Self-fulfilling traps with money as a medium of exchange

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Abstract

A model of an economy with random matching and monetary exchanges is shown to have two equilibrium regimes. In the first, agents find an opportunity for a sale in each period and there is full-employment. In the second, agents attempt to increase their balances because of a lack of opportunities for sales which is in turn induced by the saving for higher money balances, and there is unemployment. In any period of the high regime, self-fulfilling expectations can plunge the economy into the low equilibrium but a recovery from the low to the high equilibrium may not be possible.
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1 Introduction

In the current crisis, the demand for precautionary saving and the reduction of consumption has played an important contributing role. Uncertainty about employment and income raises the motive for saving and the lower demand for goods feeds into the uncertainty. The mechanism, which has some relation with the paradox of thrift, is analyzed here in a model of general equilibrium with money as the medium of exchange.

In any contemporary economy, exchanges are between goods and money. Money is liquid as it can be used to trade any good. Agents increase their money balances through sales and use these balances to buy consumption goods. Both inflows and outflows of money are subject to random micro-shocks but in a “standard” regime of economic activity, these shocks can assumed to be relatively small and agents can afford to keep a relatively low level of money inventories. Such a regime depends on individual expectations. If agents expect that opportunities for sales are subject to a larger uncertainty, they reduce their consumption to accumulate more money as a precaution. But the reduction of consumption by some agents may increase the sale uncertainty of others and raise the demand for money. The higher demand for money (liquidity) may be self-fulfilling.

In this paper, the sudden increase of the demand for money shifts the economy from an equilibrium with a regime of high consumption to another equilibrium with a regime of low consumption where agents attempt to accumulate higher money balances and there is insufficient aggregate demand and output.

Money is valuable because agents are spatially separated. The spatial separation of agents has been the foundation of models of money since Samuelson (1958). A important assumption in that model is that agents cannot enter in bilateral credit agreements because they are temporally separated in different generations and meet only once. That fundamental property has been embodied by Townsend (1980) in a setting with infinitely lived agents who are paired along the two opposite lanes of a “turnpike”, each selling and consuming with his *vis à vis* on alternate days and carrying money from a day of production to the next day for consumption. The central property again is that any two agents are paired at most once. That property is preserved here in an adaptation of the Townsend model in which agents are in a continuum of mass one and matched pair wise in each period with two other agents, one customer for the produced good and one supplier of the consumption good.

There is no search in the model, because search introduces separate issues of efficiency and is not the essential property for the existence of money[1]. The search process is implicit in Green and Zhou (2002)[2] because agents may not find a good to their taste of consumption in their current match and thus may find a randomly long time gap between opportunities for sales and consumption.

The focus on the essential property of money as in the turnpike model of Townsend will enable us to use the model as a simple analytic representation of a liquidity crisis as one of two possible equilibria.

In the full-employment equilibrium, there is not more than one period between a sale and a consumption. Money is necessary for consumption but because a stable and high inflow of cash is expected, a relatively low level of money balance is sufficient to maintain a high level of consumption. No agent is cash constrained for consumption and producers can always sell.

In an equilibrium with unemployment, a producing agent cannot sell when he is matched with an agent who has no money. Because of the probability of no sale, agents attempt to accumulate money. But the higher balances for some agents must result in smaller or no balances for others because the endogenous money price of goods is the same in the two equilibria and the total quantity of money is not affected by the regime of activity.

In the model presented here, when the economy is in a full-employment stationary equilibrium, a negative shock of expectations is sufficient to push the economy to an equilibrium with unemployment: the fear of smaller opportunities for sale induces agent to keep money: if they do not have an urgent need to consume they choose to save, but this act of saving reduces the opportunity of another agent to sell his production. The two equilibria with and without full employment are not symmetric:

[1] The role of money or credit with search has been analyzed by Diamond and Yellin (1990), Diamond (1990), Shi (1995), Trejos and Wright (1995), among others.

[2] See also Green and Zhou (1998), Zhou (1999a).

in the stationary equilibrium with unemployment a jump of optimism may not be sufficient to nudge the economy into a recovery: in full employment all agents who don't have a high need for consumption can shift to saving. In the economy with unemployment, agents who are liquidity constrained cannot jump to consumption even if they become optimistic about the future.

The model is presented in Section 2. In order to simplify the analysis, agents are constrained in the maximum of cash they can hold, by assumption. Since the higher demand for cash is what generates an inefficient equilibrium, the restriction should not limit the validity of the properties. The assumption is relaxed later in the paper.

The stationary equilibrium with full employment and low demand for money is presented in Section 3. In general, the optimal consumption rule depends on the dynamics of the economy. The dynamics of the two regimes of high and low consumption are first analyzed in Section 4. In the following section, these two regimes are shown to be equilibria under suitable parameter conditions. Section 6 shows that the steady state of the low regime with unemployment may be a trap out of which no optimism about the future can lift the economy. Section 7 discusses some technical properties and some remarks on policy are presented in the concluding section.

2 The model

There is continuum of infinitely lived agents, indexed by $i \in [0,1)$. The utility of agent i in period t is $u(x_{it}, \theta_{it})$, where $\theta_{it} \in \{0,1\}$ are i.i.d. random variables that represent shocks to the utility of consumption. $P(\theta_{it} = 1) = \alpha$ which is exogenous and known to all agents. When $\theta_{it} = 1$, agent i has a higher need to consume in period t than when $\theta_{it} = 0$. If $\theta_{it} = 0$, the agent is in period t of the *low type*, and if $\theta_{it} = 1$, the agent is in period t of the *high type*.

To simplify the exposition, we assume that the utility function in period t is given by

$$u(x_t, \theta) = \begin{cases} 1 \quad , & \text{if } x_t \geq 1, \\ \\ -c\theta_t, & \text{if } x_t < 1. \end{cases} \tag{1}$$

An illustration for the high type is that the agent has to make some repair (material or bodily) to avoid a penalty c. It will be shown later that the properties of the model do not depend on the indivisibility properties of the utility function.

The welfare of agent i from any period, say period 0, is the discounted expected sum of the utilities of consumption in the future periods:

$$U_i = E\Big[\sum_{t\geq 0} \beta^t u(x_{it}, \theta_{it})\Big], \qquad \text{with} \qquad \beta = \frac{1}{1+\rho} < 1, \tag{2}$$

Agents produce goods which they sell, and consume goods produced by others. In each period, an agent meets two other agents, one to sell to, one to buy from, according to a process of random matching: agent i produces a good in period t and is matched with the consumer $j = \phi_t(i)$ such that

$$\phi_t(i) = \begin{cases} i + \xi_t, & \text{if } i + \xi_t < 1, \\ i + \xi_t - 1, & \text{if } i + \xi_t \geq 1. \end{cases}, \tag{3}$$

where $\xi_t \in (0,1)$ are i.i.d. random variables[3]. The process implies that agent i can consume the good produced by agent $\phi_t^{-1}(i)$. The matching of agents does not depend on their money holding or on the value of θ_{it}.

