. By induction there is a sequence of vertices $\{\boldsymbol{c}^{(t)}\}_{t \geqslant 0}$ such that

$$x_j = (1-\delta_j)\sum_{t=0}^{\tau} \delta_j^t c_j^{(t)} + \delta_j^{\tau+1} y_j^{\tau+1} \quad \forall j,\ \forall \tau.$$

Since $\delta_j < 1$ and S is bounded, we have $\| \delta_j^{\tau+1} y_j^{\tau+1} \| \to 0$ as $\tau \to \infty$. Hence any point $\boldsymbol{x}$ in a self-accessible set S has a representation $x_j = (1-\delta_j)\sum_{t \geqslant 0} \delta_j^t c_j^{(t)}$. Restating this in the context of repeated games, any point lying in a set that is self-accessible for a given $\boldsymbol{\delta}$ vector can be represented as the coordinate-wise discounted average of a sequence of pure action payoffs for that $\boldsymbol{\delta}$ vector.[14]

3 A Numerical Example

Next, we presents a numerical example to underscore the computational advantages of using self-accessibility in supporting specific payoff vectors in the symmetric discounting case. Although the results presented later are more general, and are valid for any number of players, in order to abstract away from ancillary issues, we choose a simple, asymmetric version of the Prisoner's dilemma game.

3.1 Achieving the Nash Bargaining Point with Symmetric Discounting

Each of two players simultaneously choose one of two actions (A) and (N) with the payoff matrix displayed below.

	A	N
A	$(4, 2)$	$(9, 0)$
N	$(0, 7)$	$(5, 5)$

Henceforth, we refer to the payoff vectors $(4, 2)$, $(9, 0)$, $(5, 5)$ and $(0, 7)$ as $\boldsymbol{c}^1, \boldsymbol{c}^2, \boldsymbol{c}^3$ and $\boldsymbol{c}^4$ respectively. The unique dominant strategy equilibrium (A, A) is inefficient. The (efficient) Nash Bargaining payoff vector (where the set of possible agreement payoffs is the feasible set and $\boldsymbol{c}^1$ is the disagreement point) is $\boldsymbol{n} = (5.700, 4.125)$, which is a convex combination of $\boldsymbol{c}^3$ and $\boldsymbol{c}^2$ with weights .825 and .175 respectively (see Figure 1). We will

[14] To make this sequence well-defined, whenever a point in S can be decomposed in more than two ways with more than one choice of current action, we can use some pre-assigned arbitrary ordering among the vertices to decide which current action to choose.

like to obtain this payoff vector in a SPNE.

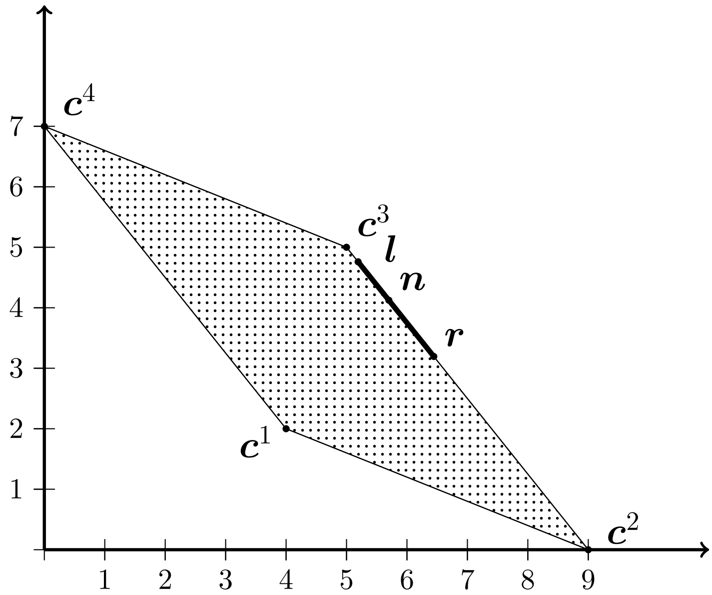

Figure 1: Realizable Payoffs in an Asymmetric Prisoner's Dilemma Game

Let us first examine the realizability of $\boldsymbol{n}$. Choose a one-dimensional closed ball $S \subset co(\{\boldsymbol{c}^2, \boldsymbol{c}^3\})$ containing $\boldsymbol{n}$, with extreme points $\boldsymbol{l}$ and $\boldsymbol{r}$ ($\boldsymbol{l}$ being closer to $\boldsymbol{c}^3$). Let $\boldsymbol{l} = \lambda(5,5) + (1-\lambda)(9,0) = (9-4\lambda, 5\lambda)$ and $\boldsymbol{r} = \mu(5,5) + (1-\mu)(9,0) = (9-4\mu, 5\mu)$ with the requirements

$$1 \geqslant \lambda \geqslant .825 \quad \text{and} \quad 0 \leqslant \mu \leqslant .825 \tag{3.1}$$

to ensure that S contains $\boldsymbol{n}$. We shall find a cutoff $\underline{\delta}$ above which S is self-accessible (relative to $\{\boldsymbol{c}^2, \boldsymbol{c}^3\}$).

Any $\boldsymbol{x} \in S$ can be written as $\theta(5,5) + (1-\theta)(9,0) = (9-4\theta, 5\theta)$, with $\mu \leqslant \theta \leqslant \lambda$. Let $\delta(\boldsymbol{x}, \boldsymbol{c}^3)$ be the lowest value of δ in $[0,1]$ such that $\boldsymbol{x} = (1-\delta)\boldsymbol{c}^3 + \delta\boldsymbol{y}$ for some point $\boldsymbol{y} \in S$. Since the farthest continuation payoff within S is $\boldsymbol{r}$, $\delta(\boldsymbol{x}, \boldsymbol{c}^3)$ solves $\boldsymbol{x} = (1-\delta)\boldsymbol{c}^3 + \delta\boldsymbol{r} = (1-\delta)(5,5) + \delta(9-4\mu, 5\mu)$, and therefore $\delta(\boldsymbol{x}, \boldsymbol{c}^3) = (1-\theta)/(1-\mu)$. Similarly, define $\delta(\boldsymbol{x}, \boldsymbol{c}^2)$ as the the lowest value of δ in $[0,1]$ such that $\boldsymbol{x} = (1-\delta)\boldsymbol{c}^2 + \delta\boldsymbol{y}$ for some point $\boldsymbol{y} \in S$; using an analogous argument this is seen to be θ/λ. If the discount factor δ is at least as much as $\delta^*(\boldsymbol{x}) := \min\{\delta(\boldsymbol{x}, \boldsymbol{c}^2), \delta(\boldsymbol{x}, \boldsymbol{c}^3)\}$, then $\boldsymbol{x}$ can be attained by playing one of the vertices $\boldsymbol{c}^2$ or $\boldsymbol{c}^3$ with the continuation payoff lying in S.[15] Finally, note that the maximum of $\delta^*(\boldsymbol{x})$ as $\boldsymbol{x}$ varies over S is achieved for a point $\bar{\boldsymbol{x}}$ where $\delta(\bar{\boldsymbol{x}}, \boldsymbol{c}^2) = \delta(\bar{\boldsymbol{x}}, \boldsymbol{c}^3)$, i.e. $(1-\theta)/(1-\mu) = \theta/\lambda$. Eliminating θ and then substituting in either expression gives

[15] More accurately, we mean "...by playing the actions corresponding to the vertices $\boldsymbol{c}^2$ or $\boldsymbol{c}^3$...". Here and elsewhere we indulge in this slight abuse of notation for brevity's sake.

a needed bound for S to be self-accessible:

$$\underline{\delta} := \max_{\boldsymbol{x} \in S} \delta^*(\boldsymbol{x}) = \max_{\boldsymbol{x} \in S} \min\{\delta(\boldsymbol{x}, \boldsymbol{c}^2), \delta(\boldsymbol{x}, \boldsymbol{c}^3)\} = \frac{1}{1 - \mu + \lambda}. \tag{3.2}$$

In particular, when $\delta \geqslant \underline{\delta}$ and (3.1) holds, $\boldsymbol{n}$ can be realized by playing a sequence of actions from the set $\{(A, N), (N, N)\}$.

Next, to support $\boldsymbol{n}$ in equilibrium, we wish to deter deviation from the prescribed path via a grim trigger mecahnism ` *a la* Friedman (1971): (A, A) is played forever as soon as any deviation is detected. This is achieved by two incentive compatibility constraints that ensure receiving the worst payoff in S for one period is at least as good as receiving the best payoff in the game once and being minmaxed forever afterwards:

$$9 - 4\lambda \geqslant 9(1 - \underline{\delta}) + 4\underline{\delta}, \tag{3.3}$$

$$5\mu \geqslant 7(1 - \underline{\delta}) + 2\underline{\delta}. \tag{3.4}$$

The minimum $\underline{\delta}$ satisfying (3.1), (3.2), (3.3) and (3.4) is .761, and the corresponding ball S is given by $\lambda = .952, \mu = .639$. This suggests that above a reasonable discount factor the Nash bargaining payoff $\boldsymbol{n}$ can be implemented in an SPNE where one player never plays his dominant action and the the other some times plays hers.

3.2 Expanding Possibilities With Asymmetric Discounting

If all players use the same discount factor, any discounted average payoff vector must stay inside the feasible set. What if they do not? We preview some of the results to follow by explaining how they apply in the context of the current example.

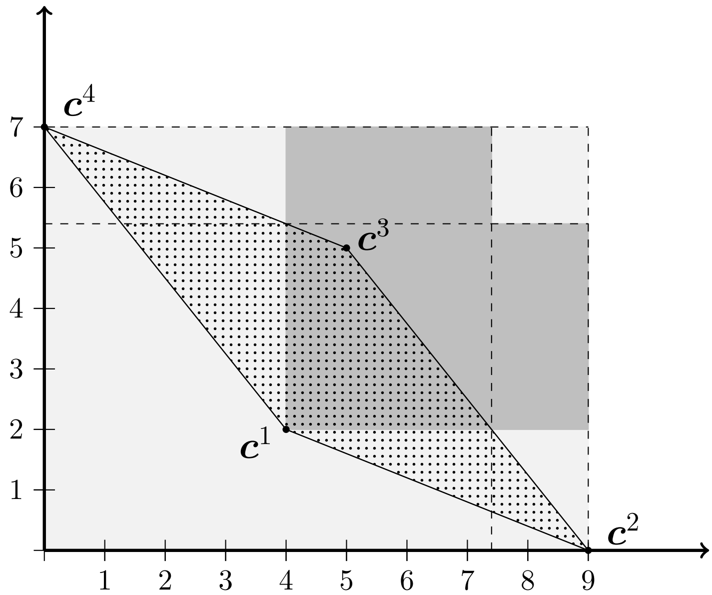

Figure 2: Realizable and Supportable Payoffs with Asymmetric Discounting

Theorem 3 will later show that any payoff vector in $int(re(F))$ (which in this case is the open rectangle $(0,9) \times (0,7)$) can be realized for large enough discount factors if we are allowed to choose the *relative* patience of the two players by fixing the ratio of their discount rates. Which of these are equilibrium payoffs and for what kind of discount factors? Theorem 5 demonstrates that there are some points in the open rectangle $(0,9) \times (0,7)$ that can not be supported in equilibrium no matter what the discount factors are. These are points in the small lightly shaded rectangle in the north-east of the figure; a payoff vector such as $(8.9, 6.9)$ where both *both* players receive close to their maximum payoffs is ruled out.

On the positive front, Theorem 2 will show that points in the interior of F^* (the dotted *and* darkly shaded region) can be supported in equilibrium for large enough discount factors and arbitrary discount rate ratios. Moreover, using ideas developed in Theorem 6 we can show that if both players are sufficiently patient and player 2 's discount rate is *sufficiently lower* relative to player 1, then it is possible to support points in the open rectangle $(4,9) \times (2,5.4)$. The coordinates of this latter rectangle are arrived at by letting player 1's payoff to range between his minmax payoff (4) and the maximum he can receive in the game (9), whereas player 2's payoff is allowed to range between his minmax (2) and the maximum he can receive *subject to giving player 1 his minmax amount*. Analogously, if both players are sufficiently patient and player 1 is sufficiently more patient relative to player 2, points in the open rectangle $(4,7.4) \times (2,9)$ can be supported (7.4 is the maximum player 1 can receive subject to giving player 2 his minmax amount). Points that are common to both rectangles may be supported by (large enough) discount factors exhibiting a wide variety of relative patience. To summarize then, the dotted region in Figure 2 is the feasible set, points in $(0,9) \times (0,7)$ are realizable payoffs for some discount factors, while the interior of the darkly shaded region are supportable payoffs for some discount factors.

4 The Case of Symmetric Discounting

This section discusses how self-accessibility simplifies the arguments of FM 1991's main result, makes discount factor bounds computable and delivers some new results as well.

4.1 Self Accessibility Under Symmetric Discounting

The main building block of the FM 1991 is Sorin's Lemma which addresses the question of realizability. The lemma states the following: Suppose $\boldsymbol{x} \in \mathbb{R}^n$ is in the convex hull of $C = \{\boldsymbol{c}^1, \boldsymbol{c}^2, \ldots, \boldsymbol{c}^{\mathrm{L}}\}$. Then, for all $\delta \geqslant 1 - 1/L$ there exists a sequence $\{\boldsymbol{c}^{(t)}\}_{t=0}^{\infty}$ in C such that $\boldsymbol{x} = (1-\delta)\sum_{t=0}^{\infty} \delta^t \boldsymbol{c}^{(t)}$.[16] Armed with this result, FM then tackle the problem

[16]Actually the bound stated in this lemma is not tight. Using Caratheodory's Theorem it can be shown that the exact tight bound is $1-1/m$ where $m = \min\{L, n+1\}$, rather than $1-1/L$. Also, in his original 1986 paper, Sorin, using a result from Fenchel (1929) obtains the bound $1 - \frac{1}{n}$ when mixed actions are allowed.

of ensuring that continuation payoffs stay close enough to the original payoff (so as to maintain individual rationality) via their Lemma 2, which makes critical use of Sorin's lemma but has a complex argument and offers little computational guidance.

Using our terminology, Sorin's Lemma shows that the entire feasible set is self-accessible relative to its extreme points for large enough discount factors. However, we can show that for any set of points in $\mathbb{R}^n$, affine balls contained in the relative interior of the convex hull of that set are self-accessible (relative to its extreme points and for large enough discount factors). When a target payoff is in F^*, by placing it at the center of a 'small' ball, we can thus achieve both realizability and (*period-wise*) individual rationality in a single step. This approach bypasses the need for Sorin's Lemma altogether and as a bonus, we can also easily compute a relevant discount factor bound.

Proposition 1. *Suppose $C' \subset C = g(A)$ is a set of points in $\mathbb{R}^n$ where $X = Co(C')$ need not be full-dimensional. Let $S = B_X(\boldsymbol{o}, r) \subset relint(X)$ be some affine ball with center $\boldsymbol{o}$ and radius $r > 0$. Then $\exists\ \underline{\delta} \in (0,1)$ such that S is self-accessible relative to C' for any vector $\delta\boldsymbol{\iota}$ with $\delta \geqslant \underline{\delta}$. This $\underline{\delta}$ is computable by solving a nonlinear maximization problem with linear/quadratic objective and constraint functions.*

As this proposition is central to the computability of discount factor bounds, we provide some intuition behind the construction of $\underline{\delta}$. Fix a closed ball $S = B_X(\boldsymbol{o}, r)$ in the relint of $co(C')$. Take any point $\boldsymbol{x}$ in S. Let $\delta(\boldsymbol{x}, \boldsymbol{c})$ be the smallest value of $\delta \in [0,1]$ satisfying the dynamic programming decomposition $\boldsymbol{x} = (1-\delta)\,\boldsymbol{c} + \delta\boldsymbol{y}$ for some $\boldsymbol{y} \in S$. The geometrical interpretation of this function is as follows: consider the line connecting $\boldsymbol{c}$ and $\boldsymbol{x}$; it cuts the surface of the ball at two points, one that is on the same side of $\boldsymbol{x}$ where $\boldsymbol{c}$ is and one on the other side (they could be same). Call this latter point $\boldsymbol{y}$ ($\boldsymbol{y}$ could be $\boldsymbol{x}$ itself). We can think of $\boldsymbol{x}$ as a $(1-\delta) : \delta$ convex combination of $\boldsymbol{c}$ and $\boldsymbol{y}$. It is this δ that is $\delta(\boldsymbol{x}, \boldsymbol{c})$. It is not hard to find a formula for this function, and as expected, it is continuous in $\boldsymbol{x}$.

Now we assert that $\delta(\boldsymbol{x}, \boldsymbol{c}) < 1$ for some vertex $\boldsymbol{c}$. If $\boldsymbol{x}$ is in $relint(S)$, any vertex $\boldsymbol{c} \in C'$ works but if $\boldsymbol{x}$ is on its (relative) boundary of S, not all vertices do. However, since S lies in the (relative) interior of the convex hull of C', we can use any point in C' that is separated from the ball by the supporting hyperplane to S at $\boldsymbol{x}$. In the accompanying figure, $C' = \{\boldsymbol{c}^1, \boldsymbol{c}^2, \boldsymbol{c}^3\}$ and X is the triangular region with those three points acting as vertices (it could be a face of the feasible set). The shaded region is S; $\boldsymbol{x}$ is a point in S. If we try to extend a line from $\boldsymbol{c}^i$ to $\boldsymbol{x}$ to the farthest point in S, we reach $\boldsymbol{y}^i$. Here, $\delta(\boldsymbol{x}, \boldsymbol{c}^1) = 1$, $\delta(\boldsymbol{x}, \boldsymbol{c}^2) = \frac{\|\boldsymbol{x}-\boldsymbol{c}^2\|}{\|\boldsymbol{y}^2-\boldsymbol{c}^2\|}$ and $\delta(\boldsymbol{x}, \boldsymbol{c}^3) = \frac{\|\boldsymbol{x}-\boldsymbol{c}^3\|}{\|\boldsymbol{y}^3-\boldsymbol{c}^3\|}$.

Let $\delta^*(\boldsymbol{x}) := \min_{\boldsymbol{c} \in C} \delta(\boldsymbol{x}, \boldsymbol{c})$. For our figure, it turns out that $\delta^*(\boldsymbol{x}) = \delta(\boldsymbol{x}, \boldsymbol{c}^3)$. Now consider maximizing δ^* over S; Weierstrass's theorem guarantees that the maximum is attained. Since $\delta^* < 1$ throughout, this maximum $\underline{\delta}$ is less than unity and the convexity of S ensures that S is self-accessible above $\underline{\delta}$. Lastly, one can show that the two-stage nested optimization problem we just described can be written as one large optimization problem which is the NLP referred to in Proposition 1.

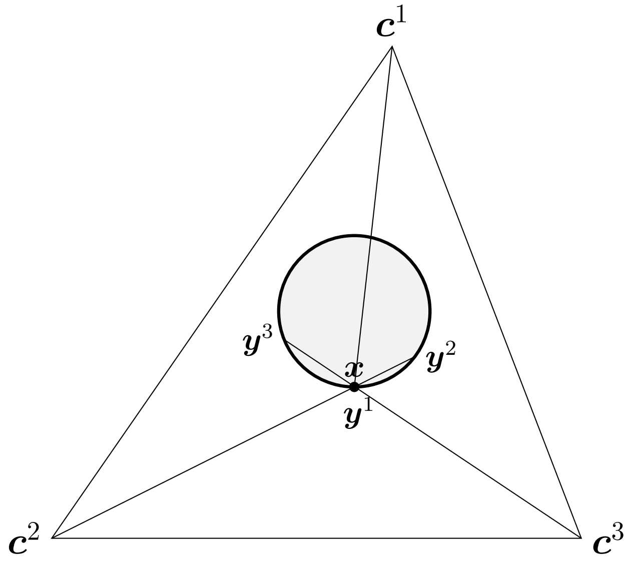

Figure 3: How $\delta^*(\boldsymbol{x})$ is constructed.

