Implementation of Majority Voting Rules*

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Abstract
I study implementation by agenda voting, a straightforward mechanism which is widely used in practice—particularly for legislative decision-making. The main result of the paper establishes that any neutral majority voting rule which satisfies two necessary conditions identified in prior work (McKelvey and Niemi [1978]; Moulin [1986]) as well as a significantly weakened version of Sen’s α [1971] can be implemented by sophisticated voting (Farquharson [1957/1969]) on an agenda. Since these sufficient conditions are satisfied by almost every majority voting rule discussed in the literature, the main result shows that the Copeland Set is effectively the only well-known majority voting rule that is not agenda implementable. The main result also helps clarify what can be implemented via dominance solvable voting and backward induction, two related solutions concepts which are broadly appealing though poorly understood.

JEL Codes: C72, D71, D72, D78. Keywords: Majority voting rules, implementation, voting agendas, tournament solutions, sophisticated voting, dominance solvable voting schemes, backward induction.

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1 Introduction

In this paper, I study the implementation of majority voting rules in the classical environment of complete information and strict voter preferences. When there are two candidates, majority voting defines a social choice rule that is both uniquely appealing from a normative standpoint (May [1952]) and straightforward to implement in dominant strategies. For more candidates, the pairwise majority relation can be used to define a variety of majority voting rules, called tournament solutions (Laslier [1997]), that preserve some appealing properties (like neutrality) associated with the two-candidate case. Some well known tournament solutions include the Copeland Set [1951], the Slater Set [1961], the Uncovered Set (Miller [1977]; Fishburn [1977]), the Banks Set [1985], the Minimal Covering Set (Dutta [1988]), the Tournament Equilibrium Set (Schwartz [1990]), and the Bipartisan Set (Laffond, Laslier, and Le Breton [1993]). Since majority voting rules are anonymous (and, thus, non-dictatorial), the Gibbard [1973] and Satterthwaite [1975] result implies that no such rule can be implemented in dominant strategies when there are more than two candidates.

The goal of this paper is to determine what can be said about the implementation of majority voting rules using a simple and practical mechanism. I address this question by focusing on sophisticated agenda voting (Farquharson [1957/1969]).

The basic idea of sophisticated agenda voting is to leverage the strategyproofness of majority voting between two candidates into situations where there are more candidates. Formally, an agenda defines a binary game tree where the set of viable candidates is progressively reduced through a sequence of majority votes. For the class of extensive form games defined by agendas, sophisticated voting is tantamount to backward induction (McKelvey and Niemi [1978]) or, equivalently, the iterated deletion of weakly dominated strategies in the associated strategic (normal form) game (Moulin [1979]).

Intuitively, the idea is that, in any stage game, the forward-looking electorate effectively chooses between the equilibrium candidates of the two subgames. Accordingly, voters have a dominant strategy to “vote for their preferred candidate” in every stage.

While agenda voting has been studied for more than fifty years, surprisingly little is known about what can be implemented. In recent surveys, Palfrey [2002] described

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1 Technically, there can be no ties – which requires a tie-breaking rule or an odd number of voters.

2 If the equilibrium candidates of the two subgames are identical, the voters’ choice is immaterial.
the characterization of implementation by agenda as “a major question in social choice theory” (p. 2293) while Moore [1992] indicated that it is “one of the most fascinating problems in implementation theory” (p. 238).3

The main contribution of the paper is to resolve this long-standing question. More specifically, I show that every candidate-neutral majority voting rule which satisfies the two necessary conditions identified in prior work (McKelvey and Niemi [1978]; Moulin [1986]) as well as a significantly weakened version of Sen’s $\alpha$ [1971] can be implemented by agenda. Because the sufficient conditions are satisfied by every tournament solution mentioned above (except the Copeland Set), the characterization establishes that virtually all of the neutral majority voting rules discussed in the literature can be implemented by agenda.

1.1 Motivation

While majority voting rules can be implemented using a variety of different solution concepts (discussed at greater length in Section 1.3 below), there are several compelling reasons to focus on agenda voting.