The matching process embodies the absence of a double coincidence of wants and implies that a agent has zero probability of find the same match in a future period. This process is a variation of Townsend's turnpike that fits a circle of infinite diameter with random pairing of atomistic agents between the two lanes . Agents cannot establish credit with each other. There is fiat money: agents sell a real produced good for money and buy real goods with money.

In order to simplify the demand for money, it is assumed that each agent is like a two-headed household: at the beginning of period t, say a day, one head of household i goes out to the market with some cash (if there is any in the household) to buy a consumption good from a randomly matched supplier $\phi_t^{-1}(i)$. The second head stays at home to get the customer $\phi_t(i)$, and if that customer buys, he produces and sells one unit of the good at no cost. The two heads meet at the end of the day to consume if a purchase has been made and to take stock of the money balance. Goods are not storable. The case where of a single person who buys and sells with a random order during the day may be analyzed in later work.

To summarize, in each period t we have the following sequence of events:

1. Each agent i first learns his type, *i.e.* the value of $\theta_{i,t}$. The probability of the high type ($\theta_{i,t} = 1$) is equal to $\alpha \in (0,1)$.

[3] One could use other matching functions ϕ_t provided that they satisfy the property that for any subset $\mathcal{H}$ of $[0,1)$, $\mu(\mathcal{H}) = \mu\Big(\phi_t(\mathcal{H})\Big)$, where μ is the Lebesgue-measure on $[0,1)$. The property is required for a uniform random matching of all agents.

2. For each agent (the two-headed household), the buyer brings a quantity of money m to the market. He does not have to bring the entire money that is held at the beginning of the period. In order to have strictly optimal strategies, it is assumed that carrying money to the market conveys a vanishingly small cost. The decision about m has to take place at the beginning of the day, before the eventual production and sale during the day.

3. For each agent, the seller produces either 0 or 1 (since agents demand 0 or 1 in the utility function). The production is cost-free[4]. The seller posts a price p and there is no bargaining. One can assume that the seller produces instantly, only after he knows whether he has a buyer.

3 Steady state equilibrium with full-employment

Assume that each agent has a quantity of money m at the beginning of the period. Any price $p < m$ defines an equilibrium with full employment. In that equilibrium each agent brings a quantity p to the market in each period. Given the posted prices p, and the very small cost of carrying money, an amount of money $\tilde{m} > p$ is strictly less preferred. Any amount $\tilde{m} < p$ leads to no purchase/consumption and is strictly inferior to the amount p if the agent has a sufficient money balance at the beginning of the period. If the agent's money balance is strictly smaller than p at the beginning of the day, knowing that the posted prices are p and that he cannot buy, he will not carry any money to the market.

Sellers know that in equilibrium the buyers carry the amount $\tilde{m} = p$ to the market. No seller finds it optimal to post a price strictly smaller than p and posting a price higher than p would result in no sale since no buyer brings more than p to the market.

Lemma 1 *Assume that the quantity of money of agents is bounded below by $\bar{m}$. Then any price $p \leq \bar{m}$ determines a steady state with full employment.*

There is a continuum of equilibrium price levels. That property is the same as in Green-Zhou (1998, 2002) because of the similar price determination. The property of a continuum of price equilibria is of no special interest here. The main property is that for all these prices there is full employment. All the equilibria with different p have the same real allocation of resources which is the socially optimal allocation.

[4]One could assume that the agent could produce $x \in [0, 1]$, costlessly, with the maximum capacity of 1, in order remove the indivisibility of the utility function.

In a full employment equilibrium, money is needed for transaction, but there is no precautionary motive since agents are sure that they will be able to sell and replenish their cash at the end of the period.

Suppose now that the economy is in an equilibrium that may have unemployment (an equilibrium that still needs to be defined). Assume that all producers post the price p that is constant and independent of time. If a agent decides to consume in a given period, by the same argument as for the previous Lemma, he brings to the market a quantity $\tilde{m} = p$. The value $\tilde{m}$ is the same for all agents and constant over time. In a rational expectations equilibrium, producers know the value of $\tilde{m}$ and the optimal price to post is $p = \tilde{m}$. A producer gets no additional sale from deviating and posting a price below $\tilde{m}$ because any customer knows that he would meet that producer with probability 0. We will therefore consider equilibria with a price that is constant over time and is normalized to 1.

All money holdings in interval $I_k = [k, k+1)$ generate the same opportunities for trade. $m \in I_k$ defines the *state* $\mathbf{k}$ of an agent. In state 0, an agent is liquidity constrained and cannot consume. We first make the technical assumption that money holdings are bounded: there is N arbitrary subject to $m < N$. This assumption restricts the hoarding capability of agents. It is not harmful in a model that generates an equilibrium with excess hoarding, and it will be removed in Section 7. Let $\Gamma(t)$ be the vector of the distribution of agents at the beginning of period t across states

$$\Gamma(t) = (\gamma_0(t), \gamma_1(t), \dots, \gamma_N(t))'.$$

Any distribution of money must satisfy the two constraints:

$$\sum_{0}^{N} \gamma_k(t) = 1, \qquad \sum_{k=1}^{N} k\gamma_k(t) = M. \tag{4}$$

In a steady state with full employment, there is no need for precautionary saving toward consumption in future period because each agent knows that at the end of the day, a sale is made for sure and the cash balance that has been reduced in the morning by the expenditure for consumption is replenished at the end of the day for the next one. In general, the demand for money will depend on the type of the agent (high or low) and the opportunities of future sales as determined by the path of the probabilities of making a sale in period t. That path depends on the consumption function of agents in the future. We consider two consumption functions that define each a regime, high or low. We will show later that under some conditions, each of the two regimes is an equilibrium.

4 Dynamics in two regimes

In the high regime, by definition, all the agents who are not liquidity constrained (*i.e.*, in state 0) consume. This regime defines the highest possible consumption level. In the low regime, if an agent is not in a high state (with a higher need for consumption), and is not in state 0 (with a liquidity constraint) or state N (the maximum level of money holding), that agent saves. The purpose of saving is to self-insure for a later day with a higher consumption need against a lack of sale opportunities. It makes little sense *a priori* to save when there is a current higher consumption need, because of the discounting of the future. Hence the definition of the consumption in a low regime. The optimality of the two consumption functions will depend on the utility of money balances that depends in turn on the future dynamics of the economy. It will be analyzed in the next section.