Once the affine ball $B_X(\boldsymbol{o}, r)$ is self-accessible relative to a set of vertices C' for a discount factor vector $\boldsymbol{\delta} = \delta\boldsymbol{\iota}$, as was explained in section 2.2, any vector $\boldsymbol{x}$ in that ball can be expressed as a discounted average of a sequence of vertices from C'; we let $\{\boldsymbol{a}^{(t)}(\boldsymbol{x}, B_X(\boldsymbol{o}, r), \delta)\}_{t=0}^{\infty}$ denote this sequence.

One of the biggest advantage of the self-accessibility approach over the FM 1991 approach is that we are very easily able to provide a *computable* uniform bound on discount factors that guarantee realizability of each point in any geometrically well-described compact set while keeping continuation payoffs within a certain small fixed distance of the original point. Non-constructive arguments such as every open cover of a compact set has some finite subcover are not required. As is well-known, these uniform bounds may become relevant if unbeservable mixed strategies are needed to minmax a player, because post-punishment plays are not known beforehand; they are calculated based on the realizations of mixed actions during the punishment period. Extending ideas that are used to prove Proposition 1, our next result shows that we can easily find a bound on discount factors that makes a *collection* of balls self-accessible where each has a certain fixed radius (say ω) and a center that lies within a fixed ball with a different radius (say $\bar{r}$); in Figure 4, these balls are colored dark grey, while the ball within which their centers lie is colored light grey.

Proposition 2. *Let $\boldsymbol{x} = \sum_{l=1}^{K} \lambda^l \boldsymbol{c}^l$ with $\lambda^l > 0$ for each l. Let $C' = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_K\}$ and $X = co(C')$. Let $\bar{r} > 0$ and $\omega > 0$ be such that $B_X(\boldsymbol{x}, \bar{r} + \omega) \subset F^* \cap relint(X)$. Then, we can find $\underline{\delta} < 1$ such that for any $\delta \in (\underline{\delta}, 1)$, any $B(\boldsymbol{x}', \omega)$ where $\boldsymbol{x}' \in B_X(\boldsymbol{x}, \bar{r})$ is self-accessible relative to C' for $\boldsymbol{\delta} = \delta\boldsymbol{\iota}$. Furthermore, $\underline{\delta}$ is computable by solving a nonlinear maximization problem.*

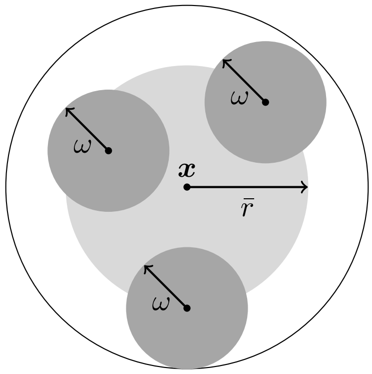

Figure 4: Finding a bound for which a collection of balls is self-accessible

4.2 A Constructive Folk Theorem via Self-Accessibility

As soon as we observe that affine balls with small enough radii are self-accessible, there hardly remains any difference between the two problems where we are trying to support a payoff vector with and without PRDs. If, assuming the extence of PRDs, one can find a discount factor bound (say $\underline{\delta}_1$) for which incentive compatibility conditions are *strict*, then all we have to do in the PRD-less case is to work with small enough balls around points which could potentially be played in the first case, find bounds which will make these balls self-accessible and then we can support the same target payoff vector in the second case using a grand bound which is the maximum of all these bounds and $\underline{\delta}_1$.

This insight drives the main result of this section which is a fully constructive version of the main result in FM 1991. It differs from the original version in three important aspects. First, because of our reliance on self-accessibility, all paths, whether on or off equilibrium become recursively computable. Second, Propositions 1 and 2 and the fact that our method of proof simplifies incentive compatibility conditions, reducing them to only two *linear* inequalities, allow us to compute a critical discount factor bound needed to support a given payoff vector in $F^* \backslash \underline{\partial} F$. Third, our proof reveals that the number of punishment periods during which a deviating player is minmaxed need not become arbitrarily large as δ goes to one. While in FM 1991, the punishment length is of the order of $(-ln\ \delta)^{-1}$, in our proof, we fix it once and for all. We believe that this is a desirable feature of our strategies, practically relevant when there is a possibility of players being susceptible to involuntary mistakes or trembles with minute probabilities.

Theorem 1. *Let F^* be full-dimensional and let $\boldsymbol{v} \in F^* \backslash \underline{\partial} F$. Then, there exists $\underline{\delta} \in (0,1)$ such that for all $\delta \in (\underline{\delta}, 1)$ there is an SPNE that does not use a PRD and has discounted average payoff $\boldsymbol{v}$. This bound is computable using the NLPs provided by Propositions 1 and 2 and two linear inequalities. All paths, on and off equilibrium, are recursively computable as well. Punishment (minmaxing) period lengths are not δ-dependent.*

A detailed proof is provided in the appendix for completeness's sake and also because, the proof applies with little change to a similar theorem in the asymmetric discounting setting. It relies on the standard architecture of equilibrium strategies introduced in FM 1986 characterized in terms of 3 phases, which is well-understood in the literature. It may be instructive here though to consider what self-accessibility brings to the table that allows us to keep the number of punishment periods δ-independent. To that end, consider the situation where Player i has deviated, has been minmaxed, and play now has shifted into the so-called Phase III(i) where players are supposed to receive the (continuation) payoff vector $\boldsymbol{u}$.[17] If one had access to PRD's, one would prescribe a path where in every period, an action generating $\boldsymbol{u}$ would be played. Without PRDs however, players play a sequence of actions that generates $\boldsymbol{u}$ as a discounted average, while the continuation payoffs stay, say, ε-close to $\boldsymbol{u}$. But now suppose, after this path is started, in the very next period, i's continuation payoff becomes $u_i - \varepsilon$. Then, if the number of punishment periods is a constant independent of the discount factor, a sufficiently patient i might want to re-start his own punishment by deviating! This is why FM 1991 needs to let the punishment period become unboundedly large as the discount factor approaches one. In our proof, the target Phase III(i) payoff is $\boldsymbol{u} - \varepsilon \boldsymbol{e}^i$, the lowest point (from i's perspective) in a self-accessible ball (since there is no requirement that a target payoff must be in its center). Thus the perverse situation described above where a patient player wants to restart his own punishment never arises in our case and δ-independent punishment periods can indeed be devised to wipe out the gains from deviation.

5 Asymmetric Discounting: Supporting Points in $int(F^*)$

In this section, we show that any point in the inerior of F^* is both realizable and supportable provided the discount factors are large enough. We begin by showing that the natural counterpart of Sorin's lemma in the asymmetric case does not hold which negates Lehrer and Pauzner's conjecture and makes the FM 1991 approach to the problem invalid. Next we show that if the discount rate ratios are not fixed, not even small balls are self-accessible for large enough discount factors. However, if they are, closed balls can be shown to be self-accessible for computable sets of discount factors. Lastly, we show that for any ball, there is a discount factor bound, such that for a discount factor vector where each component exceeds this bound, one can always find a self-accessible ellipsoid within the ball that is self-accessible relative to that specific discount factor vector. This last result leads to an asymmetric counterpart of Theorem 1.

5.1 Two Negative Results

One might hope that the following 'global' extension of Sorin's lemma to unequal discounting holds: If $C \subset \mathbb{R}^n$ is a finite set there exists $\underline{\delta} \in [0,1)$ such that if $\delta_j \geqslant \underline{\delta}$ for $1 \leqslant j \leqslant n$, and $\boldsymbol{x} \in co(C)$, there exists a sequence of points $\{\boldsymbol{c}^{(t)}\}_{t=0}^{\infty}$ in C for which

[17] It is during this phase that players $j \neq i$ are rewarded for participating in i's punishment phase.

$x_j = (1-\delta_j)\sum_{t=0}^{\infty} \delta_j{}^t\, c_j^{(t)}$. It is easy to see that as stated, the conjecture cannot be true: for example, when $n = L = 2$ (whereupon $co(C)$ is a one-dimensional set with just two vertices) we need the two discount factors to be equal in order to realize points in $co(C)$. Is this then an artifact of $co(C)$ not being full-dimensional or $\boldsymbol{x}$ lying on the boundary rather than in the interior of $co(C)$? Are points that defy the desired representation non-generic? Unfortunately, the problem runs deeper.

Counterexample 1. Let $n = 2$, with $C = \{(1,0),(0,0),(0,1)\}$. Let any $\underline{\delta} \in (0,1)$ be given. We will show the existence of an open set in $co(C)$ and a δ_1, δ_2 pair both at least as large as $\underline{\delta}$ such that no point in that open set is representable using the vertices in C and the given discount factors. To that end, suppose, one can find real numbers $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2$, and an integer T with the following properties:

$$0 < \varepsilon_2 < \varepsilon_1 < 1, \tag{5.1}$$

$$\underline{\delta} = \delta_2 < \delta_1 < 1, \tag{5.2}$$

$$\delta_2{}^T < \varepsilon_2, \tag{5.3}$$

$$(1-\delta_1)\,\delta_1{}^{T-1} > \varepsilon_1. \tag{5.4}$$

We assert that the point $(1-\varepsilon_1, \varepsilon_2)$, which is in $int(co(C))$ by (5.1), is not realizable for discount factors (δ_1, δ_2). To prove this, we first prove inductively that if

$$x_1 = 1 - \varepsilon_1 = (1-\delta_1)\sum_{t=0}^{\infty} \delta_1{}^t\, x_1^{(t)},$$

then $\boldsymbol{x}^{(0)} = \boldsymbol{x}^{(1)} = \ldots = \boldsymbol{x}^{(T-1)} = (1,0)$, i.e. $(1,0)$ must be played for the first T periods. If $\boldsymbol{x}^{(0)} \neq (1,0)$, then even if $(1,0)$ were to be played in each subsequent period, x_1 could be at most δ_1. This would mean $1 - \varepsilon_1 \leqslant \delta_1$, or $\varepsilon_1 \geqslant (1-\delta_1)$. However, (5.4) rules this out. If $\boldsymbol{x}^{(0)} = (1,0)$ but $\boldsymbol{x}^{(1)} \neq (1,0)$, then $x_1 \leqslant (1-\delta_1) + (\delta_1)^2$, which implies $(1-\delta_1) + (\delta_1)^2 \geqslant 1 - \varepsilon_1$; from this it follows that $\varepsilon_1 \geqslant (1-\delta_1)\,\delta_1$, which violates (5.4). Proceeding this way, $(1,0)$ must be played at least the first T times. But then $x_2 \leqslant (\delta_2)^T$, which violates (5.3) if $x_2 = \varepsilon_2$.

It remains to show that one can indeed satisfy the properties (5.1) – (5.4) by judicious choice of $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2$, and T. For integer t, notice that the function

$$\frac{\underline{\delta}^t}{\frac{1}{t}\left(1-\frac{1}{t}\right)^{t-1}} \to 0 \tag{5.5}$$

as t goes to infinity, and hence there exists an integer T for which a) the expression above is strictly below 1, and b) $\frac{T-1}{T} > \underline{\delta}$. Now, if we define $\delta_1 = \frac{T-1}{T}$, then, for $t = T$ the denominator in (5.5) , $\frac{1}{T}\left(1-\frac{1}{T}\right)^{T-1}$ is $(1-\delta_1)\delta_1{}^{T-1}$. Hence, if $\delta_2 = \underline{\delta}$, we can choose an open set of $\varepsilon_1, \varepsilon_2$ pairs such that $0 < \delta_2^T < \varepsilon_2 < \varepsilon_1 < (1-\delta_1)\delta_1{}^{T-1}$ and we have fulfilled all our requirements. This shows that, no matter how high the discount factors are forced to be, if we are allowed to choose them unequal, we can find an open set of points in a full-dimensional feasible set of payoffs that cannot be realized without PRDs. ■

It then seems natural to invoke self-accessibility. In analogy with the equal-discounting case, one might conjecture the following 'local' property. If $C \subset \mathbb{R}^n$ is a finite set with an n-dimensional convex hull, for any $\boldsymbol{x}$ in the interior of $co(C)$ there exists a quantity $r > 0$ and a cutoff $\underline{\delta}$ such that the ball $B(\boldsymbol{x}, r)$ is self-accessible for discount factor vectors of the form $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n)$ with $\delta_i \geqslant \underline{\delta}$. The following shows that this conjecture is false as well.

Counterexample 2. Let $n = 2$, with $C = \{(-1, 1), (-1, -1), (1, 1), (1, -1)\}$. Take any ball $B((0, 0), r)$ with $r < 1$. Suppose that the conjecture holds for $0 < \underline{\delta} < 1$. In that case, if $\boldsymbol{\delta} = (\delta_1, \delta_2) \geqslant (\underline{\delta}, \underline{\delta})$ and $\boldsymbol{x}$ is in the ball, there exists $\boldsymbol{c} \in C$ such that if we define $\boldsymbol{y}(\boldsymbol{\delta})$ via the equation

$$x_i = (1 - \delta_i)\, c_i + \delta_i\, y_i(\boldsymbol{\delta}) \text{ for } i = 1, 2, \tag{5.6}$$

then, $\boldsymbol{y}(\boldsymbol{\delta}) \in B((0, 0), r)$. Write $\underline{\delta} = 1/(1 + \theta)$ and choose $\delta_1 = \underline{\delta}$, and $\delta_2 = 1/(1 + k_2\theta)$ for $k_2 \in (0, 1)$; this ensures that $\delta_2 \geqslant \underline{\delta}$. Now specifically let us consider the point $\boldsymbol{x} = (0, r)$ and ask which $\boldsymbol{c} \in C$ will make the $\boldsymbol{y}(\boldsymbol{\delta})$ given via (5.6) lie in the ball. It is easy to see that the vertices $(1, -1)$ and $(-1, -1)$ are ruled out. By symmetry, $(1, 1)$ works if and only if $(-1, 1)$ works. For $\boldsymbol{c} = (1, 1)$, equation (5.6) gives $\boldsymbol{y}(\boldsymbol{\delta}) = (-\theta, r + k_2\theta(r - 1))$. Hence, $\boldsymbol{y}(\boldsymbol{\delta}) \to (-\theta, r)$ as $k_2 \to 0$; for any given θ, this is strictly outside the ball. Hence there exists a θ, k combination for which $\boldsymbol{y}(\boldsymbol{\delta})$ is outside the ball. ■

5.2 Two Positive Results

We now present two positive results on self-accessibility with asymmetric discounting. The first focuses on the self-accessibility of a *fixed* ball and uses the same parametrization of the discount factors as in Sugaya (2015) where δ_i is written as $1/(1 + k_i\theta)$ with $\boldsymbol{k}$ fixed.[18] This result will be used in the next section to examine which points are realizable. The second demonstrates the existence of *flexible*, $\boldsymbol{\delta}$-dependent self-accessible sets but places no restrictions on discount rate ratios (and hence, relative patience). It will be used in this section to support points inside $int(F^*)$.

Proposition 3. *Let $C \subset \mathbb{R}^n$ be finite, and let $X = co(C)$ be full-dimensional and contain in its interior the ball $B(\boldsymbol{o}, r)$ with $r > 0$. For any $\boldsymbol{k} \in \mathbb{R}^n_{++}$, there exists $\bar{\theta}(\boldsymbol{o}, r, \boldsymbol{k}) > 0$ such that for any $\theta \in (\,0,\, \bar{\theta}(\boldsymbol{o}, r, \boldsymbol{k})\,]$, the ball $B(\boldsymbol{o}, r)$ is self-accessible relative to C for any $\boldsymbol{\delta}$ satisfying $\delta_i = 1/(1 + k_i\theta)$ for each i. Furthermore, $\bar{\theta}(\boldsymbol{o}, r, \boldsymbol{k})$ is continuous in all its arguments.*

The technique for proving Proposition 3 is similar to that of Proposition 1 though neither follows directly from the other. In particular, note that the current proposition

[18] Normalizing k_1 to 1, the $\boldsymbol{k}$ vector captures the ratios of discount rates, i.e. *relative* patience among players. With that fixed, one can let θ tend to 0 so as to simultaneously make all players become *absolutely* very patient.

cannot handle affine balls - it must work with full-dimensional balls since, for arbitrary $\boldsymbol{k}$, the continuation payoff vector need not be in the original affine ball, no matter how small θ is. Also note that it is possible to provide a NLP that would enable us to compute $\bar{\theta}$ in the proposition. The details have been omitted and are available from the authors on request. Our next proposition starts out with a ball that is self-accessible for the equal discount factor $\underline{\delta}$, and then for any discount factor vector $\boldsymbol{\delta}$ where each $\delta_i \geqslant \underline{\delta}$, proposes a new, ellipsoidal self-accessible set.[19]

Proposition 4. *Let $C \subset \mathbb{R}^n$ be finite, and let $X = co(C)$ be full-dimensional and contain in its interior the ball $B(\boldsymbol{o}, r)$ with $r > 0$. Assume that $B(\boldsymbol{o}, r)$ is self-accessible realtive to C for the discount factor vector $\underline{\delta}\boldsymbol{\iota}$ where $\underline{\delta} \in (0,1)$. For any $\boldsymbol{\delta}$ such that $\delta_i \in [\underline{\delta}, 1)$ for $i = 1, \ldots, n$, there exists an ellipsoid $E(\boldsymbol{o}, r, \boldsymbol{\delta}, \underline{\delta}) \subset B(\boldsymbol{o}, r)$ given by center $\boldsymbol{o}$ and semi-axes lengths $\frac{1-\delta_i}{1-\underline{\delta}} r$ such that $E(\boldsymbol{o}, r, \boldsymbol{\delta}, \underline{\delta})$ is self-accessible relative to C for $\boldsymbol{\delta}$.*

The proposition implies that for any $\boldsymbol{v} \in E(\boldsymbol{o}, r, \boldsymbol{\delta}, \underline{\delta})$, there is a sequence of pure actions that realizes $\boldsymbol{v}$ when the discount factor vector $\boldsymbol{\delta}$ is used; we let $\{\boldsymbol{a}^{(t)}(\boldsymbol{v}, E(\boldsymbol{o}, r, \boldsymbol{\delta}, \underline{\delta}))\}_{t=0}^{\infty}$ denote that sequence.

5.3 Supporting Points in $int(F^*)$

We now present a constructive folk theorem with asymmetric discounting and without PRDs for points in $int(F^*)$. The proof of this result uses Proposition 4 and requires only minor re-writing of the proof of Theorem 1. The self-accessibility led approach thus allows us to provide a unified treatment of folk theorems with or without PRDs and with or without symmetric discounting (as long as we are interested in points in $int(F^*)$).