For one, sequential mechanisms, like agenda voting, are relatively straightforward since they rely only on the agents’ ability to do backward induction (Moore [1992]). As an added virtue, the restrictive structure of agenda voting rules out a variety of artificial features, like randomization and nuisance strategies (e.g. integer games and bad outcomes), that are used as implementation tricks with other solution concepts. Despite the fact that it is an appealing way to decentralize choice, surprisingly little is known about implementation by agenda. While McKelvey and Niemi [1978] and Moulin [1986] have identified necessary conditions for implementation (see Laslier [1997] for a survey), there has been little progress in terms of sufficient conditions since the partial result obtained by Srivastava and Trick [1996] (discussed below).

No less compelling is the fact that agenda voting is widely used in practice, particularly for legislative decision-making (Riker [1958]; Farquharson [1969]; Sheplese and Weingast [1982]; Rasch [2000]; Schwartz [2008]). There is a massive literature, dating back to Black [1948, 1958], whose goal is to understand what political outcomes can be achieved with agenda voting. Even though a wide variety of agendas are used in practice, the literature has focused on understanding what can be implemented with

particular kinds of agendas (Miller [1977, 1980, 1983]; Sheplse and Weingast [1984]; Banks [1985, 1989]; Ordeshook and Schwartz [1987]; Coughlan and Le Breton [1999]). While this work has yielded a number of key insights, it has not been able to provide a broad understanding of what can be achieved with agenda voting.

A third reason to study agenda voting is to clarify what can be achieved with less restrictive solution concepts. Of particular interest are the dominance solvable voting schemes studied by Moulin [1979, 1980, 1983, 1984] and implementation via backward induction (characterized by Golberg-Gurvich [1986] and Herrero-Srivastava [1992], and later shown to generalize sophisticated agenda voting by Dutta and Sen [1993]). While both solutions concepts are appealing for the broadly same reasons as agenda voting, relatively little is understood about what can be implemented with either one. For dominance solvable voting, the problem is the lack of an axiomatic characterization (see e.g. Moulin [1994]). For backward induction, the issue is more that the existing characterization is cumbersome and quite difficult to check in practice (see e.g. Moore [1992] and Palfrey [2002]).

1.2 Overview of the Results

The paper provides five groups of results related to implementation by agenda. Taken together, these results extend our understanding of what can be achieved with a straightforward and restrictive voting mechanism which is frequently used in practice. Not only is it possible to implement a wide variety of majority voting rules by agenda, but it is also possible to approximate a range of rules that cannot be implemented. More broadly, the results also help clarify what can be implemented with the less restrictive solution concepts of dominance solvable voting and backward induction.

Necessary and Sufficient Conditions

The main group of results addresses two separate but related notions of implementation by agenda. For single-valued social choice rules, I provide an exact characterization of the majority voting rules that are implemented by agenda when the “seeding” of the candidates on the terminal nodes is fixed. The second notion of implementation allows the seeding of candidates to vary. By permuting the candidates associated with a particular seeding, one obtains a re-seeded agenda which has the same structure as the original agenda but nonetheless changes how candidates are
paired for comparison.\textsuperscript{4} For multi-valued rules, I provide sufficient conditions which ensure that the candidates selected by the majority voting rule are precisely those chosen by sophisticated voting on some seeding of a fixed agenda.

\textit{Single-Valued Rules:} I show that a majority voting rule can be implemented by a seeded agenda if and only if, for every pair of majority relations, there exists a seeded agenda which selects the candidate indicated by the voting rule for both majority relations. Like a variety of other solution concepts (e.g. Nash implementation), the result establishes that a rule can be implemented when it satisfies a condition defined over pairs of voter profiles.

The result extends the work of Srivastava and Trick [1996], who provide necessary and sufficient conditions for “pairwise implementation” (on restricted domains consisting of two voter profiles). In the same paper, they conjecture that their \textit{pairwise condition} might be sufficient to ensure that a majority voting rule can be implemented by seeded agenda.\textsuperscript{5} The characterization given here establishes their conjecture.