4.1 The high regime

At the beginning of the first period, period 0, the distribution of money, $\Gamma(0)$, is given. Since all agents except those in state 0 consume, and the matching is independent of the money balance, each agent faces the same probability $\pi(t)$ of not making a sale in period t, and being unemployed. The probability $\pi(t)$ is equal to the fraction of agents in state 0, $\gamma_0(t)$. The evolution of the distribution of money is given by

$$\Gamma(t+1) = H(\pi_t).\Gamma(t), \qquad \text{with} \qquad \pi_t = \gamma_0(t). \tag{5}$$

with the transition matrix

$$H(\pi) = \begin{pmatrix} \pi & \pi & 0 & 0 & 0 & \dots \\ 1-\pi & 1-\pi & \pi & 0 & 0 & \dots \\ 0 & 0 & 1-\pi & \pi & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & 0 & 0 & 1-\pi & \pi \\ 0 & \dots & 0 & 0 & 0 & 1-\pi \end{pmatrix}. \tag{6}$$

For example, in the first line, the agents who are in state 0 at the end of period t are in that state either (i) because they met another agent in state 0, with probability π, and that determines H_{11}, or (ii) because they were in state 1 at the beginning of period t, consumed but met with a buyer with no cash and therefore they get no inflow of cash which determines $A_{12} = \pi$. Likewise for the other elements of the matrix H.

From Lemma 1, given the price $p = 1$, there is a steady state full-employment equilibrium if and only if the aggregate quantity of money, M, is at least equal to one. The next result (proven in the Appendix) shows that in this case, the high regime where all non liquidity-agents consume converges to full employment.

Proposition 1

Let M be the aggregate quantity of money.

- *If $M \geq 1$ and all agents who are not liquidity-constrained consume, for any initial distribution of the money M, the rate of unemployment converges to zero. At the limit no agent is liquidity-constrained.*
- *If $M < 1$, and all agents who are not liquidity-constrained consume, the distribution of money converges to $(\gamma_0^*, \gamma_1^*, \dots, \gamma_N^*)$, with $\gamma_0^* = \pi^*$, $\gamma_1^* = 1 - \gamma_0^*$ and $\gamma^* k = 0$ for $k > 1$. At the limit, the rate of unemployment is $\pi^* = 1 - M$.*

The limit distribution of money balances depends on the initial distribution. As a particular case, any distribution with full employment and $\gamma_0 = 0$ is invariant through time.

The case with three states

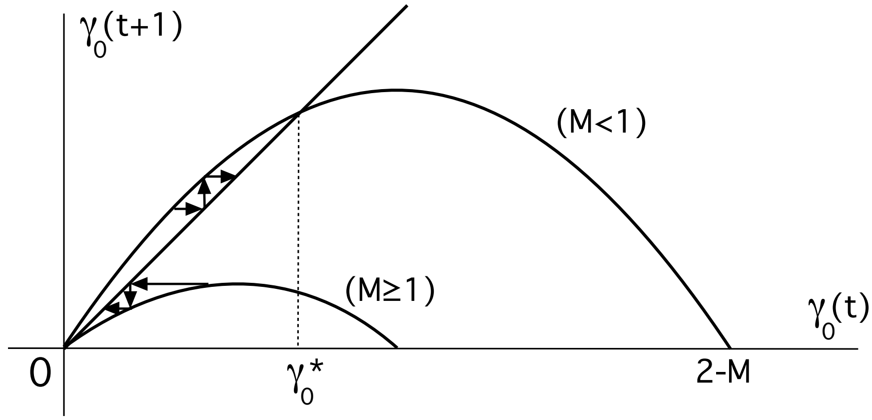
When there are three states, the quantity of money is bounded by 2. Using the equations of the quantity of agents and of money, $\sum \gamma_k = 1$ and $\sum_{k=1}^{2} k\gamma_k = M$, the dynamics can be expressed in function of $\gamma_0(t)$:

$$\gamma_0(t+1) = \gamma_0(t)(\gamma_0(t) + \gamma_1(t)),$$

which is equivalent to

$$\gamma_0(t+1) = \gamma_0(t)(2 - M - \gamma_0(t)). \tag{7}$$

The evolution of $\gamma_0(t)$ is represented in Figure 1 for the cases $M \geq 1$ and $M < 1$. When $1 \leq M < 2$, at the limit, $\gamma_0^* = \pi^* = 0$ and the distribution of money in states 1 and 2 is determined uniquely by the unit mass of agents and the quantity of money. When M increases, γ_2^* increases and γ_1^* decreases.



When $M < 1$, the economy converges to a steady state with unemployment and the fraction γ_0 of liquidity constrained agents tends to a positive value.

When $M > 1$, the economy converges to full-employment with no liquidity constrained agents in the limit.

Figure 1: Dynamics of the liquidity constrained agents in the regime of high consumption (three states)

4.2 The low regime

In the low regime, consumption originates in the fraction α of the agents in states 1 to $N-1$ and all agents in state N. As the fraction of agents who consume is $1-\pi_t$,

$$\pi(t) = 1 - \alpha \sum_{k=1}^{N-1} \gamma_k(t) - \gamma_N(t). \tag{8}$$

which can be written as

$$\pi(t) = 1 - \alpha(1-\gamma_0(t)) - (1-\alpha)\gamma_N(t). \tag{9}$$

The value of $\pi(t)$ is equal to zero if and only if $\gamma_0(t) = 0$ and $\gamma_N(t) = 1$. That can occur only in the initial period and if $\gamma_N(0) = 1$.

The quantity of money M is bounded by N. If $M = N$, all the agents are in the state N and consume: there cannot be a low regime. Since N can be chosen arbitrarily, we can always have $M < N$.

Proposition 2

In the low regime, (with $M < N$), there is unemployment in all periods: $\pi(t) > 0$ for all $t \geq 1$.

The evolution of the distribution of money is now given by

$$\Gamma(t+1) = L(\pi_t).\Gamma(t), \qquad \text{with} \qquad \pi_t = \gamma_0(t), \tag{10}$$

where the transition matrix $L(\pi)$ takes a form that depends on N.

For $N = 2$,

$$L(\pi) = \begin{pmatrix} \pi & \pi\alpha & 0 \\ & & \\ 1-\pi & (1-\alpha)\pi + \alpha(1-\pi) & \pi \\ & & \\ 0 & (1-\pi)(1-\alpha) & 1-\pi \end{pmatrix}. \tag{11}$$

For example in the first line, the mass of liquidity constrained agents $\gamma_0(t+1)$ comes from the constrained agents who meet another constrained agent in period t and the

$$\gamma_0(t+1) = (\gamma_0(t) + \alpha\gamma_1(t))(1-c).$$
$$\gamma_1(t+1) = c\gamma_0(t) + \Big((1-\alpha)(1-c) + \alpha c\Big)\gamma_1(t) + (1-c)\gamma_2(t).$$
$$\gamma_2(t+1) = c((1-\alpha)\gamma_1(t) + \gamma_2(t)).$$

For $N \geq 3$,

$$L(\pi) = \begin{pmatrix} \pi & \alpha\pi & 0 & 0 & \dots & 0 \\ 1-\pi & a & \alpha\pi & 0 & \dots & 0 \\ 0 & b & a & \alpha\pi & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & b & a & \alpha\pi & 0 \\ 0 & \dots & 0 & b & a & \pi \\ 0 & 0 & 0 & \dots & b & 1-\pi \end{pmatrix}, \tag{12}$$

with

$$\begin{cases} a = (1-\alpha)\pi + \alpha(1-\pi), \\ \\ b = (1-\pi)(1-\alpha), \end{cases}$$

and where the middle lines are omitted for $N = 3$.