Theorem 2. *Let F^* be full-dimensional. For any $\boldsymbol{v} \in int(F^*)$, let $\underline{\delta}$ be the discount factor bound computed in Theorem 1 such that $\boldsymbol{v}$ can be supported as SPNE when players use the discount factor vector $\delta\boldsymbol{\iota}$ with $\delta \geqslant \underline{\delta}$. Then, $\boldsymbol{v}$ is also an SPNE payoff when the discount factor vector $\boldsymbol{\delta} \geqslant \underline{\delta}\boldsymbol{\iota}$ is used. As in Theorem 1, all paths on and off equilibrium are recursively computable and punishment (minmaxing) periods are independent of the discount factors.*

We remark that the unbridled freedom to have players with any configuration of relative patience works only for supporting points inside F^*; it may not work for points outside it, as the analysis in Section 7 will show. The Fact below extends Theorem 2; its justification follows from the proofs of Theorems 1 and 2 as well as the rich continuity properties that discount factor bounds are endowed with in Theorem 1 thanks to self-accessibiliy.

Fact 1. *Let F^* be full-dimensional and let $\boldsymbol{u}$ and $r > 0$ be such that $B(\boldsymbol{u}, r) \in int(F^*)$. Then there exists a uniform $\underline{\delta} > 0$ such that if $\boldsymbol{\delta}$ is such that $\delta_i \geqslant \underline{\delta}$ for each i, every*

[19] We owe a special debt to Costas Cavounidis for pointing us in the direction of this result and suggesting how it can improve a previous version of Theorem 2.

$\boldsymbol{v} \in B(\boldsymbol{u}, r)$ is a SPNE payoff for $\boldsymbol{\delta}$. Hence, for a fixed $\boldsymbol{k}$ vector, there exists a bound $\bar{\theta}$, such that if $\theta \leqslant \bar{\theta}$, and $\delta_i = 1/(1 + k_i\theta)$, every $\boldsymbol{v} \in B(\boldsymbol{u}, r)$ is an SPNE payoff.

6 Going Beyond the Feasible Set: Realizability

In non-cooperative game theory, contracts are assumed to be unenforceable, which is why we are only interested in equilibrium outcomes, i.e. outcomes that respect incentive-compatibility restrictions. However, if contracts are actually enforceable, identifying the set of realizable payoffs becomes important, especially if it is a bigger set than the set of equilibrium payoffs. But typically, that is indeed the case when discounting is asymmetric, even in the limit as all discount factors tend to one, without being necessarily equal. We now present a result that shows that if we are allowed to choose the players' discount rate ratios, then, provided these players are patient enough, in the repeated game they can *simultaneously* get in a realizable payoff virtually anything they can individually get in the stage game feasible set.

Theorem 3. *Let F be full-dimensional. Then*

$$\bigcup_{\boldsymbol{\delta} \in (0,1)^n} \mathcal{F}(\boldsymbol{\delta}) \supset int(re(F))$$

More precisely, given any payoff vector $\boldsymbol{v} \in int(re(F))$ we can compute positive numbers $k_1, \ldots, k_n$ (not necessarily unique) and a $\bar{\theta}$ such that if $\delta_i = \frac{1}{1+k_i\theta}$ with $\theta < \bar{\theta}$, then, $\boldsymbol{v} \in \mathcal{F}(\boldsymbol{\delta})$. Furthermore, for any given payoff vector $\boldsymbol{u} \in int(F)$, and any $\nu > 0$, one can ensure that along the path that realizes $\boldsymbol{v}$, after a finite sequence of actions, all continuation payoff vectors lie in a ν-neighborhood of $\boldsymbol{u}$.

To obtain some intuition behind this result, suppose, $n = 2$ and we are examining the realizability of $\boldsymbol{v} \in int(re(F))$. If for some $\boldsymbol{k}$, and low enough θ, after playing a certain sequence of vertices initially, the next continuation payoff (needed to realize $\boldsymbol{v}$) enters a small fixed ball $B(\boldsymbol{u}, r)$ inside $int(F)$, then we are done thanks to Proposition 3, because that proposition shows that *any* payoff in that ball is realizable for *any* $\boldsymbol{k}$ and low enough θ. What kind of vertices should the players play? Of course, vertices that nudge the continuation payoff from $\boldsymbol{v}$ to $\boldsymbol{u}$. For $n = 2$, there is a vertex $\boldsymbol{c}^{(1)}$ that nudges Player 1's payoff in the right direction ($c_1^{(1)}$ being on the opposite side of v_1 from u_1) and there is another vertex $\boldsymbol{c}^{(2)}$ which does the same for Player 2. A potential pitfall is that while we are nudging one player's payoff in the 'right' direction, the other player's payoff could be moving in the 'wrong' direction. One can show that by playing $\boldsymbol{c}^{(1)}$ for T_1 periods and $\boldsymbol{c}^{(2)}$ for T_2 periods where T_1 and T_2 are suitably chosen functions of θ and by choosing $\boldsymbol{k}$, we can indeed achieve our goal (for θ small). This involves solving a nonlinear simultaneous equation system involving the limiting trajectory (as θ tends to 0). Once this is done, the n-player version of the problem can be recursively tackled.

For $n = 2$ (but not for $n \geqslant 3$!), Theorem 3 can be sharpened as follows.

Theorem 4. *For two-player games, if F is full-dimensional,*

$$\bigcup_{\boldsymbol{\delta}\in(0,1)^2} \mathcal{F}(\boldsymbol{\delta}) = int(re(F)) \bigcup F.$$

7 Going Beyond the FSIR Set: Supportability

The $n+1$ phase path described in the proof of Theorem 3 unfortunately does not guarantee that the continuation payoff for each player at each period satisfies individual rationality, even when the target payoff $\boldsymbol{v}$ is strictly individually rational and the eventual continuation payoff is some point in F^*. We need additional conditions on $\boldsymbol{v}$.

A permutation π is a 1-1 correspondence between I and itself; by π_i we mean $\pi(i)$. We represent π by simply stating the vector $(\pi_1, \ldots, \pi_n)$. The permutation $(1, \ldots, n)$ is called the 'natural permutation' or 'natural order'. An interpretation of these permutations is now suggested: π simply maps the ranks of discount factors into player 'names'; thus, in a 5-player game, if $\pi_2 = 4$, that means player 4's discount factor is the second-lowest. The inverse function maps names to the ranks; thus if $\pi^{-1}(j) \leqslant \pi^{-1}(i)$, we understand that player i is at least as patient as player j. This interpretetion will be useful to keep in mind for the two definitions to follow; the proofs of Theorem 5 and 6 will later validate this interpretetion.

7.1 The Diagonal Conditions

In the definitions below C is any set of points in $\mathbb{R}^n$, $F = co(C)$ and $\boldsymbol{w}$ is some point in $re(C)$. Of course, these objects have their standard interpretations in the context of a game: pure action payoffs, feasible set and minmax point.

Definition A payoff vector $\boldsymbol{v} \in re(C)$ is said to satisfy the weak diagonal condition (WD) if there exists a permutation π, such that $\forall\, i$, $\exists$ a vector $\boldsymbol{u}^i \in F$ with the property

$$u^i_{\pi_i} = v_{\pi_i}$$

and

$$u^i_j \geqslant w_j \quad \text{if } \pi^{-1}(j) \leqslant \pi^{-1}(i).$$

Definition A payoff vector $\boldsymbol{v} \in int(re(C))$ is said to satisfy the strict diagonal condition (SD) if there exists a permutation π, such that $\forall\, i$, $\exists$ a vector $\boldsymbol{u}^i \in int(F)$ with the property

$$u^i_{\pi_i} = v_{\pi_i}$$

and

$$u^i_j > w_j \quad \text{if } \pi^{-1}(j) \leqslant \pi^{-1}(i).$$

When $\boldsymbol{v}$ satisfies the first (second) definition we say that it satisfies WD (SD) in the order π. Also, the set of all points satisfying WD (SD) for a given π will be denoted as $W(\pi)(S(\pi))$. Note that full dimensionality of F^* is needed for $S(\pi)$ to be non-empty, but not so for $W(\pi)$. As for the relation between these sets, it is easy to see that for any π, $cl(S(\pi)) \subset W(\pi)$ and assuming full-dimensionality of F^*, $int(W(\pi)) = S(\pi)$.

In a 2-player game, $\boldsymbol{v}$ satisfies WD in some order if it is weakly individual rational and if, from that point, we can draw a line parallel to one of the axes and make that line intersect F^+. This implies that the the darkly shaded region (including its boundary) in Figure 2 in Section 3.2 is the set of points that satisfy WD for some permutation. Specifically, for example, the point $\boldsymbol{v} = (8, 5)$ satisfies WD in the natural order because $\boldsymbol{u}^1$ can be chosen to be $(8, 1)$ and $\boldsymbol{u}^2$ can be chosen to be $(5, 5)$. For games with 3 or more players, the following Fact, stated without proof, is helpful in discerning which points will satisfy WD or SD (for some order).

Fact 2. *$\boldsymbol{v}$ satisfies WD for permutation π iff $v_i \in [\,\underline{v}_i(\pi), \bar{v}_i(\pi)]$ $\forall i = 1, \ldots, n$, where for any given permutation π, $\underline{\boldsymbol{v}}(\pi)$ and $\bar{\boldsymbol{v}}(\pi)$ are defined as follows:*

$$\underline{v}_i(\pi) := \min\{v_i : \boldsymbol{v} \in F \text{ such that } v_j \geqslant w_j \ \ \forall j \ \text{ such that } \pi^{-1}(j) \leqslant \pi^{-1}(i)\}.$$

$$\bar{v}_i(\pi) := \max\{v_i : \boldsymbol{v} \in F \text{ such that } v_j \geqslant w_j \ \ \forall j \ \text{ such that } \pi^{-1}(j) \leqslant \pi^{-1}(i)\}.$$

Example (The "stand-out" game): We present a three-player game that illustrates Fact 2's usefulness. In this "stand-out game", each of three players can play two actions: R (right) or W (wrong). All players get 0 if 0, 2 or 3 players play R. Only if exactly one player plays R, that player gets $1 + 2\eta$ while each of the other two players receive $-\eta$ where η is some positive number. Thus, $C = \{(0, 0, 0), (1 + 2\eta, -\eta, -\eta), (-\eta, 1 + 2\eta, -\eta), (-\eta, -\eta, 1 + 2\eta)\}$. In this game the only way to obtain the superior payoff of $1 + 2\eta$ is to uniquely 'stand out' (by doing the right thing) whereupon the other players are 'shamed' into getting a negative payoff of $-\eta$. Note that for each player, R weakly dominates W. However, if all players play this weakly dominant strategy the total utility is 0, while each of the three action vectors $(R, W, W), (W, R, W)$ and (W, W, R) gives the players a total utility of 1. Each player can be minmaxed by the other two players playing R, and hence $\boldsymbol{w} = (0, 0, 0)$.

Since $\boldsymbol{w}$ is in the feasible set, clearly $\underline{v}_i(\pi) = 0$ $\forall i$ and $\forall \pi$. Now notice that the Pareto frontier in the feasible set is $co(\{(1 + 2\eta, -\eta, -\eta), (-\eta, 1 + 2\eta, -\eta), (-\eta, -\eta, 1 + 2\eta)\})$ and all points on this frontier belong to the same plane as that of the unit simplex. Hence, if we were to maximize one player's payoff in F while giving the other two a non-negative payoff, we can give him at most 1. If we were to maximize his payoff while giving one of the others at least 0 and putting no other restriction on the third, apart from requiring that we stay inside F, we can give him at most $1 + \eta$. Finally, if we tried to maximize this player's payoff without any restriction on the other two's payoffs apart from requiring that we stay in F, we can give him at most $1+2\eta$. Hence, for the natural permutation π, $\bar{\boldsymbol{v}}(\pi) = (1+2\eta, 1+\eta, 1)$ and $W(\pi) = [0, 1+2\eta] \times [0, 1+\eta] \times [0, 1]$.

We end our discussion of the diagonalizability conditions by noting that a vector's diagonalizability (weak or strict) is 'inherited' by its subvectors in a sense made precise by the Fact below, stated without proof. It will be used in the proof of Theorem 6.

Fact 3. *If $\boldsymbol{v}$ satisfies weak or strict diagonalizability under the natural order with respect to C (given the n-dimensional null vector as $\boldsymbol{w}$), then $\boldsymbol{v}_{-n}$ satisfies the same with respect to $Proj_{-n}(C)$ (using the $n-1$ dimensional null vector as $\boldsymbol{w}$).*

7.2 Necessity and Weak Diagonalizability

Theorem 5. *Consider an n-player game and let $\mathcal{P}$ denote the set of all permutations of $1, \ldots, n$. Then,*

$$\bigcup_{\pi \in \mathcal{P}} W(\pi) \supset \bigcup_{\boldsymbol{\delta} \in (0,1)^n} \mathcal{V}(\boldsymbol{\delta})$$

i.e., for any n-player game if $\boldsymbol{v}$ is a SPNE payoff vector for some discount factor vector $\boldsymbol{\delta}$, then $\boldsymbol{v} \in W(\pi)$ for some permutation π.

Note that no full-dimensionality assumption is required in the above theorem. Its proof proceeds by asking the counterfactual question: What would the payoff profile look like if the players played the proposed equilibrium path but all of them used one of the player's discount factor to evaluate their normalized payoff? It turns out that the $\boldsymbol{u}$'s needed for the WD property fall right out of these freshly evaluated payoff vectors.

Going back to the Stand-out game, Theorem 5 tells us that any payoff vector that does not lie in the union of the six cubes of the form $[0, a] \times [0, b] \times [0, c]$ where a, b, c are some arrangements of the 3 numbers $1, 1+\eta$ and $1+2\eta$, can never be an equilibrium payoff. Thus, for example, if $\eta = .5$, we can conclude that the payoff vector $(.5, 1.7, 1.9)$ cannot be an equilibrium payoff vector because two of its coordinates lie above 1.5 - it does not matter what the other coordinate is. However, this payoff vector is realizable by virtue of Theorem 3. The proof of Theorem 5 also rules out arbitrary associations of equilibrium payoffs with relative ordering of patience among the players; for example, although the payoff vector $(1.9, 1.4, .9)$ is an equilibrium payoff vector, it is conformable with one and only one ranking of player patience: the one given by the natural order.

7.3 Sufficiency and Strict Diagonalizability

Weak diagonalizability, however, cannot be a sufficient condition for a payoff vector to be an equilibrium payoff vector, even when we have full dimensionality of the FSIR set. This is easy to see - just consider the two player game with the given payoff matrix:

	L	R
U	$(1,0)$	$(0,0)$
D	$(0,0)$	$(0,1)$

The payoff vector $(1,1)$ belongs to $W(\pi)$ for each π, but it is not even realizable let alone be supportable for any discount factor vector. Strict diagonalizability, however, 'works'.

Theorem 6. *Consider an n-player game with a full-dimensional FSIR set. Let $\mathcal{P}$ denote the set of all permutations of $1,\ldots,n$. Then,*

$$\bigcup_{\pi\in\mathcal{P}} S(\pi) \subset \bigcup_{\boldsymbol{\delta}\in(0,1)^n} \mathcal{V}(\boldsymbol{\delta})$$

More specifically, if $\boldsymbol{v} \in S(\pi)$ for some π, it is possible to determine $\boldsymbol{k} >> \mathbf{0}$ (not necessarily uniquely), and $\bar{\theta} > 0$ such that if $\theta \in (0,\bar{\theta})$ and $\delta_i = \frac{1}{1+k_i\theta}$ for each i, it is possible to specify a SPNE strategy profile that supports $\boldsymbol{v}$ for those discount factors.

The proof of Theorem 6 uses several complex ideas; hence, for the reader's benefit, we now provide the intuition behind our methods in some detail.

We will borrow a technique from the proof of Theorem 3: Devise discount factor parameters and an action sequence so as to create a path for the continuation payoffs that will start with the target payoff and will end inside a suitable set (hereafter we will call this the pre-entry path). In the case of Theorem 3 the set we wished to enter was $int(F)$, while in this case it would be $int(F^*)$. Once we are able to enter $int(F^*)$, Fact 1 takes over. There are two new challenging tasks here: first to create a pre-entry path that stays strictly individual rational throughout and second, to design an equilibrium strategy that takes care of incentive compatibilty along the pre-entry play.

Where strict diagonalizability helps is with the first task. To see this for the two-player case, suppose our target payoff $\boldsymbol{v}$ is in $S(\pi)$ where π is the natural order. Then, strict diagonalizability of $\boldsymbol{v}$ guarantees us the existence of a vector $\boldsymbol{u}^2$ in $int(F^*)$ where $\boldsymbol{u}_2^2 = v_2$. Also, there must exist a vertex $\boldsymbol{c}$ such that $sgn(v_1 - c_1) = sgn(u_1^2 - v_1)$. Each time this vertex is played the continuation payoff of Player 1 shifts towards u_1^2. If we play this vertex T times where T is $\lceil \frac{b}{\theta} \rceil$), by choosing a suitable $b > 0$ and θ small, limiting arguments used in the proof of Theorem 3, show us that it is possible to take the first coordinate of the continuation payoff arbitraily close to u_1^2. On top of this if k_2 is chosen small, the second coordinate of the continuation payoff barely changes from v_2 during these T periods and the continuation payoff vector after Tperiods gets close to $\boldsymbol{u}^2$ and hence, enters $int(F^*)$.

For three or more players, unfortunately, extending this idea runs into difficulties. Suppose $n = 3$ and $\boldsymbol{v} \in S(\pi)$ for the natural order π. Now, there is a point in $int(F^*)$, namely $\boldsymbol{u}^3$, the third element of which is v_3. A small neighborhood of this point suggests itself as the location of entry into $int(F^*)$. This is what we might want to do to accomplish our goal: while keeping Player 3 extremely patient throughout, first we 'fix' Player 1's payoff by moving it to u_1^3 (via the play of some suitable vertex). We keep Player 2 patient relative to Player 1, so that his payoff does not fall below his minmax during this first phase. Next, we fix Player 2's payoff to u_2^3 by playing another suitable vertex and try and enter $int(F^*)$. The problem with this strategy is that when we are trying to fix Player 2's payoff, Player 1 can find his payoff getting 'unfixed'! We could try and fix both players' payoffs simultaneously following the method used in the proof of Theorem 3, but then, we would not have any guarantee that the path will maintain individual rationality for these players even though $v_1 > 0, v_2 > 0, u_1^3 > 0$ and $u_2^3 > 0$.

We, therefore, need a new tool: how to move one strictly individually rational payoff vector to another's neighborhood without violating (strict) individual rationality along the path. This can indeed be done as long as the two payoff vectors are both inside $int(F)$. Our next proposition, called The Capsule Lemma shows exactly how it can be done and also provides an upper bound on the number of periods needed to do it. Its proof offers the remarkable insight that the notion of self-accessibility is not just useful for keeping continuation payoffs tethered to a point; it can also be used to take them for a 'walk' inside $int(F^*)$!

In m-dimensional Euclidean space, let $[\boldsymbol{y}, \boldsymbol{z}]$ represent the line segment joining the points $\boldsymbol{y}$ and $\boldsymbol{z}$ and let $\mathcal{C}(\boldsymbol{y}, \boldsymbol{z}, r)$ denote the set of points that are at most distance r from $[\boldsymbol{y}, \boldsymbol{z}]$, i.e.

$$\mathcal{C}(\boldsymbol{y}, \boldsymbol{z}, r) = \bigcup_{\boldsymbol{o} \in [\boldsymbol{y}, \boldsymbol{z}]} B(\boldsymbol{o}, r)$$

From now on, we will refer to sets of these types as 'capsules'.