The proof of the result employs algebraic tools (due to Maroti [2002]) to extend Srivastava and Trick’s pairwise implementation result to the full domain of voter profiles. Technically, the approach is somewhat unconventional. For sufficiency results in implementation, the proof is typically carried out by constructing a mechanism which implements all rules with the prescribed features. In contrast, the proof given here is obtained by algebraic methods which are partly non-constructive. The basic idea is that extensive form games can be “added” together at the root. The strength of this approach is that the equilibrium of the new game can be determined from the equilibria of the original games. While the intuition is straightforward, no other work in implementation (with the exception of Golberg and Gurvich [1986]) leverages the algebraic structure of extensive form games.

\textit{Multi-Valued Rules:} The main results of the paper identify three sets of conditions which are sufficient for the implementation of majority voting rules by agenda. These conditions are closely related to the two necessary conditions identified in previous work. The first necessary condition, \textit{Condorcet Consistency} (COND), requires that every candidate selected by the voting rule “indirectly defeats” every other candidate through a sequence of majority comparisons (McKelvey and Niemi [1978]).

\textsuperscript{4}Notice that the re-seeding occurs at the level of the candidates \textit{rather than} the terminal nodes.\textsuperscript{5}They also posit that a weaker condition – pairwise implementation for every pair of \textit{adjacent} majority relations (that differ only in terms of the ranking of two alternatives) – might be sufficient.
other words, the candidate must belong to the Condorcet Set (a tournament solution independently proposed by Ward [1961], Good [1971], Schwartz [1972], and Smith [1973]). The second necessary condition, Weak Composition Consistency (WCOM), relates to pairs of tournaments which differ only on a common component (Moulin [1986]). Formally, a component is a subset of candidates who bear the same majority relationship to all other candidates. Basically, WCOM states that changes to the majority relation on a component only affects which candidates are selected from that component. Intuitively, changes in the majority relationship among “comparable candidates” does not influence the selection of any other candidate.

The simplest characterization establishes that a significantly weakened version of Sen’s α [1971], called Component α (COM-α), together with the two necessary conditions, is sufficient for implementation by agenda. Formally, COM-α requires that every selected candidate in a given component must also be selected when the ballot is restricted to that component. Intuitively, the removal of candidates who are “not comparable” to a particular candidate cannot prevent that candidate from being selected by the majority voting rule.

By mildly strengthening either necessary condition while preserving the other, one obtains two additional sets of sufficient conditions. In particular, one can strengthen COND to Strong Condorcet Consistency (SCOND) by requiring that Condorcet Consistency hold within components. Alternatively, one can strengthen WCOM to the Composition Consistency (COM) property proposed by Laffond, Lainé, and Laslier [1996]. Since it is generally known which tournament solutions satisfy COM (Laffond, Lainé, and Laslier [1996]), the characterization with COM and COND is arguably more practical. However, it is formally less general than the characterization with WCOM and SCOND. In terms of generality, the characterization with COM-α is nested between these two characterizations.

Some Popular Tournament Solutions

The second group of results applies the sufficient conditions identified in the main results to show that six of the most well known tournament solutions can be implemented by agenda: the Slater Set, the Uncovered Set, the Banks Set, the Minimal Covering Set, the Tournament Equilibrium Set, and the Bipartisan Set.

This significantly extends our understanding of what can be implemented by agenda. Previously, the only tournament solutions known to be implemented by this
mechanism were the Condorcet Set and the Banks Set. In a series of papers, Miller [1977, 1980, 1983] showed that both solutions are implemented by agendas frequently used by legislatures: the former by the *simple agenda* common in the European legal tradition; and, the latter by the *amendment agenda* popular in common law jurisdictions. Besides Coughlan and Le Breton [1999] (discussed below), the only other notable result was that the Copeland Set cannot be implemented (Moulin [1986]).