The dynamics of the economy are completely specified by equation (10) where the matrix $L(\pi)$ is defined in (11) or (12) and π_t is given in (8).

The stationary economy

Let e be the row-vector with $N+1$ components equal to 1. One verifies that $e.L(\pi) = e$ for any π. Fix a value of π. The matrix $L(\pi)$ has an eigenvalue equal to 1 and that eigenvalue is of order 1. There is a unique vector Γ^* such that $B(\pi).\Gamma^* = \Gamma^*$ and $e.\Gamma^* = 1$. The vector Γ^* defines a stationary distribution of money holdings that depends on the probability π. The total amount of money is equal to

$$M = \sum_{k=1}^{N} k\gamma_k^*.$$

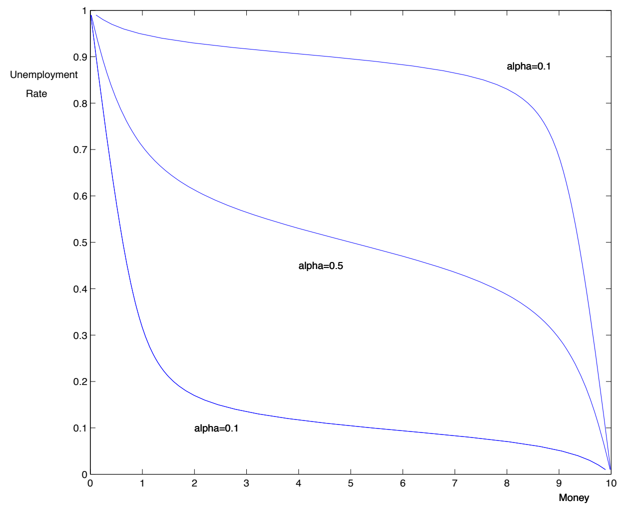


Figure 2: Money and unemployment in the stationary economy under the low regime ($N = 10$)

We thus have a function from π to the total amount of money in the economy. The next result which is proven in the Appendix is intuitive: there is a positive relation between money and employment.

Proposition 3

When the distribution of money is stationary in the low regime with $M < N$, the rate of unemployment is a strictly decreasing function of the quantity of money.

The proposition is illustrated in Figure 2 for three values of α. When the quantity of money tends to N, which is the maximum value of individual money holding in the example, the rate of unemployment tends to zero. Asymptotically, all agents have money holding in the state 10. As in the previous section, we analyze in more details the case of three states with $N = 2$.

The case with three states

The distribution of agents across the states of money balances has to satisfy the two conditions of the mass of agents equal to one and the total quantity of money equal to M, which must be smaller than 2 for the existence of a low regime. With three states, there is only one degree of freedom and the economy can be represented by the evolution of one variable, chosen here to be $\gamma_0(t)$.

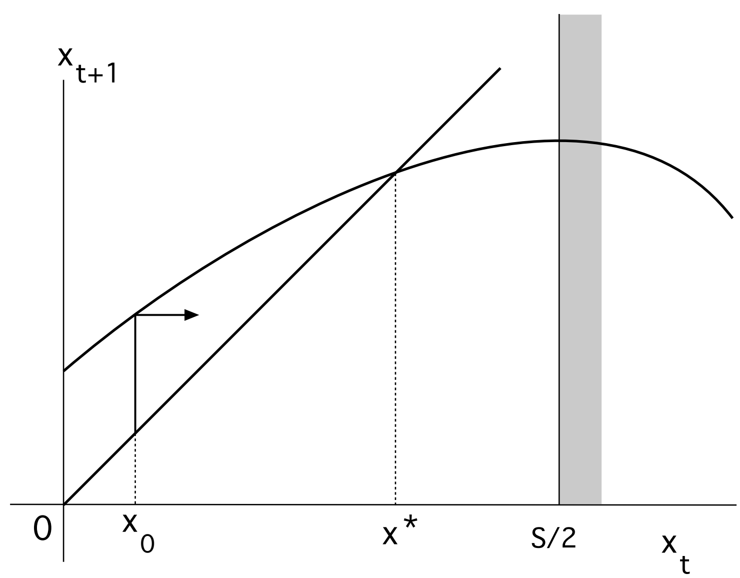
Let $S = 2 - M$. From the constraints on the distribution of money $(\gamma_0(0), \gamma_1(0), \gamma_2(0))$, we have $-2\gamma_0(t) + S = \gamma_1(t) \geq 0$. Hence, any initial value for the fraction of agents in state 0, $\gamma_0(0)$, must satisfy the condition

$$\gamma_0(0) \leq S/2. \tag{13}$$

The analysis which are presented in the Appendix, shows that for $t \geq 0$,

$$\begin{aligned} \gamma_0(t+1) &= P(\gamma_0(t)), \qquad \text{with} \\ P(x) &= -(1-2\alpha)^2 x^2 + (1-2\alpha)^2 Sx + \alpha(1-\alpha)S^2. \end{aligned} \tag{14}$$

The polynomial $P(x)$ reaches its maximum for $x = S/2$. One verifies[5] that $P(S/2) < S/2$. Since $P(x)$ is increasing on the interval $[0, S/2]$, there is for $x > 0$ a unique value x^* such that $P(x^*) = x^*$ and $x^* < S/2$. For any admissible value of x_0 which must be in the interval $[0, S/2]$ by (13), the sequence $x_{t+1} = P(x_t)$ converges to x^* monotonically. The evolution of x_t is represented in Figure 3.



$\gamma_0(t) = x_t$, $S = 2 - M$.

Figure 3: Dynamics of the fraction of liquidity-constrained agents in the low regime

[5] $P(S/2) = \left(\frac{(1-2\alpha)^2}{4} + \alpha(1-\alpha)\right)S^2$. Since $S < 1$, this expression is strictly smaller than $S/2$.

During the transition, the value of $\gamma_0(t)$ varies monotonically towards its steady state. The unemployment rate, $\pi(t)$, is a linear function of $\gamma_0(t)$

$$\pi(t) = -(1-2\alpha)\gamma_0(t) + (1-\alpha)S, \tag{15}$$

and it converges to the limit π^* that is determined by the equation

$$M = \psi(\pi^*) = 2 - \frac{\pi^*\Big(1-\pi^*(1-2\alpha)\Big)}{1-\alpha-\pi^*(1-2\alpha)}. \tag{16}$$

The function $\psi(\pi)$ has a negative derivative and is strictly decreasing, as shown already in Proposition 3. Let ϕ be its inverse function with $\pi = \phi(M)$. One verifies that

$$\phi(0) = 1, \quad \phi(2) = 0, \quad \phi(1.5-\alpha) = 0.5. \tag{17}$$

We have the following result.