Proposition 5. (*The Capsule Lemma*) *Let D be a finite set in m-dimensional Euclidean space. For $\boldsymbol{y}, \boldsymbol{z}$ in $int(co(D))$, let $r > 0$ be such that the capsule $\mathcal{C}(\boldsymbol{y}, \boldsymbol{z}, r)$ is in the interior of $co(D) \cap \mathbb{R}^m_{++}$, which is assumed to be m-dimensional. Then,*

a) For any $\boldsymbol{u} \in B(\boldsymbol{y}, r)$ and any vector $\boldsymbol{k} >> \boldsymbol{0}$, there exists $\theta^\dagger(\boldsymbol{k}) > 0$ such that for any $\theta \in (0, \theta^\dagger(\boldsymbol{k}))$ we can find a finite sequence of points $\{\boldsymbol{c}^t\}_{t=0}^{T-1}$ in D and a finite sequence of points $\{\boldsymbol{x}^t\}_{t=0}^{T}$ in $\mathcal{C}(\boldsymbol{y}, \boldsymbol{z}, r)$ such that
i) $\boldsymbol{x}^0 = \boldsymbol{u}$,
ii) $\boldsymbol{x}_i^{t+1} - \boldsymbol{c}_i^t = (1 + k_i\theta)(\boldsymbol{x}_i^t - \boldsymbol{c}_i^t)$ for $i = 1, \ldots, m$, $t = 0, \ldots, T-1$ and
iii) $\boldsymbol{x}^T \in B(\boldsymbol{z}, r)$.

b) Furthermore, there exists a $\theta^\ddagger(\boldsymbol{k}) \leqslant \theta^\dagger(\boldsymbol{k})$, a strictly positive numbers m_1 and a strictly negative number m_2 (both dependent on $\boldsymbol{k}$) such that when $\theta \in (0, \theta^\ddagger)$, the T given in part a) is less than or equal to $\left\lceil \frac{||y-z||}{r - \sqrt{m_1\theta^2 + m_2\theta + r^2}} \right\rceil$.

Theorem 6's proof uses a combination of strict diagonalizability, Capsule Lemma and Fact 3 to design a (continuation payoff) path that starts at the target payoff $\boldsymbol{v}$ and ends inside $int(F^*)$. We cannot effect this transition with one application of the Capsule Lemma because for that to work, the entire pre-entry path needs to be inside $int(F)$ which of course we may not assume. However, we *can* use the Capsule lemma to first change only Player 1's payoff, then Player 1 and 2's payoffs, then Player 1, 2 and 3' s payoffs etc., each time operating inside the interior of F^* of the relevant subset of players. The strict diagonalizability condition provides us the 'anchors' which serve as landmarks for the pre-entry path around which our capsules are fashioned. It is then shown that with appropriate choice of the k_i's and θ small, we can indeed enter $int(F^*)$ while keeping each continuation payoff vector strictly individual rational all along the pre-entry path.

The equilibrium strategy now can be qualitatively described. Play starts and continues on the pre-entry action path until one of the players unilaterally deviates whereupon he is minmaxed by the other players for a certain number of periods. *After minmaxing, play 'returns' to the same point on the pre-entry action path where the deviation took place.* At the completion of the pre-entry path, players play an action sequence corresponding to a point which shifts the target post-entry equilibrium continuation payoff by a) an 'adjustment' term to make any punishing player indifferent among his minmaxing pure strategies, and b) a 'reward' to them for participating in the punishment. During pre-entry, any deviation from a punishment phase or a new deviation after play has returned to the 'pre-entry' path is treated as if a fresh deviation just took place.

Why does this "stick now (for the deviant), carrot later (for the punisher)" strategy ensure incentive compatibility for patient players? With rising patience, bad outcomes for a finite number of periods still wipes out one period gains (if nothing else changes). That is how the 'stick now' threat keeps players from deviating. The prospective punishers may also suffer during the punishment periods but they are compensated by a reward coming later. Now it is true that as they become more patient, the time taken to to get one's reward also grows infinitely large, but the Capsule Lemma ensures that these times don't grow too fast, and in fact, the PDV of the reward tends to a limit. With finite punishment periods, this ensures that patient players do not balk from punishing a deviant player.

8 Conclusion

This paper provides a unified treatment of discounted repeated games with perfect monitoring and without PRDs. The scope of our inquiry follows a logical chain, successively allowing for wider target payoff sets and less restrictive discounting structures. The glue that holds all the results together is the simple geometrical notion: self-accessibility. The analysis culminates in Theorem 6, where we show that any point $\boldsymbol{v}$ satisfying the Strict Diagonal Condition is an SPNE payoff for some possibly asymmetric discounting profile. This easy to check condition translates into the following: there exists an ordering π of the players such that for player $\pi(i)$ there is a payoff vector in the interior of the feasible set at which $\pi(i)$ gets the payoff $v_{\pi(i)}$ and everyone before $\pi(i)$ in the ordering gets more than their respective minmax values. Our result can be viewed as a new folk theorem for repeated games with unrestricted discounting patterns that is built on fully constructive foundations.

Appendix: All Proofs

Proof of Proposition 1

Existence of $\underline{\delta}$: We assume wlog that the center of the ball is the origin. Also wlog, we assume that $C' = \{\boldsymbol{c}^1, \dots \boldsymbol{c}^K\}$ is in fact, the set of extreme points of X.

Fix $\boldsymbol{c} \in C'$, $\boldsymbol{x} \in S$ and let $\delta(\boldsymbol{x}, \boldsymbol{c})$ be defined as the solution of the following problem[20]:

$$\text{Min } \delta \in [0,1] \text{ subject to } \boldsymbol{x} = (1-\delta)\boldsymbol{c} + \delta\, \boldsymbol{y}, \text{ for some } \boldsymbol{y} \in S.$$

We now characterize $\delta(\boldsymbol{x}, \boldsymbol{c})$. Note that if $\boldsymbol{y}$ satisfies the equation $\boldsymbol{x} = (1-\delta(\boldsymbol{x}, \boldsymbol{c}))\boldsymbol{c} + \delta(\boldsymbol{x}, \boldsymbol{c})\, \boldsymbol{y}$, then $\boldsymbol{y}$ must be at the boundary of the ball. Hence, it must be that $\boldsymbol{y}.\boldsymbol{y} = r^2$ which implies $\boldsymbol{x} - (1-\delta(\boldsymbol{x}, \boldsymbol{c}))\boldsymbol{c}$'s dot product with itself is $\delta^2 r^2$. Upon rearranging this shows that $\delta(\boldsymbol{x}, \boldsymbol{c})$ must be a root of the following quadratic equation in δ :

$$\delta^2(\boldsymbol{c}.\boldsymbol{c} - r^2) + 2\delta\boldsymbol{c}.(\boldsymbol{x} - c) + (\boldsymbol{x} - \boldsymbol{c}).(\boldsymbol{x} - \boldsymbol{c}) = 0. \tag{8.1}$$

For any $\boldsymbol{c}$, given that the left hand side of (8.1) is a *convex* quadratic (since $\boldsymbol{c}.\boldsymbol{c} - r^2 > 0$), with a strictly positive value at 0 (since $(\boldsymbol{x} - \boldsymbol{c}).(\boldsymbol{x} - \boldsymbol{c}) > 0$) and a non-positive value at 1 (since $\boldsymbol{x}.\boldsymbol{x} - r^2 \leqslant 0$), there are two roots: one is greater than 0 and less than or equal to 1 while the other is greater than or equal to 1. We are seeking the smaller root, which is continuous in $\boldsymbol{x}$, making $\delta(\boldsymbol{x}, \boldsymbol{c})$ continuous in $\boldsymbol{x}$.

Furthermore, we assert that for one of the $\boldsymbol{c}$'s, the smaller root must be *strictly* less than 1. Clearly this will be true if the quadratic at 1 is strictly negative, i.e. $\boldsymbol{x}.\boldsymbol{x} - r^2 < 0$. So assume that $\boldsymbol{x}.\boldsymbol{x} = r^2$. It now suffices to show that the slope of the quadratic at 1 is strictly positive for some $\boldsymbol{c}$, which will be true if for at least one l, $\boldsymbol{c}^l.\,\boldsymbol{x} > r^2$. If not, then

$$\boldsymbol{c}^l.\,\boldsymbol{x} \leqslant r^2 \quad \text{for } \; l = 1, \dots, K \tag{8.2}$$

If that is the case, we claim that each of these inequalities must actually be an equality. To see this note that since $\boldsymbol{x}$ is in the relative interior of X, $\boldsymbol{x} = \sum_{l=1}^K \lambda^l\, \boldsymbol{c}^l$ where λ^l's are strictly positive weights summing to 1. Multiplying each inequality in (8.2) by λ^l and summing over l, on the left hand side we will have $\left(\sum_{l=1}^K \lambda^l \boldsymbol{c}^l\right).\,\boldsymbol{x} = \boldsymbol{x}.\boldsymbol{x}$ while on the right hand side we will have $\left(\sum_{l=1}^K \lambda^l\right) r^2 = r^2$ and since these two are equal, the claim follows. But now, if for each l, $\boldsymbol{c}^l.\,\boldsymbol{x} = r^2$, since the center of the ball (the origin) can also be expressed as a convex cobination of the vertices, i.e. $\boldsymbol{o} = \sum \theta^l \boldsymbol{c}^l$ for a set of weights θ^l summing to one, this will imply $\boldsymbol{o}.\,\boldsymbol{x} = 0 = r^2$, a contradiction. Hence, for every $\boldsymbol{x} \in B_X(\boldsymbol{0}, r)$, there exists a vertex $\boldsymbol{c}$ such that the quadratic in (8.1) has a strictly positive slope at 1, and hence for that vertex, $\delta(\boldsymbol{x}, \boldsymbol{c}) \in (0,1)$.

Let $\delta^*(\boldsymbol{x}) := \min\{\delta(\boldsymbol{x}, \boldsymbol{c}) : \boldsymbol{c} \in C'\}$, with the minimum attained at $c^*(\boldsymbol{x})$.[21]Clearly, δ^* is continuous, being the minimum of continuous functions and lies in $(0,1)$. Finally, define $\underline{\delta} := \max\{\delta^*(\boldsymbol{x}) : \boldsymbol{x} \in S\}$. Since S is compact, this maximum is attained at some $\boldsymbol{x}^*$. Since for any $\boldsymbol{x}$, $\delta^*(\boldsymbol{x}) \in (0,1)$ we must have $\underline{\delta} \in (0,1)$. It is now easily verifiable that for this $\underline{\delta}$ and any common discount factor above this value S is self-accessible relative to C'.

[20]For the first part of the proof, to keep the notation simple, when a function depends on the location of the ball, we will drop the center and the radius as arguments (thus, for example $\delta(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, r)$ will be simply referred to as $\delta(\boldsymbol{x}, \boldsymbol{c})$).

[21]To achieve well-definition, in case of ties, use any arbitrary preference ordering among the vertices.

Computability of $\underline{\delta}$: Having described the problem of determining $\underline{\delta}(\boldsymbol{o}, r)$ as the nested problem

$$\underset{\boldsymbol{x} \in B_X(\boldsymbol{o}, r)}{\text{Max}} \quad \underset{\boldsymbol{c} \in C'}{\text{Min}} \quad \delta(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, r)$$

we note that the 'inside' problem can be stated as a maximization, rather than a minimization problem as shown:

$$\begin{aligned} &\text{Max} \quad \delta \\ &\textit{subject to} \\ &\delta \leqslant \delta(\boldsymbol{x}, \boldsymbol{c}^l, \boldsymbol{o}, r) \quad \forall \boldsymbol{c}^l \in C'. \end{aligned}$$

For affine balls $B_X(\boldsymbol{o}, r)$ in the relative interior of X, an explicit formula for $\delta(\boldsymbol{x}, \boldsymbol{c}^l, \boldsymbol{o}, r)$ exists in terms of the smaller root of equation (8.1); a numerically simpler way to characterize that root is to just require that the slope of the quadratic is less than or equal to zero, in addition to stating that the quadratic vanishes. For any given $\boldsymbol{o}$ (not necessarily the origin), this leads to the NLP below the solution of which gives us $\underline{\delta}(\boldsymbol{o}, r)$:

$$\begin{aligned}
&\text{Max} \quad \delta \\
&\textit{subject to} \\
&\delta \leqslant \delta^l \quad \forall\, l = 1, \ldots, K \qquad (1) \\
&(\delta^l)^2\{(\boldsymbol{c}^l - \boldsymbol{o}).(\boldsymbol{c}^l - \boldsymbol{o}) - r^2\} + 2\delta^l(\boldsymbol{c}^l - \boldsymbol{o}).(\boldsymbol{x} - \boldsymbol{c}^l) + \|\boldsymbol{x} - \boldsymbol{c}^l\|^2 = 0 \quad \forall\, l = 1, \ldots, K \qquad (2) \\
&(\delta^l)\{(\boldsymbol{c}^l - \boldsymbol{o}).(\boldsymbol{c}^l - \boldsymbol{o}) - r^2\} + (\boldsymbol{c}^l - \boldsymbol{o}).(\boldsymbol{x} - \boldsymbol{c}^l) \leqslant 0 \quad \forall\, l = 1, \ldots, K \qquad (3) \\
&(\boldsymbol{x} - \boldsymbol{o}).(\boldsymbol{x} - \boldsymbol{o}) \leqslant r^2 \qquad (4) \\
&\boldsymbol{x} = \textstyle\sum_{l=1}^K \lambda^l \boldsymbol{c}^l \qquad (5) \\
&\textstyle\sum_{l=1}^K \lambda^l = 1 \qquad (6) \\
&\lambda^l \geqslant 0 \quad \forall\, l = 1, \ldots, K \qquad (7).
\end{aligned}$$

In the NLP, constraints (2) and (3) characterize the $\delta(\boldsymbol{x}, \boldsymbol{c}^i, \boldsymbol{o}, r)$'s (for $i = 1, \ldots, K$) while constraint (1) finds the minimum of these. Of course, the minimization is also over $\boldsymbol{x}$ and constraints (4) - (7) ensure that each feasible $\boldsymbol{x}$ belongs to the affine ball $B_X(\boldsymbol{o}, r)$.[22] ■

Proof of Proposition 2

The function $\underline{\delta}(\boldsymbol{o}, r)$ defining a discount factor bound that makes $B_X(\boldsymbol{o}, r)$ self-accessible is continuous in both its arguments - as follows from the continuity of $\delta(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, r)$ in $(\boldsymbol{o}, r)$, the proof of Proposition 1 and a straightforward application of the Maximum Theorem. The set $B_X(\boldsymbol{x}, \bar{r})$ being compact (wherein the centers of the smaller balls must lie), the existence of the required

[22]Note that if the ball was full-dimensional, we could have dropped constraints (5) - (7).

uniform bound follows. Now we can state the required NLP.

$$\begin{aligned}
&\text{Max } \delta \\
&\textit{subject to} \\
&\delta \leqslant \delta^l \quad \forall\, i = 1, \ldots, K \qquad (1) \\
&(\delta^l)^2\{(\boldsymbol{c}^l - \boldsymbol{x}')(\boldsymbol{c}^l - \boldsymbol{x}') - \omega^2\} + 2\delta^l(\boldsymbol{c}^l - \boldsymbol{x}')(\boldsymbol{x}'' - \boldsymbol{c}^l) + \|\boldsymbol{x}'' - \boldsymbol{c}^l\|^2 = 0 \quad \forall\, l = 1, \ldots K \qquad (2) \\
&(\delta^l)\{(\boldsymbol{c}^l - \boldsymbol{x}')(\boldsymbol{c}^l - \boldsymbol{x}') - \omega^2\} + (\boldsymbol{c}^l - \boldsymbol{x}')(\boldsymbol{x}'' - \boldsymbol{c}^l) \leqslant 0 \quad \forall\, l = 1, \ldots, K \qquad (3) \\
&(\boldsymbol{x} - \boldsymbol{x}').(\boldsymbol{x} - \boldsymbol{x}') \leqslant \bar{r}^2 \qquad (4) \\
&(\boldsymbol{x}' - \boldsymbol{x}'').(\boldsymbol{x}' - \boldsymbol{x}'') \leqslant \omega^2 \qquad (5) \\
&\boldsymbol{x}' = \textstyle\sum_{l=1}^K \lambda^l \boldsymbol{c}^l \qquad (6) \\
&\boldsymbol{x}'' = \textstyle\sum_{l=1}^K \theta^l \boldsymbol{c}^l \qquad (7) \\
&\textstyle\sum_{l=1}^K \lambda^l = 1 \qquad (8) \\
&\textstyle\sum_{l=1}^K \theta^l = 1 \qquad (9) \\
&\lambda^l, \theta^l \geqslant 0 \quad \forall\, l = 1, \ldots, K \qquad (10)
\end{aligned}$$

The construction of the given NLP follows directly from the proof of Proposition 1 and the observation that the problem of finding a uniform bound as $\boldsymbol{x}'$ ranges in the set $B_X(\boldsymbol{x}, \bar{r})$ adds another layer of nesting to the optimization problem in that proposition, making the current problem representable as

$$\underset{\boldsymbol{x}' \in B_X(\boldsymbol{x}, \bar{r})}{\text{Max}} \quad \underset{\boldsymbol{x}'' \in B_X(\boldsymbol{x}', \omega)}{\text{Max}} \quad \underset{\boldsymbol{c} \in C'}{\text{Min}} \quad \delta(\boldsymbol{x}'', \boldsymbol{c}, \boldsymbol{x}', \omega).$$

■

Proof of Theorem 1

Because F^* is full-dimensional and $\boldsymbol{v} \in F^* \backslash \underline{\partial} F$, there exists $\boldsymbol{v}'$ such that it is in $int(F^*)$ and $\boldsymbol{v}' << \boldsymbol{v}$. First, we define two constants Δ and N that we need to define the equilibrium strategy.[23]

We choose $\Delta > 0$ such that

$$B(\boldsymbol{v}', 4\sqrt{n-1}\Delta) \subset int(F^*). \tag{8.3}$$

Further, if $\boldsymbol{v}$ is not a payoff vector associated with a pure strategy profile, and can be expressed as $\sum_{l=1}^K \lambda^l \boldsymbol{c}^l$ where each $\lambda^l > 0$, Δ should be small enough so that

$$B_X(\boldsymbol{v}, \Delta) \ \subset \ relint(X) \tag{8.4}$$

where $X = co\{\boldsymbol{c}^1, \ldots, \boldsymbol{c}^K\}$. Note that from (8.3), it follows that $v_i' > \Delta$. Now define $N \in \mathbb{N}$ by

$$N = \left\lceil \max_i \frac{M}{v_i' - \Delta} \right\rceil, \tag{8.5}$$

[23] 2Δ serves as the 'reward' for the punishers; while Δ serves as the radius of all self-accessible balls we will be dealing with in this proof. N is the number of minmaxing periods.

implying that $N+1 > M/(v_i' - \Delta)$ for all i. Let $\underline{\delta}_1$ be such that for $\delta \geqslant \underline{\delta}_1$,

$$\delta^N \geqslant \frac{N}{N+1}. \tag{8.6}$$

The last two inequalities guarantee the following inequality which will be critical later:

$$1 + \delta + \ldots + \delta^N \geqslant \frac{M}{v_i' - \Delta} \quad \forall i\ . \tag{8.7}$$

Next, given N, there is a $\underline{\delta}_2$, such that for $\delta \geqslant \underline{\delta}_2$,

$$\delta^N \geqslant \frac{M}{M+\Delta} \quad \forall i. \tag{8.8}$$

Note that this implies, for $\delta \geqslant \underline{\delta}_2$,

$$\delta^N \geqslant \frac{v_i'}{v_i' + \Delta} \geqslant \frac{v_i' - \Delta}{v_i' + \Delta} \quad \forall i. \tag{8.9}$$

If $\boldsymbol{v}$ is a payoff vector for a pure action profile, let $\underline{\delta}_3 = 0$. Otherwise, $B_X(\boldsymbol{v}, \Delta)$ referred to in (8.4) is self-accessible for all δ larger than some bound; let $\underline{\delta}_3$ be that bound computable via Proposition 2. Lastly define

$$\boldsymbol{v}'(i) := (v_1' + 2\Delta, \ldots, v_{i-1}' + 2\Delta,\ v_i',\ v_{i+1}' + 2\Delta, \ldots, v_n' + 2\Delta)$$

Consider the set of all (full-dimensional) balls the centers of which are at most $\sqrt{n-1}\ \Delta$ away from $\boldsymbol{v}'(i)$, each with radius Δ. It may be checked because of (8.3), each of these small balls are fully contained in the interior of F^*. Using Proposition 2, a uniform bound can be computed such that each of these balls is self-accessible when the common discount factor is as large as the bound. Call this bound δ_{4i}. Now define

$$\underline{\delta} = \max(\underline{\delta}_1,\ \underline{\delta}_2,\ \underline{\delta}_3,\ \max_i \underline{\delta}_{4i}). \tag{8.10}$$

For any discount factor δ exceeding this bound, $\boldsymbol{v}$ may be supported by a three-phase strategy which we now describe.