Proposition 4

In a low regime, with $N = 2$ and $M < 2$, the distribution of money converges to a stationary distribution and the unemployment rate converge to a limit that is a decreasing function $\phi(M)$ such that $\phi(0) = 1$, $\phi(1) = \pi_1^* > 0$, and $\phi(2) = 0$.

(i) On the dynamic path, $\gamma_0(t)$ is a monotone function of time and the unemployment rate, $\pi(t)$, is a decreasing (increasing) function of $\gamma_0(t)$ when $\alpha < 1/2$, ($\alpha > 1/2$).

(ii) In the special case where $\alpha = 1/2$, the distribution of money and the unemployment rate are constant for all periods with $\gamma_0 = (S/2)^2$.

The previous analyses covers all plausible cases of the consumption function when $N = 2$. Agents in state 0 cannot consume and if they make a sale, they save; agents in state 2 consume since they cannot accumulate more money, by assumption. The consumption functions are differentiated by the behavior of agents in state 1. Since it is absurd to think that the high type would save and the low type consume, we have covered all cases. In the next two sections, it is assumed that there are three states for individual money balances and $N = 2$. That restriction is lifted in Section 7.

5 Optimal consumption functions

So far, we have considered how the distribution of money and the unemployment rate varies under the different consumption functions. We now analyze which consumption is optimal. Let $U_k(t)$ be the utility of an agent in state k who consumes, with $k = 1, 2$. That utility is the same for the high and the low types. Let U_0 the utility of an agent in state 0 at the beginning of the period *before* he learns his type. (Recall that such an agent cannot consume).

We assume that if an agent is in state 1, he consumes with probability ζ and saves with probability $1 - \zeta$. (Recall that the behavior of agents in state 0 or 2 is determined). In the high regime, $\zeta = 1$ and in the low regime, $\zeta = \alpha$. We now analyze the evolution of the utility levels $U_k(t)$ in the low regime.

5.1 Equilibrium in the low regime

Consider an agent in state 2, with the highest balance, $m \in I_2$. He consumes, gets a utility of 1 from that consumption to which he adds the discounted value of the utility in the next period. By standard backward induction, his utility in period t is

$$U_2(t) = 1 + \beta\Big((1-\pi)U_2(t+1) + \pi((1-\alpha)W_1(0, t+1) + \alpha W_1(1, t+1))\Big), \quad (18)$$

where $W_1(\theta, t+1)$ is the utility of an agent at the beginning of period $t+1$ of type θ with $m \in I_1$. (The time index is omitted for π in the next three equations). For example, the agent makes in the current period no sale with probability π in which case he has $m \in I_1$ in the next period and with probability α he is of the type $\theta - 1$ in which case his utility is $W_1(1, t+1)$.

In the low regime, a low type in state 1 does not consume and has the same distribution of money at the end of the period as an agent in state 2 and consumes: $W_1(0, t+1) = U_2(t+1) - 1$. An agent with a high type in state 1 consumes and has a utility $U_1(t+1)$. Substituting in the previous equation,

$$U_2(t) = 1 + \beta\Big((1-\pi)U_2(t+1) + \pi((1-\alpha)(U_2(t+1) - 1) + \alpha U_1(t+1)\Big). \quad (19)$$

Likewise, for a consumer in state 1,

$$U_1(t) = 1 + \beta\Big((1-\pi)\Big((1-\alpha)(U_2(t+1) - 1) + \alpha U_1(t+1)\Big) + \pi U_0(t+1)\Big), \quad (20)$$

where $U_0(t+1)$ is utility in period $t+1$, before knowing the type in that period, of a liquidity-constrained agent (in state 0) who cannot consume. He pays in the present period a penalty c for not consuming if he is of the high type, which occurs with probability α. Such an agent gets out of his state and save to get in state 1 only if he meets a buyer. By backward induction,

$$U_0(t) = -\alpha c + \beta\Big((1-\pi)((1-\alpha)(U_2(t+1) - 1) + \alpha U_1(t+1)) + \pi U_0(t+1)\Big). \quad (21)$$

Equations (19), (20) and (21) determine $(U_0(t), U_1(t), U_2(t))$ as a recursive function of $(U_0(t+1), U_1(t+1), U_2(t+1))$. Let U be the vector of utilities $U = (U_0, U_1, U_2)'$. The three recursive equations form a system

$$U(t) = \beta A(\pi(t))U(t+1) + B(\pi(t)), \tag{22}$$

where $A(\pi)$ is a 3×3 matrix and $B(\pi)$ is a 3×1 vector:

$$A(\pi) = \begin{pmatrix} \pi & (1-\pi)\alpha & (1-\pi)(1-\alpha) \\ \pi & (1-\pi)\alpha & (1-\pi)(1-\alpha) \\ 0 & \pi\alpha & 1-\pi\alpha \end{pmatrix}, \qquad B(\pi) = \begin{pmatrix} -\alpha c - \beta(1-\pi)(1-\alpha) \\ 1-\beta(1-\pi)(1-\alpha) \\ 1-\beta\pi(1-\alpha) \end{pmatrix} \tag{23}$$

The value of $\pi(t)$ is a function of the distribution of money in period t, and is determined by (8).

The consumption function of the low regime is optimal under the following conditions:

(i) the utility of a low type agent who saves, $U_2 - 1$, is greater than that of consumption, U_1: $U_2(t) - U_1(t) \geq 1$;

(ii) the utility of a high type agent who consumes, U_1, is greater than the payoff of saving, $U_2 - 1 - c$. Hence, $U_2(t) - U_1(t) \leq 1 + c$.

Combining the two conditions, the necessary and sufficient condition is

$$1 \leq U_2(t) - U_1(t) \leq 1 + c. \tag{24}$$

With probability one, the inequalities are strict. We want therefore to focus on the differences

$$\begin{cases} Z_t = (X_t, Y_t)', \text{ with} \\ X_t = U_2(t) - U_1(t), \\ Y_t = U_2(t) - U_0(t). \end{cases}$$

We can write

$$\begin{cases} X_t = g.U(t), & \text{with} \quad g = \begin{pmatrix} 0 & -1 & 1 \end{pmatrix}, \\ Y_t = h.U(t), & \text{with} \quad h = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}. \end{cases}$$

Using (22),

$$\begin{cases} X_t = g.U(t) = \beta g.A(\pi_t)U(t+1) + g.B(\pi(t)), \\ \\ Y_t = h.U(t) = \beta h.A(\pi_t)U(t+1) + h.B(\pi(t)). \end{cases}$$

$$\begin{pmatrix} X_t \\ \\ Y_t \end{pmatrix} = \beta \begin{pmatrix} \alpha(1-2\pi_t) & \pi_t \\ \\ \alpha(1-2\pi_t) & \pi_t \end{pmatrix} \cdot \begin{pmatrix} X_{t+1} \\ \\ Y_{t+1} \end{pmatrix} + \begin{pmatrix} \beta(1-\alpha)(1-2\pi_t) \\ \\ 1+\alpha c + \beta(1-\alpha)(1-2\pi_t) \end{pmatrix}.$$

Hence,

$$\begin{cases} X_t = \beta\alpha(1-2\pi_t)X_{t+1} + \pi_t\beta Y_{t+1} + \beta(1-\alpha)(1-2\pi_t), \\ \\ Y_t = X_t + 1 + \alpha c. \end{cases}$$

Substituting for Y_{t+1} in the first equation,

$$X_t = \beta\Big(\alpha(1-2\pi_t) + \pi_t\Big)X_{t+1} + \beta\pi_t(1+\alpha c) + \beta(1-\alpha)(1-2\pi_t), \tag{25}$$

with the the stationary solution

$$X^* = \beta\frac{\pi^*(1+\alpha c) + (1-\alpha)(1-2\pi^*)}{1-\beta(\alpha(1-2\pi^*)+\pi^*)}. \tag{26}$$

The condition $X^* > 1$ is equivalent to

$$\frac{\pi^*\alpha c}{\rho} > 1. \tag{27}$$

The inequality has a simple interpretation: the discounted expected value of the cost of unemployment measured as the product of the probability of the high type and the penalty for not consuming in the high type must be greater than one. We assume that this condition is satisfied.

One can verify that the value of X^* in (26) is smaller than $1+c$ for any parameters and value of π^*. The inequality $X^* < 1+c$ has also a simple interpretation: it were not satisfied, a high type would prefer to save in order to reduce his cost of no consumption while a high type in the future over the cost of no consumption in the present. Because of the discounting, this cannot be true.

Dynamics

We assume that (27) is satisfied. Hence $X^* > 1$ and on the transition path, $X_t \geq 1$ for t sufficiently large.

Suppose that $X_{t+1} > 1$. Then by (25), a sufficient condition for $X_t > 1$ is

$$\pi_t \alpha c > \rho. \tag{28}$$

From Proposition 4, π_t varies monotonically on the transition path. It decreases with time if $\alpha < 1/2$ and $\gamma_0(0) < \gamma_0^*$, or $\alpha > 1/2$ and $\gamma_0(0) > \gamma_0^*$. If π_t is decreasing, then $X_t > 1$ for all t.

In the other parametric cases, π_t is increasing and $\pi_0 \alpha c > \rho$, then (28) is satisfied for all t and since for some T, $X_t > 1$ for $t > T$, by backward induction, $X_t > 1$ for all t. The condition $\pi_0 \alpha c > \rho$ with $\pi_0 = \gamma_0(0) + (1-\alpha)(\gamma_1(0) + \gamma_2(0))$. In this case, the steady state condition (27) holds. Using $\sum \gamma_k = 1$, the previous expression can be replaced by $\pi_0 = 1 - \alpha + \alpha \gamma_0(0)$.

If $\alpha = 1/2$, the economy is stationary for all periods and the necessary and sufficient condition for the optimality of consumption is (27).

Proposition 5

The low regime consumption is optimal under the following condition:

$$\begin{cases} \pi^* \alpha c > \rho, \textit{ if } \alpha \leq \frac{1}{2} \textit{ and } \gamma_0(0) < \gamma_0^*, \textit{ or } \alpha > 1/2 \textit{ and } \gamma_0(0) > \gamma_0^*, \\ \\ \pi_0 \alpha c > \rho, \quad \textit{with } \pi_0 = -(1-2\alpha)\gamma_0(0) + (1-\alpha)(2-M), \textit{ in the other cases.} \end{cases} \tag{29}$$

When the economy is initially at full employment, $\gamma_0(0) = 0$, and the previous conditions are simpler

$$\begin{cases} \pi^* \alpha c > \rho, \text{ if } \alpha \leq \frac{1}{2}, \\ \\ \pi_0 \alpha c > \rho, \quad \text{with } \pi_0 = (1-\alpha)(2-M), \text{ if } \alpha > \frac{1}{2}. \end{cases} \tag{30}$$

5.2 Equilibrium in the high regime

The consumption function of the high regime is optimal in period t if an agent in state 1 and of the low type prefers to consume over saving, that is if $X_t = U_2(t) - U_1(t) \leq 1$.

The analysis of the low regime can be used here if we replace α by 1 in the previous equations. Equation (25) takes now the form

$$X_t = \beta(1-\pi_t)X_{t+1} + \beta\pi_t(1+c). \tag{31}$$

The stationary solution is $\tilde{X}^* = \beta\pi^*(1+c)/(1-\beta(1-\pi^*))$. Since $\pi^* = 0$ (Proposition 1), $\tilde{X}^* = 0$. The consumption function is trivially optimal near the steady state. Using (31), if $X_{t+1} < 1$, a sufficient condition for $X_t < 1$ is that $c < \rho$.

Proposition 6

If $c < \rho$, the consumption in the high regime is optimal in any period for any distribution of money.

One should remark that the condition $c < \rho$ in the proposition is strong as the result applies for any distribution of money, and it could be weakened for particular distributions of money. For example, if $M \geq 1$ with a uniform distribution, a stationary equilibrium with full employment is sustainable, for any value of c (Proposition 1). By continuity, if the initial money distribution is not too different from a full-employment distribution, the high regime can be a equilibrium that converges to full employment.

6 Dynamics and Liquidity Trap

Suppose first that the economy is a stationary equilibrium with full employment. (There can be more than one distribution of money for such an equilibrium). From Proposition 5, there exists a value $\bar{c}_1$ such that if $c > \bar{c}_1$, an exogenous shift of (perfect foresight) expectations towards pessimism can push the economy into the low regime equilibrium with an employment rate that converges to a positive value. The shift to the low regime is self-fulfilling.

Suppose now that the economy is in the stationary equilibrium of the low regime with $\pi^*\alpha c > \rho$ (Proposition 5). Can an exogenous change of animal spirits lift the economy out of that state and set the economy on a path back to full employment?

Let period 0 be the first period in which the consumption is higher than in the low regime. In that period, we must have $X_0 \leq 1$. From (25) with α replaced by 1 (in the high regime) and using $X_1 \geq 0$, (more money is better),

$$X_0 \geq \beta\pi(0)(1+c),$$

where $\pi(0)$ is the unemployment rate in period 0 on the transition path of the high regime. In that regime, the only agents who do not consume are liquidity-constrained in state 0. Hence $\pi(0) = \gamma_0^*$, where γ_0^* is determined by $\gamma_0^* = P(\gamma_0^*)$ in (14).

X_0 cannot be smaller than 1 if $\beta\gamma_0^*(1+c) > 1$, which is equivalent to

$$\gamma_0^*(1+c) > 1+\rho, \tag{32}$$

with γ_0^* depending only on M (Proposition 4). If c is sufficiently large (to satisfy $\pi^*\alpha c > \rho$ and (32)), then $X_1 > 1$ and there is no first period in which agents in state 1 with a low type shift to consumption instead of saving. The stationary equilibrium in the low regime is the only equilibrium. The previous discussion is summarized by the next result.