Phase I: If $\boldsymbol{v}$ is a pure action profile, play that pure action profile forever. Otherwise, play the action sequence $\{\boldsymbol{a}^{(t)}(\boldsymbol{v}, B_X(\boldsymbol{v}, \Delta), \delta)\}_{t=0}^{\infty}$. If there is a unilateral deviation by player i in Phase I, move to Phase II(i).

Phase II(i): For each of N periods play $\boldsymbol{m}^i$, the (possibly mixed) action profile that minmaxes i. If player j unilaterally deviates from this phase (i.e. he is observed to play an action that is not in the support of m_j^i), start Phase II(j). Otherwise, at the completion of this phase, go to Phase III(i).

Phase III(i): Let $\tilde{\boldsymbol{a}}^{(t)}$, $t = 1, \ldots, N$ be the realized actions during Phase II(i). In Phase III(i), play the sequence of actions given by $\{\boldsymbol{a}^{(t)}(\boldsymbol{v}'(i) - \boldsymbol{z}^i - \Delta \boldsymbol{e}^i, B(\boldsymbol{v}'(i) - \boldsymbol{z}^i, \Delta), \delta)\}_{t=0}^{\infty}$, where $\boldsymbol{z}^i$ is an adjustment vector defined by the following two equations

$$z_j^i = \begin{cases} \frac{(1-\delta^N)}{\delta^N} r_j^i & \text{if } j \neq i \\ 0 & \text{otherwise.} \end{cases} \tag{8.11}$$

$$r_j^i = \frac{(1-\delta)}{(1-\delta^N)} \sum_{t=1}^{N} \delta^{t-1}\ g_i(\tilde{\boldsymbol{a}}^{(t)}), \tag{8.12}$$

If there is any unilateral deviation from Phase III(i) by player j, start Phase II(j).

Inequality (8.8) implies $\frac{1-\delta^N}{\delta^N}\ M \leqslant \Delta$ and since, $|r_j^i| \leqslant M$, $|z_j^i| \leqslant \Delta$ for $j \neq i$. This implies that $\boldsymbol{v}^{'}(i) - \boldsymbol{z}^i$ is at most $\sqrt{n-1}\ \Delta$ away from $\boldsymbol{v}'(i)$. Hence, given the construction of $\underline{\delta}_{4i}$ earlier, $B(\boldsymbol{v}^{'}(i) - \boldsymbol{z}^i, \Delta)$ is indeed self-accessible for discount factors above that bound. Let us now examine conditions for player i's strategy to be unimprovable.

- For unimprovability from Phase I, it suffices to have

$$(1-\delta)M + \delta^{N+1}(v_i' - \Delta) \leqslant v_i - \Delta, \tag{8.13}$$

- For unimprovability from Phase II(i) with τ periods left in the phase, it suffices to have

$$0 + \delta^{N+1}(v_i' - \Delta) \leqslant 0 + \delta^{\tau}(v_i' - \Delta) \quad \text{for } \tau = 1, \ldots, N. \tag{8.14}$$

- For unimprovability from Phase III(i), it suffices to have

$$(1-\delta)M + \delta^{N+1}(v_i' - \Delta) \leqslant v_i' - \Delta. \tag{8.15}$$

- To analyze unimprovability from Phase II(j) we note that because of the adjustment term $\boldsymbol{z}$ in Phase IIIj's target point, player i is indifferent in Phase IIj between playing any action that is in the support of $\boldsymbol{m}^j$. The question is whether he wishes to play an action which is not in the support of $\boldsymbol{m}^j$. Letting $\{\tilde{\boldsymbol{a}}^{(t)}\}_{t=1}^N$ denote the sequence of actions that are realized in Phase II(j), with τ periods left in that phase, if player i knew the last τ entries of that sequence, the following inequality would deter deviation:

$$(1-\delta)M + \delta^{N+1}(v_i' - \Delta) \leqslant (1-\delta)[g_i(\tilde{\boldsymbol{a}}^{(N-\tau+1)}) + \ldots + \delta^{\tau-1} g_i(\tilde{\boldsymbol{a}}^{(N)})] + \delta^{\tau}(v_i' + 2\Delta - z_i^j). \tag{8.16}$$

  Using (8.11) and (8.12), the right hand side of (8.16) becomes:

$$\delta^{\tau}(v_i' + 2\Delta) - \frac{1-\delta}{\delta^{N-\tau}} \left(g_i(\tilde{\boldsymbol{a}}^{(1)}) + \ldots + \delta^{N-\tau-1} g_i(\tilde{\boldsymbol{a}}^{(N-\tau)}) \right), \tag{8.17}$$

  which is bounded from below by $\delta_i^{\tau} \left(v_i' + 2\Delta - \frac{1-\delta^N}{\delta^N} M \right)$. Hence by (8.8), inequality (8.16) is satisfied for any sequence of $\boldsymbol{a}$'s if

$$(1-\delta)M + \delta_i^{N+1}(v_i' - \Delta) \leqslant \delta^N \left(v_i' + \Delta \right). \tag{8.18}$$

- Lastly, for unimprovability from Phase IIIj it suffices to have:

$$(1-\delta)M + \delta^{N+1}(v_i' - \Delta) \leqslant v_i' + 2\Delta - z_i^j - \Delta \tag{8.19}$$

  for all possible values of z_i^j. Because $|z_i^j| \leqslant \Delta$, this is satisfied if the following holds:

$$(1-\delta)M + \delta^{N+1} v_i' \leqslant v_i'. \tag{8.20}$$

Examination of these conditions shows that while (8.14) is trivially true, (8.15) directly implies (8.13) and (8.20). Because of (8.9), it also implies (8.18). Thus to ensure all incentive compatibility conditions one only needs to satisfy equation (8.15), which is (8.7). Thus the given strategy profile is indeed an SPNE. ■

Proof of Proposition 3

Without loss of generality, we can and henceforth do discard points in C that are not extreme points of X, and relabel it as $C' = \{c^1, \cdots, c^{L'}\}$. For $\boldsymbol{x} \in B(\boldsymbol{o}, r)$, and $\boldsymbol{c} \in C'$ define a vector $\boldsymbol{y}(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, r, \boldsymbol{k}, \theta)$ where

$$y_i(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, r, \boldsymbol{k}, \theta) := \theta k_i(x_i - c_i) + x_i \quad \text{for } i = 1, \dots, n. \tag{8.21}$$

In terms of discount factors, the above is just $\frac{1}{\delta_i} x_i - \frac{1-\delta_i}{\delta_i} c_i$, i.e. it is the i'th coordinate of the'continuation point' given the target $\boldsymbol{x}$, the current action $\boldsymbol{c}$ and the discount factor vector $\boldsymbol{\delta}$. Let

$$\begin{aligned} f(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, r, \boldsymbol{k}, \theta) &:= ||\boldsymbol{y}(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, r, \boldsymbol{k}, \theta) - \boldsymbol{o}||^2 - r^2 \\ &= \sum_{i=1}^{n} (\theta\, k_i(x_i - c_i) + (x_i - o_i))^2 - r^2. \\ &= \theta^2 \sum_{i=1}^{n} k_i^2 (x_i - c_i)^2 + 2\theta \sum_{i=1}^{n} k_i(x_i - c_i)(x_i - o_i) + (\sum_{i=1}^{n} (x_i - o_i)^2 - r^2) \end{aligned} \tag{8.22}$$

Because of the full-dimensionality assumption, $f \leqslant 0 \implies \boldsymbol{y}(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, r, \boldsymbol{k}, \theta) \in B(\boldsymbol{o}, r)$. To prove the proposition, we will show that there exists $\bar{\theta}(\boldsymbol{o}, r, \boldsymbol{k}) > 0$ such that if $0 < \theta \leqslant \bar{\theta}(\boldsymbol{o}, r, \boldsymbol{k})$, then for every $\boldsymbol{x} \in B(\boldsymbol{o}, r)$, there exists a $\boldsymbol{c}$ such that $f \leqslant 0$.

The expression in (8.22) is a strictly convex quadratic in θ with $f(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, r, \boldsymbol{k}, 0) \leqslant 0$ (and so has at least one non-negative real root). Let $\theta(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, r, \boldsymbol{k})$ denote its larger root, which is continuous in $(\boldsymbol{x}, \boldsymbol{o}, r, \boldsymbol{k})$.

For $\boldsymbol{x} = \boldsymbol{o}$, for every $\boldsymbol{c}$, $f(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, r, \boldsymbol{k}, 0) = -r^2 < 0$ and hence, the larger root is strictly positive and hence, for every $\boldsymbol{c}$, $f \leqslant 0$ for $\theta \in (0, \theta(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, r, \boldsymbol{k})]$.

For $\boldsymbol{x} \neq \boldsymbol{o}$, if we can show that there exists a $\boldsymbol{c}$ such that $\frac{\partial f}{\partial \theta} < 0$ at $\theta = 0$ then we can assert that for that $\boldsymbol{x}$ there exists a $\boldsymbol{c}$, such that $\theta(\, \boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, r, \boldsymbol{k}) > 0$ and for $\theta \in (0, \theta(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, r, \boldsymbol{k})]$, f is non-positive. It suffices to show that that for every $\boldsymbol{x}$, there exists a $\boldsymbol{c}$ such that $\sum_{i=1}^{n} k_i(x_i - c_i)(x_i - o_i) < 0$, or

$$\sum_{i=1}^{n} k_i(x_i - o_i)x_i < \sum_{i=1}^{n} k_i(x_i - o_i)c_i \tag{8.23}$$

Consider the hyperplane $H = \{\boldsymbol{y} : \boldsymbol{p}\boldsymbol{y} = \alpha\}$, where $p_i = k_i(x_i - o_i)$ and $\alpha = \sum_{i=1}^{n} k_i(x_i - o_i)x_i$ (since $\boldsymbol{x} \neq \boldsymbol{o}$, $\boldsymbol{p}$ is a non-zero vector). If inequality (8.23) is false for every vertex $\boldsymbol{c}$, that would mean that every vertex lies on one side of the hyperplane, while clearly $\boldsymbol{x}$ is situated *on* that hyperplane. This can not be true since $\boldsymbol{x}$ is in a ball which lies in the *interior* of $co(C')$.

The foregoing analysis implies $\theta^*(\boldsymbol{x}, \boldsymbol{o}, r, \boldsymbol{k}) := \max_{\boldsymbol{c} \in C} \theta(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, r, \boldsymbol{k})$ is a strictly positive number, and is continuous in $\boldsymbol{k}, \boldsymbol{o}, r$ and $\boldsymbol{x}$ (being the maximum of continuous functions), and furthermore, for every $\boldsymbol{x} \in B(\boldsymbol{o}, r)$, if $\theta \in (0, \theta^*(\boldsymbol{x}, \boldsymbol{o}, r, \boldsymbol{k})]$, $f \leqslant 0$. Finally, define the required bound as

$$\bar{\theta}(\boldsymbol{o}, r, \boldsymbol{k}) := \min_{\boldsymbol{x} \in B(\boldsymbol{o}, r)} \theta^*(\boldsymbol{x}, \boldsymbol{o}, r, \boldsymbol{k}). \tag{8.24}$$

Since $B(\boldsymbol{o}, r)$ is compact, this minimum is achieved at some $\boldsymbol{x}$, is strictly positive-valued, and because of the Maximum Theorem is continuous in $\boldsymbol{o}$, r and $\boldsymbol{k}$. Clearly it satisfies the desired requirement of the bound. ■

Proof of Proposition 4

Define $r_i := \frac{1-\delta_i}{1-\underline{\delta}} r$ for each i. With these being the lengths of the semi-axes of the desired ellipsoid, the latter can be written as: $E(\boldsymbol{o}, r, \boldsymbol{\delta}, \underline{\delta}) = \{\boldsymbol{x} : \sum_{i=1}^n \frac{(x_i - o_i)^2}{r_i^2} \leqslant 1\}$. It is easy to see that $E(\boldsymbol{o}, r, \boldsymbol{\delta}, \underline{\delta})$ is contained in $B(\boldsymbol{o}, r)$ and can be written as $f(B(\boldsymbol{o}, r))$ where f is a 1-1 correspondence from $B(\boldsymbol{o}, r)$ to $E(\boldsymbol{o}, r, \boldsymbol{\delta}, \underline{\delta})$ given by

$$f_i(\boldsymbol{x}) = o_i + (x_i - o_i) \frac{1-\delta_i}{1-\underline{\delta}} \qquad \forall i. \tag{8.25}$$

Now let $\boldsymbol{x} \in E(\boldsymbol{o}, r, \boldsymbol{\delta}, \underline{\delta})$; to prove the proposition, we need to show that there exists $\boldsymbol{c} \in C$ such that $\boldsymbol{y} \in E(\boldsymbol{o}, r, \boldsymbol{\delta}, \underline{\delta})$ where

$$y_i = \frac{1}{\delta_i} x_i - \frac{1-\delta_i}{\delta_i} c_i \qquad \forall i. \tag{8.26}$$

To see this, let $\boldsymbol{x} = f(\boldsymbol{x}')$ where $\boldsymbol{x}' \in B(\boldsymbol{o}, r)$. By equation (8.25),

$$(1-\underline{\delta}) x_i = (1-\delta_i) x_i' + (\delta_i - \underline{\delta}) o_i \quad \forall i. \tag{8.27}$$

and because of the self-accessibility of $B(\boldsymbol{o}, r)$ for $\underline{\delta}\boldsymbol{\iota}$, there exists $\boldsymbol{c}$ such that

$$\sum_{i=1}^n \frac{1}{r^2} \left(\frac{1}{\underline{\delta}} \left[x_i' - (1-\underline{\delta}) c_i \right] - o_i \right)^2 \leqslant 1 \tag{8.28}$$

Using this particular $\boldsymbol{c}$ in the edfintion of $\boldsymbol{y}$ in (8.26),

$$\begin{aligned}
\frac{y_i - o_i}{r_i} &= \frac{1}{r} \frac{1-\underline{\delta}}{1-\delta_i} \left\{ \frac{1}{\delta_i} [x_i - (1-\delta_i) c_i] - o_i \right\} && (8.29) \\
&= \frac{1}{r} \left\{ \frac{1}{\delta_i} \left[\frac{1-\underline{\delta}}{1-\delta_i} x_i - (1-\underline{\delta}) c_i \right] - \frac{1-\underline{\delta}}{1-\delta_i} o_i \right\} && (8.30) \\
&= \frac{1}{r} \left\{ \frac{1}{\delta_i} \left[x_i' + \frac{\delta_i - \underline{\delta}}{1-\delta_i} o_i - (1-\underline{\delta}) c_i \right] - \frac{1-\underline{\delta}}{1-\delta_i} o_i \right\} && (8.31) \\
&= \frac{1}{r} \left\{ \frac{1}{\delta_i} \left[x_i' - (1-\underline{\delta}) c_i \right] - \frac{\underline{\delta}}{\delta_i} o_i \right\} && (8.32) \\
&= \frac{1}{r} \left\{ \frac{\underline{\delta}}{\delta_i} \left(\frac{1}{\underline{\delta}} \left[x_i' - (1-\underline{\delta}) c_i \right] - o_i \right) \right\} && (8.33)
\end{aligned}$$

where we have used (8.27) to go from (8.30) to (8.31). Now, equation (8.28) and the fact that $\delta_i \geqslant \underline{\delta}$ for each i, allow us to conclude that $\sum_{i=1}^n \left(\frac{y_i - o_i}{r_i} \right)^2 \leqslant 1$ and we are done. ■

Proof of Theorem 2

The reader is requested to refer once again to the proof of Theorem 1 as we point out the parallels and dissmilarities between that proof and the current one. We choose $\boldsymbol{v'}$, Δ, N, $\boldsymbol{v}_i^{'}$, $\underline{\delta}_1, \underline{\delta}_2, \underline{\delta}_3, \underline{\delta}_{4i}$ and hence, $\underline{\delta}$ exactly as before. This guarantees that equation (8.7) is valid with δ being replaced by δ_i (since δ_i is at least as large as $\underline{\delta}$ and hence $\underline{\delta}_1$), Thus, we have

$$1 + \delta_i + \ldots + {\delta_i}^N \geqslant \frac{M}{v_i' - \Delta} \quad \forall i\,. \tag{8.34}$$

Similarly, we may argue, since $\delta_i \geqslant \underline{\delta} \geqslant \underline{\delta}_2$,

$$\delta_i^N \geqslant \frac{M}{M + \Delta} \quad \forall i. \tag{8.35}$$

Define $\Delta_i := \Delta \frac{1-\delta_i}{1-\underline{\delta}}$. From the previous inequality it also follows that

$$\delta_i^N \geqslant \frac{v_i'}{v_i' + \Delta} \geqslant \frac{v_i' - \Delta_i}{v_i' + \Delta} \quad \forall i.. \tag{8.36}$$

Next, we describe the strategies which follow the standard three-phase pattern used previously. Since $\boldsymbol{\delta} \geqslant \underline{\delta}\boldsymbol{\iota}$, and $B(\boldsymbol{v}, \Delta)$ is self-accessible for $\underline{\delta}\boldsymbol{\iota}$, via Proposition 4, we know that the ellipsoid $E(\boldsymbol{v}, \Delta, \boldsymbol{\delta}, \underline{\delta}) \subset B(\boldsymbol{v}, \Delta)$ is self-accessible for $\boldsymbol{\delta}$. In Phase I, it is prescribed that the players play the sequence $\{\boldsymbol{a}^{(t)}(\boldsymbol{v}, E(\boldsymbol{v}, \Delta, \boldsymbol{\delta}, \underline{\delta}))\}_{t=0}^{\infty}$. Note that at anytime during this phase, the worst continuation payoff for player i is $v_i - \Delta_i$. Phase II(i)'s play does not change at all. To describe Phase III(i), define the quantities z_j^i and r_j^i's as before except to use δ_j rather that δ in their expressions given by equations (8.11) and (8.12). With these new definitions in place, now in Phase III(i), let the players play the action sequence $\{\boldsymbol{a}^{(t)}(\boldsymbol{v'}(i) - \boldsymbol{z}^i - \Delta_i \boldsymbol{e}_i, E(\boldsymbol{v'}(i) - \boldsymbol{z}^i, \Delta, \boldsymbol{\delta}, \underline{\delta}))\}_{t=0}^{\infty}$. The transitions between the phases follow the same pattern as before.