Proposition 7

There exists a value $\bar{c}$ such that if $c > \bar{c}$,

(i) if the economy is at or near the full-employment stationary equilibrium, a shift of expectations can push the economy to a low regime path with an unemployment rate that converges to a strictly positive value;

(ii) if the economy is at the stationary equilibrium of the low regime with positive unemployment, that is the unique equilibrium.

The previous result shows the existence of a liquidity trap equilibrium. In that equilibrium, agents attempt to accumulate money balances because of the uncertainty of future exchanges. There is an asymmetry between the high regime with full employment and the low regime that leads to a liquidity trap. In any period, a switch from the high to the low regime can occur, but if the economy has been sufficiently long in a low regime, the economy may not switch back to a path toward full employment and the low regime may be the only equilibrium with a permanent positive unemployment rate.

7 Extensions

Unbounded money holdings

We now show that the assumption of an arbitrary upper-bound on money holding can be removed. If the total quantity of money M is greater than 1, there can be a full-employment equilibrium. In that equilibrium, the distribution of money is stationary but somewhat irrelevant. The interesting case is an equilibrium with unemployment. The next result, which is proven in the Appendix, shows that the upper-bound on money holdings that was assumed in the previous section is not a serious restriction.

Proposition 8

In an equilibrium with unemployment and a stationary distribution of money, the distribution of individual money balances has a bounded support.

To prove the result, note first that if there is K such that all agents in a state $k > K$, (of a low and a high type) consume, then the distribution is obviously bounded by K. Assume therefore that there is an infinite and increasing sequence k_j such that the low type agents in state k_j prefer to save. Let $f(n)$ be the number of elements of sequence $\{k_j\}$ with $k_j < n$. We therefore assume that $f(n)$ tends to infinity if n tends to infinity.

Choose a value N, arbitrarily large. Generalizing equations (22) and (23),

$$(I - \beta A)U = B, \tag{33}$$

where I is the N-identity matrix and

$$A(\pi) = \begin{pmatrix} \pi & (1-\pi)\alpha & b & 0 & \dots & 0 \\ \pi & (1-\pi)\alpha & b & 0 & \dots & 0 \\ 0 & \pi & (1-\pi)\alpha & b & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \pi & (1-\pi)\alpha & b \\ 0 & 0 & 0 & \dots & \pi\alpha & 1-\pi\alpha \end{pmatrix}, \quad B(\pi) = \begin{pmatrix} -\alpha c - \beta b \\ 1 - \beta b \\ \vdots \\ 1 - \beta b \\ 1 - \beta\pi(1-\alpha) \end{pmatrix} \tag{34}$$

Define the vector $\Delta = (U_0, U_1 - U_0, U_2 - U_1, \dots, U_N - U_{N-1})'$. By definition of k_n, and $\Delta_{k_n} \geq 1$.

$$\Delta = J.U, \qquad \text{with} \qquad J = \begin{pmatrix} 1 & 0 & 0 & \dots \\ -1 & 1 & 0 & \dots \\ 0 & -1 & 1 & 0 \\ \vdots & \vdots & \ddots & \ddots \\ 0 & \dots & -1 & 1 \end{pmatrix}$$

Using (33),

$$B = (I - \beta A).J^{-1}.\Delta, \tag{35}$$

with

$$J^{-1} = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 1 & 1 & 0 & \dots \\ 1 & 1 & 1 & \dots \\ \vdots & \vdots & \ddots & \ddots \\ 1 & \dots & 1 & 1 \end{pmatrix}$$

Let $v = J^{-1}.\Delta$. From the definitions of v and $f(k)$,

$$v_k \geq n(k) + U_0.$$

Since $U_0 \geq -\alpha c/(1-\beta)$ (the level of utility if an agent never consumes),

$$v_k \geq f(k) - \frac{\alpha c}{1-\beta}.$$

From (35) and the definition of the matrix A,

$$B_N = \pi\alpha v_{N-1} + (1-\pi\alpha)v_N \geq \pi\alpha f(N-1) + (1-\pi\alpha)f(N) - \frac{\alpha c}{1-\beta}.$$

But $B_N < 1$. Since $f(N)$ tends to infinity with N, we have a contradiction. The support of the distribution cannot be unbounded.

Algorithm for the stationary equilibrium in the low regime

From Proposition 8, one can suggest the following algorithm to determine the stationary equilibrium in the low regime. Let the quantity of money M be given. If $N = M$, there is no unemployment.

1. Set first $N = M + 2$ to look for an equilibrium where the maximum money holding is $M + 1$. Compute the stationary distribution of money balances that is the eigenvector to the eigenvalue 1 of the matrix $L(\pi)$ that satisfies (4), with π given in (8).

2. Determine the vector Δ given by

$$\Delta = J.(I - \beta A)^{-1}B.$$

3. If $\Delta_k \geq 1$ for $1 \leq k \leq M+1$ and $\Delta_{M+2} \leq 1$, then we have found the equilibrium and the maximum money holding is $M+1$. If $\Delta_{M+2} > 1$, then the maximum of money holding cannot be bounded by $M+1$ when there is no exogenous bound at $M+1$. Go to step 1 where N is increased by one unit and repeat the steps until $\Delta_N \leq 1$.

8 Conclusion

When the economy is in a liquidity trap, a uniform lump-sum distribution of money can make the switch to a high regime possible. However, if expectations remain of a low regime, that regime may still be an equilibrium. The policy has some effect however because we have seen that in the stationary equilibrium of the low regime, the rate of unemployment is inversely related to the money supply. There is probably a sufficient quantity of money expansion that can eliminate the unemployment[6].

The reduction of the price level by policy to a new value that is still an equilibrium value has the same effect as an expansion of money in this model. That equivalence may not hold if agents are able to borrow from financial institutions, an issue that will examined later.

[6]If individual money balances are bounded by some number N, we have seen that if $M \geq N$, there is full-employment. Without upper-bound, the property presumably holds also.

APPENDIX: Proofs

Proposition 1

Assume that $M > 1$. The sum of the masses of agents in all states is equal to 1 and the vector $\Gamma(t)$ with $\sum_{k=0}^{N} \gamma_k(t) = 1$ belongs to the simplex of dimension N that is compact. Suppose that has an accumulation point Γ^* with $\gamma_0^* > 0$, then $\pi(t) > \epsilon$ for some finite $\epsilon > 0$ and for an infinite number of values of t. Inspection of the matrix $A(\pi)$ shows that in this case, $\gamma_N^* = 0$ which implies that $\gamma_{N-1}^* = 0$ and so on: $\gamma_k^* = 0$ for $k \geq 2$. Because $M = \gamma_1^* \leq 1$, we have a contradiction. The sequence $\pi(t) = \gamma_0(t)$ converges to 0.