It now remains to verify the incentive-compatibility conditions which are exactly the same as before except that each occurrence of δ is now subscripted with an i and *some* occurrences of Δ are subscripted with an i.

- For unimprovability from Phase I, it suffices to have

$$(1 - \delta_i)M + {\delta_i}^{N+1}(v_i' - \Delta_i) \leqslant v_i - \Delta_i, \tag{8.37}$$

- For unimprovability from Phase II(i) with τ periods left in the phase, it suffices to have

$$0 + {\delta_i}^{N+1}(v_i' - \Delta_i) \leqslant 0 + {\delta_i}^{\tau}(v_i' - \Delta_i) \quad \text{for } \tau = 1, \ldots, N. \tag{8.38}$$

- For unimprovability from Phase III(i), it suffices to have

$$(1 - \delta_i)M + {\delta_i}^{N+1}(v_i' - \Delta_i) \leqslant v_i' - \Delta_i. \tag{8.39}$$

- Unimprovability from Phase II(j), after identical analysis undertaken before, is assured by:

$$(1 - \delta_i)M + \delta_i^{N+1}(v_i' - \Delta_i) \leqslant \delta_i^N \left(v_i' + \Delta\right). \tag{8.40}$$

- Lastly, for unimprovability from Phase III(j) it suffices to have:

$$(1-\delta_i)M + \delta_i^{N+1} v_i' \leqslant v_i'. \tag{8.41}$$

As in the previous proof, the nontrivial inequalities (8.37), (8.41) are directly guaranteed by (8.39) while (8.40) is guaranteed by (8.39) because of (8.36). (8.39) is itself guaranteed by (8.34) and hence.... ■

Proof of Fact 1

First, we prove an analogous 'uniform' version of Theorem 1: Let F^* be full-dimensional. Suppose $\boldsymbol{u}$ and $r > 0$ be such that $B(\boldsymbol{u}, r) \subset int(F^*)$. Then, there exists a uniform discount factor bound $\underline{\delta}$ such that when $\delta \in [\underline{\delta}, 1)$, every point $\boldsymbol{v} \in B(\boldsymbol{u}, r)$ is an SPNE payoff for $\boldsymbol{\delta} = \delta \boldsymbol{\iota}$.

To see this, consider the set $V(\varepsilon) = \{\boldsymbol{v}' : \boldsymbol{v}' = \boldsymbol{u}' - \varepsilon \boldsymbol{\iota}, \boldsymbol{u}' \in \underline{\partial} B(\boldsymbol{u}, r)\}$ where $\underline{\partial} B(\boldsymbol{u}, r)$ is the lower boundary of $B(\boldsymbol{u}, r)$. Clearly, there exists a ε small enough, say $\bar{\varepsilon}$ such that $V(\bar{\varepsilon})$ is inside $int(F^*)$. For every point $\boldsymbol{v}$ in $B(\boldsymbol{u}, r)$, we can then find a $\boldsymbol{v}' \in V$ such that $\boldsymbol{v}' << \boldsymbol{v}$ (here and in the rest of this proof, we use the same notation we used in the proof of Theorem 1). In addition, there is also a uniform $\Delta > 0$ such that for every such pair of pair $\boldsymbol{v}$ and $\boldsymbol{v}'$ the conditions (8.3) and (8.4) hold. Now N, a uniform number of punishment periods can be chosen as $\left\lceil \max_{\boldsymbol{v}' \in V(\bar{\varepsilon})} \max_i \frac{M}{v_i' - \Delta} \right\rceil$. Having defined N, $\underline{\delta}_1$ and $\underline{\delta}_2$ can be defined as before. A uniform bounds can can be chosen for $\underline{\delta}_3$ as $\boldsymbol{v}$ varies over the compact set $B(\boldsymbol{u}, r)$ since the proof of Proposition 1 plus an application of Maximum Theorem shows that the discount factor bound found in that proposition is continuos in the center of the relevant ball. Similarly, for each i, using the proof of Proposition 2 and the compactness of $V(\bar{\varepsilon})$ over which $\boldsymbol{v}'$ varies, a uniform bound for $\underline{\delta}_{4i}$ can be found. Choosing $\underline{\delta}$ to be maximum of $\underline{\delta}_1, \underline{\delta}_2$, and the last two uniform bounds, works as a common discount factor bound for supporting all points in $B(\boldsymbol{u}, r)$.

Now Fact 1 follows from this the same way Theorem 2 follows from Theorem 1. ■

Proof of Theorem 3

Step 1. Let $\boldsymbol{k} >> \boldsymbol{0}$ and player i's discount factor be given by $\delta_i = \frac{1}{1+k_i\theta}$ where for now, θ, a positive number, is unspecified. For a given payoff vector $\boldsymbol{v}$, consider the problem of designing a path such that $\boldsymbol{v}$ is realized through an $m+1$ phase path described as follows. For given non-negative numbers $b_1, \ldots, b_m$, in phase 1, lasting for $T_1 = \lceil (b_1/\theta) \rceil$ periods, a certain vertex $\boldsymbol{c}^{(1)}$ will be played, then in phase 2, lasting for the next $T_2 = \lceil (b_2/\theta) \rceil$ periods some vertex $\boldsymbol{c}^{(2)}$ will be played, etc., and for phase m, vertex $\boldsymbol{c}^{(m)}$ will be played for $T_m = \lceil (b_m/\theta) \rceil$ periods. In the $m+1$'th phase, an action sequnce that generates a continuation payoff will be played so that the whole path indeed realizes $\boldsymbol{v}$. Call this continuation payoff $\tilde{\boldsymbol{v}}(\theta)$. Because of Proposition 3, our strategy will succeed if at least for small values of θ, at the end of the first m phases, the continuation payoff enters $int(F)$. This suggests that we need to know what $\tilde{\boldsymbol{v}} := \lim_{\theta \to 0} \tilde{\boldsymbol{v}}(\theta)$ is.

If $\boldsymbol{x}^t$ denotes the continuation payoff during the t'th period of any path, and in the t'th period vertex $\boldsymbol{c}$ is played, then for each i the following holds: $x_i^t = (1-\delta_i)c_i + \delta_i x_i^{t+1}$. Since $\delta_i = \frac{1}{1+k_i\theta}$, we can rewrite this as

$$x_i^{t+1} - c_i = (1 + k_i\theta)(x_i^t - c_i). \tag{8.42}$$

Hence, reasoning recursively, if $\boldsymbol{c}$ is played for T periods in periods $t, t+1, \ldots t+T-1$, then for each i,

$$x_i^{t+T} - c_i = (1 + k_i\theta)^T(x_i^t - c_i). \tag{8.43}$$

If $T = \lceil \frac{b}{\theta} \rceil$, where b is some non-negative number, then as θ tends to 0, the i'th coordinate of the limiting continuation payoff vector will satisfy

$$\lim_{\theta\to 0} x_i^{t+T} - c_i = e^{k_i b}(x_i^t - c_i). \tag{8.44}$$

Hence, in the context of the $m+1$ phase path discussed above if $m = 1$,

$$\tilde{v}_i - c_i^{(1)} = e^{k_i b_1}(v_i - c_i^{(1)}) \tag{8.45}$$

and therefore using the same idea twice, if $m = 2$,

$$\begin{aligned} \tilde{v}_i - c_i^{(2)} &= e^{k_i b_2}(c_i^{(1)} + e^{k_i b_1}(v_i - c_i^{(1)}) - c_i^{(2)}) \\ &= e^{k_i b_2}(c_i^{(1)} - c_i^{(2)}) + e^{k_i(b_2+b_1)}(v_i - c_i^{(1)}) \end{aligned} \tag{8.46}$$

Proceeding inductively, we conclude that for arbitrary integer m, for each i we will have

$$\tilde{v}_i - c_i^{(m)} = e^{k_i b_m}(c_i^{(m-1)} - c_i^{(m)}) + e^{k_i(b_m+b_{m-1})}(c_i^{(m-2)} - c_i^{(m-1)}) + \cdots + e^{k_i(b_m+\cdots+b_1)}(v_i - c_i^{(1)}) \tag{8.47}$$

Step 2. Let $\hat{\boldsymbol{v}}, \boldsymbol{v}$ be any two points in $int(re(F))$ such that for all i, $\hat{v}_i \neq v_i$. We claim that for any $\varepsilon > 0$, there exist positive numbers $k_1, \ldots, k_n$, $b_1, \ldots, b_n$, and vertices $\boldsymbol{c}^{(1)}, \ldots, \boldsymbol{c}^{(n)}$, such that the system

$$\begin{aligned} \hat{v}_1 - c_1^{(n)} &= e^{k_1 b_n}(c_1^{(n-1)} - c_1^{(n)}) + e^{k_1(b_n+b_{n-1})}(c_1^{(n-2)} - c_1^{(n-1)}) + \cdots + e^{k_1(b_n+\cdots+b_1)}(v_1 - c_1^{(1)}) \\ \hat{v}_2 - c_2^{(n)} &= e^{k_2 b_n}(c_2^{(n-1)} - c_2^{(n)}) + e^{k_1(b_n+b_{n-1})}(c_2^{(n-2)} - c_2^{(n-1)}) + \cdots + e^{k_1(b_n+\cdots+b_1)}(v_2 - c_2^{(1)}) \\ &\vdots \\ \hat{v}_n - c_n^{(n)} &= e^{k_n b_n}(c_n^{(n-1)} - c_n^{(n)}) + e^{k_1(b_n+b_{n-1})}(c_n^{(n-2)} - c_n^{(n-1)}) + \cdots + e^{k_n(b_n+\cdots+b_1)}(v_n - c_n^{(1)}) \end{aligned} \tag{8.48}$$

has an ε-solution, in the sense that the two sides of each equation differ by at most ε.

We give an induction-type argument to justify our claim. First, we will show that if we consider just two players, in fact an exact solution is possible. Wlog, let these be players 1 and 2; we will specifically show that there exist k_1, k_2, b_1, b_2 (all positive), and vertices $\boldsymbol{c}^{(1)}, \boldsymbol{c}^{(2)} \in C$, such that the following two equations hold:

$$\hat{v}_1 - c_1^{(2)} = e^{k_1 b_2}(c_1^{(1)} - c_1^{(2)}) + e^{k_1(b_2+b_1)}(v_1 - c_1^{(1)}) \tag{8.49}$$

$$\hat{v}_2 - c_2^{(2)} = e^{k_2 b_2}(c_2^{(1)} - c_2^{(2)}) + e^{k_2(b_2+b_1)}(v_2 - c_2^{(1)}) \tag{8.50}$$

Now choose vertices $\boldsymbol{c}^{(1)}, \boldsymbol{c}^{(2)}$ satisfying the conditions below.

$$sgn(\hat{v}_1 - v_1) = sgn(v_1 - c_1^{(1)}) \tag{8.51}$$

$$sgn(\hat{v}_2 - v_2) = sgn(v_2 - c_2^{(2)}) \tag{8.52}$$

Note that since $\hat{v}_i \neq v_i$ for all i, and both $\hat{\boldsymbol{v}}$ and $\boldsymbol{v}$ are in $int(re(C))$, it is possible to find two such vertices (and they could be the same vertex), where none of the signum functions above return 0. Let us set $k_2 = 1$ and make the following substitutions: $e^{b_2} = 1 + p$ and $e^{b_1+b_2} = 1 + p + q$ in equations (8.49) and (8.50). After some cancellations, the second equation can be rewritten as

$$\hat{v}_2 - v_2 = p(v_2 - c_2^{(2)}) + q(v_2 - c_2^{(1)}) \tag{8.53}$$

while the first equation becomes

$$\hat{v}_1 - c_1^{(2)} = (1 + p)^{k_1}(c_1^{(1)} - c_1^{(2)}) + (1 + p + q)^{k_1}(v_1 - c_1^{(1)}). \tag{8.54}$$

We are interested in positive k_1, p, q that will solve this pair of equations. In (8.53), we can choose q to be very small and strictly positive such that $sgn((\hat{v}_2 - v_2) - q(v_2 - c_2^{(1)})) = sgn(\hat{v}_2 - v_2)$, and then because of (8.52), we can find a positive p that solves (8.53). From p and q, b_1 and b_2 may be extracted. Turning to equation (8.54), we note that if $k_1 \to 0$, $sgn(RHS - LHS)$ is the same as $sgn(v_1 - \hat{v}_1)$. On the other hand, as $k_1 \to \infty$, because of the positivity of p and q, $sgn(RHS-LHS)$ is the same as $sgn(v_1 - c_1^{(1)})$. Hence, by the Intermediate Value Theorem, there exists $k_1 > 0$ which solves the equation if $sgn(v_1 - \hat{v}_1) = -sgn(v_1 - c_1^{(1)})$ or $sgn(\hat{v}_1 - v_1) = sgn(v_1 - c_1^{(1)})$. But this is just (8.51).[24]

Now, we show that for any $m < n$, and for any $\varepsilon > 0$ if an ε-solution exists for $m - 1$ equation version of (8.48), where the vertices are chosen according to the rule $sgn(\hat{v}_i - v_i) = sgn(v_i - c_i^{(i)})$ for each player i then a solution exists for the m equation version as well where the additional vertex is chosen using the same rule (used for the additional player). To see this, consider (8.48) with n replaced by m; suppose we set $b_1 = 0$ in all those m equations. If we consider the second through the m-th equation of the system, they become exactly the system for an $m - 1$ player scenario (where the players are indexed 2 through m). This is because in the equation pertaining to player i ($i = 2, \ldots, m$) the sum of the last two terms $e^{k_i(b_m+\cdots+b_2)}(c_i^{(1)} - c_i^{(2)}) + e^{k_i(b_m+\cdots+b_1)}(v_i - c_i^{(1)})$ collapses to the single term $e^{k_m(b_m+\cdots+b_2)}(v_i - c_i^{(2)})$ on setting b_1 to zero (note that $\boldsymbol{c}^{(1)}$ disappears from the system as a result as well). We will call this particular system the 'revised system'. Choose $k_2, \ldots, k_m$, $\boldsymbol{c}^{(2)}, \ldots \boldsymbol{c}^{(m)}$ and $b_2, \ldots, b_m$, so that the left hand side and the right hand side of each equation in this revised system differ by at most $\frac{\varepsilon}{2}$. Next, choose b_1 small enough so that the right hand sides of the original equations for players 2 through m and right hand sides of the corresponding revised equations differ by at most $\frac{\varepsilon}{2}$, *no matter what* $\boldsymbol{c}^{(1)}$ *is*, which can indeed be ensured since the right hand sides of the original equations are continuous in b_1. Thus, we have chosen now $k_2, \ldots, k_m$ and $b_1, \ldots, b_m$ and vertices $\boldsymbol{c}^{(2)}, \ldots, \boldsymbol{c}_2^{(m)}$ such that equations 2 through m satisfy the desired property. It remains to tackle the first equation and determine k_1. Indeed k_1 can now be chosen to satisfy the equation exactly. Once again, as can be easily checked, this is an application of the Intermediate Value Theorem, as long as $\boldsymbol{c}^{(1)}$ satisfies the condition $sgn(\hat{v}_1 - v_1) = sgn(v_1 - c_1^{(1)})$. This proves the claim.

Step 3. Now, choose any $\boldsymbol{u}$ in $int(F)$ such that there exists a ν-ball around $\boldsymbol{u}$ which lies fully inside $int(F)$ (guaranteed as a consequence of the full-dimensionality assumption). Let $\hat{\boldsymbol{v}}$ be such that the distance between $\hat{\boldsymbol{v}}$ and $\boldsymbol{u}$ is at most $\nu/3$, and for each i, $\hat{v}_i \neq v_i$. For this $\hat{\boldsymbol{v}}$ and the given $\boldsymbol{v}$, choose the k_i's and the b_i's and the $\boldsymbol{c}^i$'s using Step 2 such that for these parameters, $\tilde{\boldsymbol{v}}$, the limit

[24] Writing equation (8.54) as $a = bx_1^k + cy_1^k$ where $y > x > 1$, it is easily seen that for $k_1 \geqslant ln(\frac{|a|+|b|}{|c|})/ln(\frac{y}{x})$, sgn(RHS - LHS) =sgn(c). Now the method of bisection can be used to identify the solution to an arbitrary desired degree of precision.

point after the first n phases of the path described in Step 1 is at most $\nu/3$ away from $\hat{\boldsymbol{v}}$. Let $\bar{\theta}_1$ be such that for $\theta < \bar{\theta}_1$, the actual required continuation payoff $\tilde{\boldsymbol{v}}(\theta)$ is at most $\nu/3$ away from its limit point $\tilde{\boldsymbol{v}}$. This will ensure that when $\theta < \bar{\theta}_1$, for the k_i's chosen, and for $\delta_i = \frac{1}{1+k_i\theta}$, at the end of the first n phases, the required continuation payoff $\tilde{\boldsymbol{v}}(\theta)$ will be within a distance of ν from $\boldsymbol{u}$. For the chosen k_i's let $\bar{\theta}_2$ be the cutoff on θ that is required to make $B(\boldsymbol{u}, \nu)$ self-accessible as demonstrated in Proposition 3. Then, when $\theta < \bar{\theta} = \min(\bar{\theta}_1, \bar{\theta}_2)$, we can *simultaneously* ensure that $\tilde{\boldsymbol{v}}(\theta)$ is inside $B(\hat{\boldsymbol{v}}, \nu)$ and that there is a sequence of actions that generate $\tilde{\boldsymbol{v}}(\theta)$. Combining this last phase with the n phases described in Step 1, it follows that for discount factors given by $\delta_i = \frac{1}{1+k_i\theta}$ with $\theta < \bar{\theta}$, the $n+1$ phase path described in that step will indeed realize $\boldsymbol{v}$. ∎

Proof of Theorem 4

We first demonstrate the inclusion: $\bigcup\{\mathcal{F}(\boldsymbol{\delta}) \mid \boldsymbol{\delta} \in (0,1)^2\} \subset int(re(F)) \bigcup F$. Let $\boldsymbol{x} \in F(\boldsymbol{\delta})$ for some $\boldsymbol{\delta}$. Obviously, $\boldsymbol{x} \notin (re(F))^c$, for otherwise there exists a player i such that x_i is either strictly greater or strictly less than what player i can achieve in the stage game – an impossibility. Next we show that if $\boldsymbol{x}$ is on the boundary of $re(F)$, then $\boldsymbol{x} \in F$. In this case, there exists i such that x_i is an extremal (either maximum or minimum) payoff for player i (in the stage game). Let $\{\boldsymbol{a}^{(t)} \mid t \in \mathbb{Z}_+\}$ be the sequence of actions played to realize $\boldsymbol{x}$, and define $C_i^e := \{(g(\boldsymbol{a}^{(t)}) \mid t \in \mathbb{Z}_+\}$ to denote the set of all payoff profiles earned in any period. Since x_i is an extremal payoff of i, we must have $g_i(\boldsymbol{a}^{(t)}) = x_i$ for all t; therefore not only is it true that $x_i = (1-\delta_i)\sum \delta_i^t g_i(\boldsymbol{a}^{(t)})$, but it is also true that $x_i = (1-\delta_j)\sum \delta_j^t g_i(\boldsymbol{a}^{(t)})$ for any δ_j. For player $j = 3 - i$, of course we have $x_j = (1-\delta_j)\sum \delta_j^t g_j(\boldsymbol{a}^{(t)})$. Therefore, we can write the *vector* equality using j's discount factor: $\boldsymbol{x} = (1-\delta_j)\sum \delta_j^t g(\boldsymbol{a}^{(t)})$, where each $g(\boldsymbol{a}^{(t)}) \in C_i^e$. This implies $\boldsymbol{x} \in co(C_i^e) \subset F$.[25]

To demonstrate the other inclusion, $\bigcup\{F(\boldsymbol{\delta}) \mid \boldsymbol{\delta} \in (0,1)^2\} \supset int(re(F)) \bigcup F$, we appeal to Theorem 3 (making use of the full-dimensionality assumption) and further note that for any $\boldsymbol{x} \in F$, Proposition 1 guarantees that $\boldsymbol{x} \in \mathcal{F}(\delta\boldsymbol{\iota})$ for sufficiently large δ. ∎

Proof of Theorem 5

Let us wlog assume that $w_j = 0$ for all j. We begin by asserting that on any path for any player, if he is using discount factor δ, then provided all his continuation payoffs are nonnegative, increasing the discount factor to $\tilde{\delta} > \delta$ would keep all his continuation payoffs still nonnegative. To see this let s^t be the continuation payoff from t onwards when δ is used, i.e.

$$s^t = (1-\delta)[v^t + \delta v^{t+1} + \delta^2 v^{t+2} + \cdots] \tag{8.55}$$

where v^t is the player's actual payoff in period t. Equation (8.55) implies $\frac{s^t}{1-\delta} - v^t = \delta\frac{s^{t+1}}{1-\delta}$ and hence,

$$v^t = \frac{s^t}{1-\delta} - \delta\frac{s^{t+1}}{1-\delta}. \tag{8.56}$$

[25] When there are more than 2 players a little thought should convince the reader that this logic will not extend unless all players other than i have the same discount factor.

Similarly let $\tilde{s}^t$ be the continuation payoff from t onwards with the discount factor $\tilde{\delta}$. Using equations (8.55) and (8.56), we can write that as

$$\begin{aligned}\tilde{s}^t &= (1-\tilde{\delta})\left[\left(\frac{s^t}{1-\delta}-\frac{\delta s^{t+1}}{1-\delta}\right)+\tilde{\delta}\left(\frac{s^{t+1}}{1-\delta}-\frac{\delta s^{t+2}}{1-\delta}\right)+\tilde{\delta}^2\left(\frac{s^{t+2}}{1-\delta}-\frac{\delta s^{t+3}}{1-\delta}\right)+\cdots\right] \\ &= \left(\frac{1-\tilde{\delta}}{1-\delta}\right)\left[s^t+(\tilde{\delta}-\delta)s^{t+1}+\tilde{\delta}(\tilde{\delta}-\delta)s^{t+2}+\cdots\right] \\ &\geqslant 0 \qquad (8.57)\end{aligned}$$

Now assume for the moment that $\boldsymbol{v}$ was achieved as an equilibrium payoff vector with a discount factor vector where $\delta_1 \leqslant \delta_2 \leqslant \ldots \leqslant \delta_n$. Consider what payoff vector would realize if we stayed with the same played path but increased each of player $1, 2, \ldots, n-1$'s discount factors to δ_n. Since, $\boldsymbol{v}$ is an equilibrium payoff it is weakly individual rational, and all continuation payoffs for all players for all periods must be nonnegative as well. Hence, after this adjustment of discount factors, the payoff vector we obtain has the first $n-1$ components non-negative, the last component is the same as v_n and moreover, since with equal discounting any play must result in a payoff vector that is in F, this particular payoff vector satisfies all the requirements of $\boldsymbol{u}^n$ in the WD condition (under the natural order).

Next consider, the effect of changing all the discount factors to δ_{n-1}; this involves increasing the first $n-2$ discount factors, keeping the $(n-1)$'th discount factor same and *decreasing* the last discount factor. We cannot predict what happens to player n's payoff as a result, but we can surely claim that the first $n-2$ players' payoffs continue to remain non-negative, player $n-1$'s payoff remains at v_{n-1} and the whole payoff vector, taken together is in F. But then this new payoff vector satisfies the requirements of $\boldsymbol{u}^{n-1}$ in the WD condition. Proceeding similarly, each of the conditions imposed in WD can be seen to be satisfied. Finally, If the ordering of the discount factors is not 'natural', but it is the case that $\delta_{\pi_1} \leqslant \delta_{\pi_2} \leqslant \ldots \leqslant \delta_{\pi_n}$, for some permutation π, the same argument can be easily adapted to show that $\boldsymbol{v} \in W(\pi)$. ■

Proof of Proposition 5

Part a): In this proof we borrow notation used and results derived in the proof of Proposition 3. For a given ball $B(\boldsymbol{o}, r)$, let $\boldsymbol{y}(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, \boldsymbol{k}, \theta)$ refer to the continuation point when we decompose the current payoff vector $\boldsymbol{x}$ using the current action $\boldsymbol{c}$ while $\boldsymbol{k}, \theta$ parametrize the discount factor vector. We let $d(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, \boldsymbol{k}, \theta) := ||\boldsymbol{y} - \boldsymbol{o}||$ and $f(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, \boldsymbol{k}, \theta) = d^2 - r^2$.[26] The arguments used in Proposition 3 shows that for the fixed ball $B(\boldsymbol{o}, r)$ there exists a strictly positive-valued function $\bar{\theta}(\boldsymbol{o}, \boldsymbol{k})$, continuous in its arguments such that if $\theta \in (0, \bar{\theta}(\boldsymbol{o}, \boldsymbol{k}))$, for each $\boldsymbol{x} \in B(\boldsymbol{o}, r)$ there exists a vertex $\boldsymbol{c}^*(\boldsymbol{x}, \boldsymbol{o}, \boldsymbol{k})$ with the property $\sum_{i=1}^n k_i(x_i - c^*{}_i)(x_i - o_i) < 0$ and for that vertex, $f < 0$ and hence, $d < r$.[27] Define

$$\theta^\dagger(\boldsymbol{k}) = \min_{\boldsymbol{o} \in [\boldsymbol{y}, \boldsymbol{z}]} \bar{\theta}(\boldsymbol{o}, \boldsymbol{k}) \qquad (8.58)$$

which is well-defined and strictly positive because $\bar{\theta}$ is strictly positive and continuous in its arguments and because $[\boldsymbol{y}, \boldsymbol{z}]$ is compact. Hence, if $\theta \in (0, \theta^\dagger(\boldsymbol{k}))$, for any $\boldsymbol{o} \in [\boldsymbol{y}, \boldsymbol{z}]$ and $\boldsymbol{x} \in B(\boldsymbol{o}, r)$,

[26] $\boldsymbol{y}, d$ and f also depend on r, but we ignore this for brevity's sake as r stays fixed once we fix our capsule. This shortcut is used for other functions as well.

[27] Recall that $\boldsymbol{c}^*$ was chosen with a view to maximize the range of θ over which f stays non-positive.

there exists $\boldsymbol{c}$, such that $f < 0$ or $d < r$. Now define $\bar{d}(\boldsymbol{k}, \theta)$ by

$$\bar{d}(\boldsymbol{k}, \theta) := \max_{\boldsymbol{o} \in [\boldsymbol{y}, \boldsymbol{z}]} \max_{\boldsymbol{x} \in B(\boldsymbol{o}, r)} \min_{\boldsymbol{c} \in C} d(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, \boldsymbol{k}, \theta) \tag{8.59}$$

which is well-defined and $< r$ because of continuity of $\boldsymbol{d}$ in $\boldsymbol{x}$ and $\boldsymbol{o}$, the Maximum Theorem and compactness of the relevant feasible sets. In $\bar{d}(\boldsymbol{k}, \theta)$, we now have created a uniform bound on the distance of the continuation payoff from the center of any ball that makes up the capsule and any target point in the ball by choosing the current action to be $\hat{\boldsymbol{c}}(\boldsymbol{x}, \boldsymbol{o}, \boldsymbol{k}, \theta) := \operatorname{argmin}_{\boldsymbol{c} \in C} d(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, \boldsymbol{k}, \theta)$ (ties being broken using any arbitrary preference ordering among vertices).[28]

Now, for $\theta < \theta^{\dagger}(\boldsymbol{k})$, we recursively construct the sequences $\{\boldsymbol{c}^{(t)}\}$ and $\{\boldsymbol{x}^t\}$ via a third sequence $\{\boldsymbol{o}^t\}$. Define $\boldsymbol{o}^0 = \boldsymbol{y}$ and $\boldsymbol{x}^0 = \boldsymbol{u}$. Of course, if $\boldsymbol{u}$ is already in $B(\boldsymbol{z}, r)$, we can stop immediately with $T = 0$, so we assume that this is not the case. Let $\boldsymbol{c}^{(0)} = \hat{\boldsymbol{c}}(\boldsymbol{x}^0, \boldsymbol{o}^0, \boldsymbol{k}, \theta)$. Let $\boldsymbol{x}^1 = \boldsymbol{y}(\boldsymbol{x}^0, \boldsymbol{c}^{(0)}, \boldsymbol{o}^0, \boldsymbol{k}, \theta)$. Again, we stop with $T = 1$ if $\boldsymbol{x}^1 \in B(\boldsymbol{z}, r)$. Otherwise, we define $\boldsymbol{o}^1 = \operatorname{argmin}_{\substack{\boldsymbol{x} \in [\boldsymbol{y}, \boldsymbol{z}] \\ ||\boldsymbol{x} - \boldsymbol{x}^1|| = r}} ||\boldsymbol{x} - \boldsymbol{z}||$, and $\boldsymbol{c}^{(1)} = \hat{\boldsymbol{c}}(\boldsymbol{x}^1, \boldsymbol{o}^1, \boldsymbol{k}, \theta)$.

In general, given $\boldsymbol{x}^t, \boldsymbol{o}^t, \boldsymbol{c}^{(t)}$ we define

$$\boldsymbol{x}^{t+1} = \boldsymbol{y}(\boldsymbol{x}^t, \boldsymbol{c}^{(t)}, \boldsymbol{o}^t, \boldsymbol{k}, \theta) \tag{8.60}$$

$$\boldsymbol{o}^{t+1} = \operatorname{argmin}_{\substack{\boldsymbol{x} \in [\boldsymbol{y}, \boldsymbol{z}] \\ ||\boldsymbol{x} - \boldsymbol{x}^{t+1}|| = r}} ||\boldsymbol{x} - \boldsymbol{z}|| \tag{8.61}$$

$$\boldsymbol{c}^{(t+1)} = \hat{\boldsymbol{c}}(\boldsymbol{x}^{t+1}, \boldsymbol{o}^{t+1}, \boldsymbol{k}, \theta) \tag{8.62}$$

and we stop the recurrence with $T = t$ as soon as $\boldsymbol{x}^t \in B(\boldsymbol{z}, r)$. Indeed, we are assured of stopping in a finite number of steps because, $||\boldsymbol{o}^t - \boldsymbol{x}^{t+1}|| \leqslant \bar{d}(\boldsymbol{k}, \theta)$, $||\boldsymbol{x}^{t+1} - \boldsymbol{o}^{t+1}|| = r$ which on applying Triangle Inequality ensures that $||\boldsymbol{o}^t - \boldsymbol{o}^{t+1}||$ is at least $r - \bar{d}(\boldsymbol{k}, \theta)$. This implies $T \leqslant \left\lceil \frac{||y - z||}{r - \bar{d}(\boldsymbol{k}, \theta)} \right\rceil$ and completes the proof of Part a).

Part b): To prove this part, we need to bound $\bar{d}(\boldsymbol{k}, \theta)$ by some suitable function of θ. We start by observing that the square of the d function

$$d^2(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, \boldsymbol{k}, \theta) = \theta^2 \sum_{i=1}^{n} k_i^2 (x_i - c_i)^2 + 2\theta \sum_{i=1}^{n} k_i (x_i - c_i)(x_i - o_i) + \sum_{i=1}^{n} (x_i - o_i)^2 \tag{8.63}$$

is convex in $\boldsymbol{x}$.[29] Now for any $\boldsymbol{x} \neq \boldsymbol{o}$ in $B(\boldsymbol{o}, r)$, let $\bar{\boldsymbol{x}}$ be the point on the surface of the ball that is intersected by the ray emanating from $\boldsymbol{o}$ and going towards $\boldsymbol{x}$, i.e. $\bar{\boldsymbol{x}} = \boldsymbol{o} + \frac{r}{||\boldsymbol{x} - \boldsymbol{o}||}(\boldsymbol{x} - \boldsymbol{o})$. Then, since, $\boldsymbol{x}$ is a convex combination of $\bar{\boldsymbol{x}}$ and $\boldsymbol{o}$, we have:

$$d^2(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, \boldsymbol{k}, \theta) \leqslant \max\{d^2(\bar{\boldsymbol{x}}, \boldsymbol{c}, \boldsymbol{o}, \boldsymbol{k}, \theta), d^2(\boldsymbol{o}, \boldsymbol{c}, \boldsymbol{o}, \boldsymbol{k}, \theta)\} \tag{8.64}$$

This in turn implies that

$$\begin{aligned} d^2(\boldsymbol{x}, \hat{\boldsymbol{c}}(\boldsymbol{x}, \boldsymbol{o}, \boldsymbol{k}, \theta), \boldsymbol{o}, \boldsymbol{k}, \theta) &\leqslant d^2(\boldsymbol{x}, \hat{\boldsymbol{c}}(\bar{\boldsymbol{x}}, \boldsymbol{o}, \boldsymbol{k}, \theta), \boldsymbol{o}, \boldsymbol{k}, \theta) \\ &\leqslant \max\{d^2(\bar{\boldsymbol{x}}, \hat{\boldsymbol{c}}(\bar{\boldsymbol{x}}, \boldsymbol{o}, \boldsymbol{k}, \theta), \boldsymbol{o}, \boldsymbol{k}, \theta), d^2(\boldsymbol{o}, \hat{\boldsymbol{c}}(\bar{\boldsymbol{x}}, \boldsymbol{o}, \boldsymbol{k}, \theta), \boldsymbol{o}, \boldsymbol{k}, \theta)\} \end{aligned} \tag{8.65}$$

We will now bound each component in the max function in equation (8.65). For the first component,

[28]We note in passing that $\hat{\boldsymbol{c}}$ depends on θ and need not be the same as $\boldsymbol{c}^*$.

[29]This can be easily verified by checking the positive definiteness of the Hessian.

we note that

$$d^2(\boldsymbol{k}, \bar{\boldsymbol{x}}, \hat{\boldsymbol{c}}(\boldsymbol{k}, \bar{\boldsymbol{x}}, \boldsymbol{o}, \theta), \boldsymbol{o}, \theta) \leqslant d^2(\boldsymbol{k}, \bar{\boldsymbol{x}}, \check{\boldsymbol{c}}(\boldsymbol{k}, \bar{\boldsymbol{x}}, \boldsymbol{o}), \boldsymbol{o}, \theta) \tag{8.66}$$

where $\check{\boldsymbol{c}}(\boldsymbol{k}, \bar{\boldsymbol{x}}, \boldsymbol{o})$ minimizes $\sum_{i=1}^n 2k_i(x_i - c_i)(x_i - o_i)$ where the minimand, as argued before, is strictly negative. Now by referring to equation (8.63) we see that $d^2(\boldsymbol{k}, \bar{\boldsymbol{x}}, \hat{\boldsymbol{c}}(\boldsymbol{k}, \bar{\boldsymbol{x}}, \boldsymbol{o}, \theta), \boldsymbol{o}, \theta)$ is less than or equal to $m_1(\boldsymbol{k})\theta^2 + m_2(\boldsymbol{k})\theta + r^2$ where

$$m_1(\boldsymbol{k}) := \max_{\boldsymbol{x} \in \mathcal{C}(\boldsymbol{y}, \boldsymbol{z}, r), \boldsymbol{c} \in C} \sum_{i=1}^n k_i^2 (x_i - c_i)^2 > 0 \tag{8.67}$$

and

$$m_2(\boldsymbol{k}) := \max_{\substack{\boldsymbol{o} \in [\boldsymbol{y}, \boldsymbol{z}] \\ \boldsymbol{x} \in B(\boldsymbol{o}, r)}} \min_{\boldsymbol{c} \in C} \sum_{i=1}^n 2k_i(x_i - c_i)(x_i - o_i) < 0. \tag{8.68}$$

On the other hand, for the second of the two components in the right hand side of equation (8.65), we notice that both the θ term and the constant term in (8.63) drops out and hence, it is less than or equal to $m_1(\boldsymbol{k})\theta^2$. There exists a $\theta^\ddagger(\boldsymbol{k}, \theta) \leqslant \theta^\dagger(\boldsymbol{k}, \theta)$ when $m_2(\boldsymbol{k})\theta + r^2 \geqslant 0$, and hence, for $\theta \in (0, \theta^\ddagger(\boldsymbol{k}, \theta))$, for all $\boldsymbol{o} \in [\boldsymbol{y}, \boldsymbol{z}]$, all $\boldsymbol{x} \in B(\boldsymbol{o}, r)$ (including the centers of the balls), $\min_{\boldsymbol{c} \in C} d^2(\boldsymbol{k}, \boldsymbol{x}, \boldsymbol{c}, \boldsymbol{o}, \theta) \leqslant m_1(\boldsymbol{k})\theta^2 + m_2(\boldsymbol{k})\theta + r^2$ and hence, $\bar{d}(\boldsymbol{k}, \theta))^2 \leqslant m_1(\boldsymbol{k})\theta^2 + m_2(\boldsymbol{k})\theta + r^2$. This shows $r - \bar{d}(\boldsymbol{k}, \theta) \geqslant r - \sqrt{m_1(\boldsymbol{k})\theta^2 + m_2(\boldsymbol{k})\theta + r^2}$ and the claim follows using the bound derived on T in the proof of part a). ■

Proof of Theorem 6

Wlog we assume that $\boldsymbol{w} = \boldsymbol{0}$ and $\boldsymbol{v}$ is strictly diagonalizable for the natural order, letting $\boldsymbol{u}^1, \ldots, \boldsymbol{u}^n$ be as in the definition of the property. We will show that $\boldsymbol{v}$ is an SPNE payoff.

We will first specify the equilibrium path since the equilibrium strategy is based off it. The equilibrium path will entail a 'pre-entry path' based on a sequence of actions lasting for $T(\theta)$ periods that will transition the (required) continuation payoff from $\boldsymbol{v}$ to a point inside a ball inside $int(F^*)$, with $\boldsymbol{v}^t(\theta)$ denoting what the continuation payoff is at period t. The center of this ball will be $\boldsymbol{u}^n$ which we know, resides in $int(F^*)$. Thereafter, the equilibrium path will coincide with the path created by playing the SPNE strategy to support $\boldsymbol{v}^{T(\theta)}(\theta)$. Of course, to specify this strategy exactly, and hence the 'post-entry path' it leads to, we need to specify the radius of the ball of entry, the $\boldsymbol{k}$ vector and ensure that θ will be below a certain bound - we will do all that in due course.