Let $\tilde{\Gamma}(t)$ the vector such that $\tilde{\gamma}_k(t) = 0$ for $k = 0, 1$ and $\tilde{\gamma}_k(t) = \gamma_k(t)$ for $2 \leq k \leq N$. Define the norm $|\tilde{\Gamma}| = \sum_{k=2}^{N} |\tilde{\gamma}_k|$. One verifies that

$$|\tilde{\Gamma}(t+1)| = |A(\pi_t)\tilde{\Gamma}(t)| \leq |\tilde{\Gamma}(t)|,$$

with strict inequality if $\pi(t) > 0$. The application defined by $A(\pi_t)$ is contracting and there cannot be two distincts accumulation points of a sequence $\tilde{\Gamma}(t)$ which therefore converges to a limit. Since $\gamma_k(t) = \tilde{\gamma}_k(t)$ for $2 \leq k \leq N$, $\gamma_1(t) = 1 - \sum_{k=2}^{N} \gamma_k(t)$ and $Lim \gamma_0(t) = 0$, we have proven that $\Gamma(t)$ has a limit.

Assume that $M < 1$. Since $M = \sum k\gamma_k(t)$, $\pi(t) = \gamma_0(t) \geq 1 - M$. For $k \geq 2$, $\gamma_k(t)$ tends to 0 and $\pi(t)$ converges to $\pi^* = 1 - M$. The distribution $\Gamma(t)$ converges to $(1 - M, M)'$.

$\square$

To establish Proposition 3, we need the following Lemmata.

Lemma 2

For any distribution of money Γ, let $\tilde{\Gamma}(\pi) = L(\pi)\Gamma$, where $L(\pi)$ is the transition matrix given in (12). If $\pi' \geq \pi$, then the distribution $\tilde{\Gamma}(\pi')$ dominates the distribution $\tilde{\Gamma}(\pi)$ in the sense of first-order stochastic dominance: for any $K < N$,

$$\sum_{k=0}^{k=K} \tilde{\gamma}_k(\pi') > \sum_{k=0}^{k=K} \tilde{\gamma}_k(\pi). \tag{37}$$

We assume $N \geq 3$ since the case $N = 2$ is treated separately in ***. For $K = 1$, the inequality is verified because $\gamma_0 > 0$. For $K = 2$,

$$\tilde{\gamma}_0(\pi) + \tilde{\gamma}_1(\pi) = \gamma_0 + \Big(\pi(1-\alpha) + \alpha\Big)\gamma_1 + \alpha\pi\gamma_2,$$

which is non-decreasing in π.

Likewise, for $K \leq N-1$,

$$\sum_{k=0}^{K} \tilde{\gamma}_0(\pi) = \sum_{k=0}^{K-1} \gamma_0(\pi)\gamma_0 + \Big(\pi(1-\alpha)+\alpha\Big)\gamma_K + \zeta_K \pi \gamma_{K+1}, \tag{38}$$

where $\zeta_K = \alpha$ if $K \leq N-2$, and $\zeta_K = 1$ if $K = N-1$.

The right-hand side in (38) is non-decreasing in π.

□

Lemma 3

If $\Gamma^2 \succeq \Gamma^1$, then for any π, $L(\pi)\Gamma^2 \succeq L(\pi)\Gamma^1$.

Lemma 4

If $\Gamma^2 \succeq \Gamma^1$, then the quantity of money generated by Γ^2 is smaller than that generated by Γ_1.

Proposition 3

Let $\Gamma^*(\pi)$ the steady state distribution associated to π and take $\hat{\pi} > \pi$. Define the sequence $\hat{\Gamma}_t = L(\pi')^t\Gamma^*$ with $\hat{\Gamma}_0 = \Gamma^*$. From the previous Lemma, and the definition of Γ^*,

$$\Gamma_1 = \mathcal{T}(\Gamma^*; \hat{\pi}) \succeq \mathcal{T}(\Gamma^*; \pi) = \Gamma^* = \hat{\Gamma}_0.$$

Assume that $\hat{\Gamma}_t \succeq \Gamma_{t-1}$. From Lemma 3, $\hat{\Gamma}_{t+1} \succeq \hat{\Gamma}_t$. By the increasing stochastic dominance, the sequence $\gamma_0(t)$ is increasing and converges. The sequence $\gamma_0(t)+\gamma_1(t)$ is increasing and converges. Since $\gamma_0(t)$ converges, $\gamma_1(t)$ converges. By induction γ_k converges for any $k = 0, \ldots, N$. The vector $\hat{\Gamma}_t$ converges. It must converge to an eigenvector associated to $\hat{\pi}$. The proof is concluded by using Lemma 4.

□

The dynamics of the low regime with three states

Let I be the identity matrix of dimension 3. The matrix $I - L$ is of rank 2 and the value of γ is the unique eigenvector of L associated to the eigenvalue 1 such that

$\gamma_0 + \gamma_1 + \gamma_2 = 1$. It is a function of π (through the matrix L), and is determined by

$$\gamma_0(\pi) = \frac{\alpha\pi^2}{1 - \pi - \alpha(1 - 2\pi)}, \qquad \gamma_1(\pi) = \frac{\pi(1 - \pi)}{1 - \pi - \alpha(1 - 2\pi)}, \tag{39}$$

The agents who do not consume are in state 0 or in state 1 with the low type (with probability $1 - \alpha$). The mass of such agent is also the probability π that any seller faces no buyer:

$$\pi = \gamma_0(\pi) + (1 - \alpha)\gamma_1(\pi). \tag{40}$$

The values of the $\gamma_0(\pi)$ and $\gamma_1(\pi)$ in (39) satisfy this equation which is therefore redundant. The value of π will depend on the quantity of money as it is shown now.

The transition matrix L in the case $N = 2$ is given in (12). For any t, the quantity of money is $M = 2(1 - \gamma_0(t) - \gamma_1(t)) + \gamma_1$,

$$\gamma_1(t) = -2\gamma_0(t) + S, \qquad \text{with} \qquad S = 2 - M, \tag{41}$$

and the rate of unemployment, $\pi(t)$, is equal to $\gamma_0(t) + (1 - \alpha)\gamma_1(t)$. Hence,

$$\pi(t) = -(1 - 2\alpha)\gamma_0(t) + (1 - \alpha)S, \tag{42}$$

and

$$\begin{aligned} \gamma_0(t + 1) &= \pi(t)\Big(\gamma_0(t) + \alpha(-2\gamma_0(t) + S\Big), \\ &= \Big((2\alpha - 1)\gamma_0(t) + (1 - \alpha)S\Big)\Big((1 - 2\alpha)\gamma_0(t) + \alpha S\Big), \end{aligned}$$

or

$$\gamma_0(t + 1) = -(1 - 2\alpha)^2\gamma_0(t)^2 + (1 - 2\alpha)^2 S\gamma_0(t) + \alpha(1 - \alpha)S^2. \tag{43}$$

The value of γ_0 characterizes the distribution of money because γ_1 and γ_2 can be derived from γ_0 using the mass one of agents and the quantity M of money.

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