As was clarified in Section 7, the pre-entry path will be broken down into $n-1$ stages, each stage witnessing an application of the Capsule Lemma. In the l'th stage ($l = 1, \ldots, n-1$) we will be operating with a capsule that is situated inside the payoff space of the first l players for $T^l(\theta)$ periods. We now describe these capsules. In what follows, if $\boldsymbol{x}$ is an n-dimensional vector, the subvector consisting of its first m coordinates will be denoted as $\boldsymbol{x}[m]$.

By strict diagonalizability and Fact 3, for every $l = 1, \ldots, n-1$, $\boldsymbol{u}^l[l]$ and $\boldsymbol{u}^{l+1}[l]$ both belong to $int(F(1, \ldots, l)) \cap \mathbb{R}^l_{++}$ (recall that $F(1, \ldots, l)$ is the convex hull of $Proj_{\{1,\ldots,l\}}C$). Hence, there exists $\bar{r}_l$ such that if $r_l < \bar{r}_l$, the entire capsule $\mathcal{C}(\boldsymbol{u}^l[l], \boldsymbol{u}^{l+1}[l], r_l)$ also lies in $int(F(1, \ldots, l)) \cap \mathbb{R}^l_{++}$. This is the capsule we will work with in the l'th stage with r_l to be further specified later.

The purpose of operating the l'th stage is to change the continuation payoffs of Players 1 through l (from what they were at the end of $l-1$'th stage). But it is not that during stages 1 through l, the continuation payoffs of Player i, where $l + 1 \leqslant i \leqslant n$, will stay put. However, we can make these players relatively patient so that their payoffs will not change by much and thus we can maintain strict individual rationality for those players (since v_i was strictly positive for all i). In particular, let ε be any number below $\min_i v_i$. We will show, k_l for $l > 1$ can be chosen so that during each of the stages $1, \ldots l-1$, Player l's continuation payoff changes by at most ε/n (from the previous stage). This will ensure that at the beginning of stage l, his continuation payoff stays strictly positive. From stages l onwards we do not have to worry about the strict individual rationality of his continuation payoffs because for these stages they are in capsules within which every vector is strictly positive. What works here is that we choose a player's k *before* his payoff becomes part of any capsule, and once done, his payoffs can be transitioned through any capsule since capsules can handle any arbitrary $\boldsymbol{k}$ vector.

Given the vectors $\boldsymbol{u}^l, l = 1, \ldots, n$, we define the following *anchor* vectors: $\boldsymbol{z}^1 = \boldsymbol{v}$ and for $l = 2, \ldots, n$,

$$\boldsymbol{z}_i^l = \begin{cases} u_i^l & \text{for } i \leqslant l-1 \\ v_i & \text{for } i > l-1 \end{cases}$$

Notice that the the first $\boldsymbol{z}$ is the target payoff $\boldsymbol{v}$ is and the last $\boldsymbol{z}$ is $\boldsymbol{u}^n$, which is in $int(F^*)$. The point behind the terminology 'anchor' should be clear now: until the continuation payoff enters $int(F^*)$, the entire on equilibrium continuation payoff path will stay close to the following 'piecewise-linear' path:

$$\boldsymbol{z}^1 \longrightarrow \boldsymbol{z}^2 \longrightarrow \cdots \longrightarrow \boldsymbol{z}^n$$

In the l'th stage, all continuation payoffs $\boldsymbol{v}^t(\theta)$, will be zig-zagging around the line segment joining $\boldsymbol{z}^l$ and $\boldsymbol{z}^{l+1}$.

We need to make sure that the 'starting ball' of the $l+1$'th stage will accommodate the continuation payoffs of players $1, \ldots l$ arriving 'transformed' via the 'ending ball' of the previous capsule *as well as* the continuation payoff of Player $l+1$. This is ensured by the following relation between the radii of one capsule and the next: $r_{l+1}^2 = r_l^2 + \varepsilon^2$ where ε is an upper bound on by how much Player $l+1$'s payoff can change up until period $T^1(\theta) + \cdots + T^l(\theta)$. As our rquilibrium strategy will show, we need 'room' around the final ball of entry inside F^* to take care of off-equilibrium behavior; if the maximum of the absolute value of the adjustment term is Δ and the amount of reward for punishing players is 2Δ, we need to have a ball of center $\boldsymbol{u}^n$ and radius $r_n + 3\sqrt{n-1}\Delta$ fit inside $int(F^*)$. To summarize then, we impose the following restrictions on the sequence of capsule radii:

$$B(\boldsymbol{u}^l[l], r_l) \subset int(F(1, \ldots, l)) \cap \mathbb{R}_{++}^l \qquad \text{for } l = 1, \ldots, n-1 \tag{8.69}$$
$$B(\boldsymbol{u}^{l+1}[l], r_l) \subset int(F(1, \ldots, l)) \cap \mathbb{R}_{++}^l \qquad \text{for } l = 1, \ldots, n-1 \tag{8.70}$$
$$B(\boldsymbol{u}^n, r_n + 3\sqrt{n-1}\Delta) \subset int(F^*) \qquad \text{for some } \Delta > 0 \tag{8.71}$$
$$r_{l+1}^2 = r_l^2 + \varepsilon^2 \qquad \text{for } l = 1, \ldots, n-1, \tag{8.72}$$

where $\varepsilon \leqslant \min_i v_i$. Note that the first two constraints ensure that the capsule $\mathcal{C}(\boldsymbol{u}^l[l], \boldsymbol{u}^{l+1}[l], r_l)$ lies in $int(F(1, \ldots, l)) \cap \mathbb{R}_{++}^l$.

Let m be the minimum any player receives in any point in any of the capsules. Hence, m is also a lower bound on any player's continuation payoff at any point on the pre-entry path. Now define N, which we will use as the number of punishment periods, such that

$$N = \left\lceil \frac{M}{m} \right\rceil. \tag{8.73}$$

Next we turn our attention to permissible patterns of discount factor vectors. We start by specifying the k_i's. This is done recursively using part b) of the Capsule Lemma. Set $k_1 = 1$. Next, for any l, assuming that we already know $k_1, \dots, k_l$ we will determine k_{l+1}. Apply the Capsule Lemma to the j'th capsule $\mathcal{C}(\boldsymbol{u}^j[j], \boldsymbol{u}^{j+1}[j], r_j)$where $j \leqslant l$. Using the notation from that result let $\bar{\theta}'_{1,j}$ be $\theta^{\ddagger}(k_1, \dots, k_j)$. With $T^j(\theta)$ being the number of periods needed to execute the procedure described there, let $\{\boldsymbol{c}^{(1)}(j,\theta), \cdots, \boldsymbol{c}^{(T^j(\theta))}(j,\theta)\}$ be the vertices *in the original game* to be played to carry out the procedure.[30] Now, we assert that there exists a $\bar{\theta}_{1jl} \leqslant \bar{\theta}'_{1j}$ and a $\bar{k}_{l+1}$, such that if $k_{l+1} < \bar{k}_{l+1}$ and $\theta < \bar{\theta}_{1jl}$, the maximum absolute difference between Player l's continuation payoff at the beginning of the procedure compared to that at the end of the procedure is ε/n. To see this recall that for any fixed k_{l+1}, $v^{t+1}_{l+1}(\theta) - c^{(t)}_{l+1} = (1 + k_{l+1}\theta)(v^t_{l+1}(\theta) - c^{(t)}_{l+1})$ where $\boldsymbol{c}^{(t)}$ is the vertex played and $v^t(\theta)$ is Player l's continuation payoff for the t'th period during the operation. We can rewrite this as

$$v^{t+1}_{l+1}(\theta) - v^t_{l+1}(\theta) = (k_{l+1}\theta)(v^t_{l+1}(\theta) - c^{(t)}_{l+1}) \tag{8.74}$$

Hence,

$$\begin{aligned} |v^{t+1}_{l+1}(\theta) - v^t_{l+1}(\theta)| &= (k_{l+1}\theta)|(v^t_{l+1}(\theta) - c^{(t)}_{l+1})| \\ &\leqslant (k_{l+1}\theta)2M \end{aligned} \tag{8.75}$$

from which it follows that the absolute difference between beginning and end payoffs during stage j for Player l is at most $2k_{l+1}M\theta T^j(\theta)$ the θ-dependent part of which is bounded by the expression

$$\theta\left(\frac{\alpha_j}{r_j - \sqrt{m_{1j}\theta^2 + m_{2j}\theta + r_j}} + 1\right),$$

with α_j being $d(\boldsymbol{u}^j[j], \boldsymbol{u}^{j+1}[j])$ and the m's are constants depending on the cylinder, $k_1, \dots, k_l$ but not on k_{l+1}. The limit of the above expression as θ tends to 0, using L'Hospital's rule is the constant $\frac{2\alpha_j\sqrt{r_j}}{-m_{2j}}$ and hence choosing $\bar{k}_{l+1} < \frac{-m_{2j}\varepsilon}{4\alpha_j nM\sqrt{r_j}}$ suffices for the assertion. From the above it is clear if $k_{l+1} < \bar{k}_{l+1}$ and $\theta < \bar{\theta}_{1l} := \min_{j \leqslant l} \bar{\theta}_{1jl}$ after l stages, i.e. after $T^1(\theta) + \cdots + T^l(\theta)$ periods, Player $l+1$'s continuation payoff could not change by more than ε from its original target value v_l. If $\bar{\theta}_1 := \min_{1 \leqslant l \leqslant n-1} \bar{\theta}_{1l}$, then given the $\boldsymbol{k}$ vector we have chosen, for $\theta < \bar{\theta}_1$, the above statement is true for each player.

For the chosen $\boldsymbol{k}$ vector, from Fact 1 we know that there is another positive bound $\bar{\theta}_2$ such that when $\theta < \bar{\theta}_2$, for any $\boldsymbol{x} \in B(\boldsymbol{u}^n, r_n + 3\sqrt{n-1}\Delta)$ there is a SPNE strategy $\sigma^*(\boldsymbol{x})$ that realizes $\boldsymbol{x}$. Hence, for $\theta < min(\bar{\theta}_1, \bar{\theta}_2)$ we have designed a path that realizes $\boldsymbol{v}$. This path involves playing the following sequence of vertices along its pre-entry segment:

$$\boldsymbol{c}^{(1)}(1,\theta), \cdots, \boldsymbol{c}^{(T^1(\theta))}(1,\theta), \boldsymbol{c}^{(1)}(2,\theta), \cdots, \boldsymbol{c}^{(T^2(\theta))}(2,\theta), \cdots, \boldsymbol{c}^{(1)}(n-1,\theta), \cdots, \boldsymbol{c}^{(T^{n-1}(\theta))}(n-1,\theta)$$

followed by the path yielded by $\sigma^*(\boldsymbol{v}^{T(\theta)})$ with $T(\theta)$ being $T^1(\theta) + \cdots + T^{n-1}(\theta)$. For notational ease we will henceforth refer to the sequence of vertices on the pre-entry path simply as $\tilde{\boldsymbol{c}}^{(1)}, \dots, \tilde{\boldsymbol{c}}^{(T(\theta))}$.

Now, we can formally describe the equilibrium strategy in the language of automata (Rubinstein

[30] It is important to note that these are n-dimensional vertices. Though the capsule is in the projected space of players' payoffs on the first j coordinates, playing a vertex in the projected space requires the participation of *all* players.

1986) as shown below. There are three types of (common) states, each identifed by a set of state variables:

- A$[\tau, i, \boldsymbol{z}]$ where $1 \leqslant \tau \leqslant T(\theta)$, $0 \leqslant i \leqslant n$, $\boldsymbol{z} \in \mathbb{R}^n$
- B$[\tau, \tau', i, \boldsymbol{r}^i]$ where $1 \leqslant \tau \leqslant T(\theta)$, $1 \leqslant \tau' \leqslant N$, $1 \leqslant i \leqslant n$, $\boldsymbol{r}^i \in \mathbb{R}^n$
- C$[\boldsymbol{x}]$ where $\boldsymbol{x} \in \mathbb{R}^n$

The interpretation of an A-type state is that going forward, we have τ periods left of going through the pre-entry path, i was the last deviator (if $i = 0$, a deviation never took place), and $\boldsymbol{z}$ is the adjustment vector in the ball $B(\boldsymbol{u}^n, r_n + 3\sqrt{n-1}\Delta)$ that we will need to subtract from the equilibrium point of entry (besides giving a 'reward' to Player(s) $j \neq i$) once we are done with the pre-entry path. The interpretation of a B-type state is that we are on a punishment path where Player i, the last deviator is being minmaxed and τ' periods of minmaxing still needs to be done while τ denotes from what type of A state we have (eventually) arrived here, and the j'th component of $\boldsymbol{r}^i$ denotes the normalized payoff for $j \neq i$ based on the past realizations of the $N - \tau'$ periods of minmaxing i ($r_i^i = 0$).[31] The interpretation of a Type C state is that it is an 'absorbing' state where $\sigma^*(\boldsymbol{x})$ is played from that point onwards.

The game starts at the state A$[T(\theta), 0, \mathbf{0}]$. For any state A$[\tau, i, \boldsymbol{z}]$, $\tilde{\boldsymbol{c}}^{(T(\theta)-\tau+1)}$ is to be played next. If in the observed action profile, there is a unilateral deviation by player j, play switches to the state B$[\tau, N, j, \mathbf{0}]$. Otherwise, play switches to A$[\tau-1, i, \boldsymbol{z}]$ if $\tau > 1$ and to C$[\boldsymbol{v}^{T(\theta)} - \boldsymbol{z} + 2\Delta(\boldsymbol{\iota} - \boldsymbol{e}_i)]$ if $\tau = 1$. For any state B$[\tau, \tau', i, \boldsymbol{r}^i]$, $\boldsymbol{m}^i$ is to be played next. If j is the only player whose action is not observed to be in support of $\boldsymbol{m}_j^i$, play switches to B$[\tau, N, j, \mathbf{0}]$. Otherwise, if $\tau' > 1$, play next moves to B$[\tau, \tau'-1, i, \tilde{\boldsymbol{r}}^i]$ where

$$\tilde{r}_j^i = \begin{cases} \frac{g_j(\boldsymbol{a}) + \delta_j r_j^i + \cdots + \delta_j^{N-\tau'} r_j^i}{1 + \delta_j + \cdots + \delta_j^{N-\tau'}} & \text{if } j \neq i \\ 0 & \text{if } j = i \end{cases} \tag{8.76}$$

with $\boldsymbol{a}$ being the last action profile observed. If on the other hand, $\tau' = 1$, play switches to A$[\tau, i, \boldsymbol{z}]$ where $z_j^i = \frac{1-\delta_j^N}{\delta_j^{N+\tau}} \tilde{r}_j^i$, where $\tilde{r}_j^i$ is defined above. The behavior of the automaton at a C type stage has already been described.

We need to ensure that $|z_j^i|$ is suitably bounded otherwise the adjustment term could take us out of the last ball. Note that $|z_j^i| \leqslant \frac{1}{\delta_j^{T(\theta)}} \frac{1-\delta_j^N}{\delta_j^N} M$. The limit of ${\delta_j}^{T(\theta)}$ (as θ goes to 0) can be written as $\lim_{\theta \to 0} {\delta_j}^{T_1(\theta)} \times \cdots \times \lim_{\theta \to 0} {\delta_j}^{T_{n-1}(\theta)}$. It may be easily checked that that if $\lim_{\theta \to 0} \theta T^l(\theta) = b_l$, then $\lim_{\theta \to 0} {\delta_j}^{T_l(\theta)} = e^{-k_j b_l}$, and hence, there exists a bound $\bar{\theta}_3$, such that if $\theta < \bar{\theta}_3$, $|z_j^i| < \Delta$. Given that each 'punisher' is rewarded by the amount 2Δ, and $||\boldsymbol{v}^{T(\theta)} - \boldsymbol{u}^n|| \leqslant r_n$, this shows that $||\boldsymbol{v}^{T(\theta)} - \boldsymbol{z} + 2\Delta(\boldsymbol{\iota} - \boldsymbol{e}_i) - \boldsymbol{u}^n|| \leqslant r_n + 3\sqrt{n-1}\Delta$.

Next, we examine the requirements for incentive compatibility. As σ^* is an SPNE by construction, we only need to check for incentive compatibility at A and B type states.

If the current state is A$[\tau, 0, \mathbf{0}]$ or A$[\tau, i, \boldsymbol{z}]$ for some $\boldsymbol{z}$, and Player i did not deviate, the game will follow a certain path and he will receive a certain stage game payoff sequence. Let his normalized payoff from this be y_i. If he unileterally deviates, he receives 0 for the next N periods and therafter, he will receive exactly the same sequence of payoffs had he not deviated at all (note that he receives neither a reward nor an adjustment post entry). Unimprovability from

[31]Note: $\boldsymbol{r} \neq \boldsymbol{z}$. The latter will depend on τ.

the prescription at a Type A state in this case is then ensured by

$$(1-\delta)M + \delta^{N+1} y_i \leqslant y_i \tag{8.77}$$

or, since $y_i \geqslant m$ by

$$M/m \leqslant 1 + \delta_i + \cdots + \delta_i^N \tag{8.78}$$

which equation (8.73) assures us will hold for high enough δ_i and hence θ below a certain bound. Our next bound $\bar{\theta}_4$ is precisely this bound.

The argument for i's unimprovability from states of the form A$[\tau, j, \boldsymbol{z}]$, $(j \neq i)$ is even stronger than the argument given in favor of the argument for unimprovability in the just argued case, because by sticking to the equilibrium prescription, he would have received y_i in normalized payoff plus at least an extra amount of Δ forever after τ periods.

Next consider i's incentive to deviate from a state of the form B$[\tau, \tau', i, \boldsymbol{r}^i]$. This can't be profitable because it will simply postpone playing the same path that has a strictly positive (normalized) payoff.

Lastly, consider the prospect of i deviating from a state of the form B$[\tau, \tau', j, \boldsymbol{r}^j]$, $j \neq i$. If equilibrium prescription is followed, i will receive at worst,

$$(1-\delta_i^{\tau'}).-M + \delta_i^{\tau'}(y_i + \delta_i^{\tau}\Delta) \tag{8.79}$$

where y_i is as before. if he deviates he will receive at best

$$(1-\delta_i)M + \delta_i^{N+1} y_i \tag{8.80}$$

Hence the difference in i's payoff between conforming and deviating is

$$\begin{aligned} &(1-\delta_i^{\tau'}).-M - (1-\delta_i)M + (\delta_i^{\tau'} - \delta_i^{N+1})y_i + \delta_i^{\tau'+\tau}\Delta \\ &\geqslant (1-\delta_i^N).-M - (1-\delta_i)M + (\delta_i^N - \delta_i^{N+1})m + \delta_i^{N+T(\theta)}\Delta \end{aligned} \tag{8.81}$$

The last term in the above expression converges to the positive number $e^{-k_i b}\Delta$ while other terms go to 0 as δ_i goes to 1. Hence, there exists a positive bound $\bar{\theta}_5$, such that if $\theta < \bar{\theta}_5$, there is no incentive for i to deviate at any B$[\tau, \tau', j, \boldsymbol{r}^j]$ type state. We can now conclude that for the chosen $\boldsymbol{k}$ vector if $\theta < \min(\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3, \bar{\theta}_4, \bar{\theta}_5)$, the prescribed strategy is an SPNE. ■

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