**Some Additional Tournament Solutions**

The third group of results serves to highlight the richness of the characterizations by showing that it is possible to recombine tournament solutions satisfying the sufficient conditions to obtain additional solutions that are agenda implementable. In particular, I establish that the class of tournament solutions satisfying WCOM and COND is closed under union while the class of tournament solutions satisfying COM and SCOND is closed under intersection, composition, and stabilization (i.e. the operation of taking the minimal von Neumann-Morgenstern [1944] stable sets associated with the tournament solution, see Brandt [2011]). These results stand in contrast to the situation with dominant strategy and Nash implementation (Benoit, Ok, and Sanver [2007]; Kutlu [2008]). While Nash implementation is preserved under union, neither solution concept is preserved under intersection.

The properties of closure under intersection and composition are particularly useful for showing that some well-known refinements of the popular tournament solutions can be implemented by agenda. Most notable are the tournament solutions, called the *k-Uncovered Set* and the *k-Banks Set*, defined by taking k iterations of the underlying solution concept (see Laslier [1997]). Two other refinements worth noting are the corresponding limits of these families, the *Iterated Uncovered Set* and the *Iterated Banks Set*, obtained by letting k → ∞. The fact that these tournament solutions are agenda implementable extends the results of Coughlan and Le Breton [1999]. Using an agenda consisting of nested amendment agendas, they establish only that a sub-correspondence of the Iterated Banks Set can be implemented by agenda.6

**Approximate Implementation by Agenda**

The fourth group of results addresses approximate implementation—an issue whose study is motivated by the fact that the Copeland Set cannot be implemented by agenda. Since it is coarser than the Iterated Banks Set, the same is true for the Iterated Uncovered Set.
agenda. Given this negative result, it is natural to ask whether there is an agenda that gets “close” to the Copeland Set or, more generally, “close” to another tournament solution which cannot be implemented by agenda.

The first approximation result establishes that there is an agenda which only selects candidates whose Copeland score is at least two-thirds that of the Copeland winner(s). The constant lower bound of $2/3$ is a significant improvement on the asymptotically vanishing lower bounds identified in earlier work (Fischer, Procaccia, and Samorodnitsky [2011]; Iglesias, Ince, and Loh [2012]).

The second result, which applies more generally, addresses the possibility of approximating a tournament solution by agenda implementable solutions which are close in a set-theoretic sense. The goal is to nest the solution in question between lower and upper approximations which can be implemented by agenda. To that end, I provide necessary and sufficient conditions for a tournament solution to have a maximal sub-correspondence which satisfies COM and COND and a minimal super-correspondence which satisfies WCOM and SCOND. These approximations are (respectively) analogous to the notions of weak implementation (Thomson [1996]; Maskin and Sjöström [2002]; Benoît, Ok, and Sanver [2007]) and minimal monotonic extensions (Sen [1995]; Thomson [1999]) studied in the context of Nash implementation.

Implications for Related Solution Concepts

The final group of results describes the implications for two related solution concepts. Starting from tournament solutions that can be implemented by agenda, I establish that it is possible to define a variety of neutral or anonymous rules which can be implemented via dominance solvable voting and backward induction.

These results extend our understanding of what can be achieved with these solution concepts. With dominance solvable voting, it is practically impossible to implement a Pareto efficient social choice rule that is anonymous and neutral (Moulin [1980]): if the number of voters has a prime factor less than the number of candidates, no implementable rule satisfies all three criteria. Because it is a special case of dominance solvable voting, the same is true for implementation via backward induction (Moulin [1979]; Golberg and Gurvich [1986]). While the literature does identify a number of implementable Pareto efficient rules, most notably the neutral voting by repeated veto (Mueller [1978]; Moulin [1979]) and the anonymous voting by repeated veto.
unanimous approval (Moulin [1980, 1984]), it does not provide a broad understanding of the appealing rules which can be implemented with either solution concept.

The two results established here describe general families of anonymous/neutral rules that can be implemented via dominance solvable voting and backward induction. The anonymous “fixed seeding” rules identified are directly related to agenda voting. Given an agenda which implements a tournament solution, it is possible to implement, via backward induction, the single-valued social choice rule which selects the sophisticated voting outcome for a fixed seeding of the agenda (Moulin [1979]; Dutta and Sen [1993]). The neutral “tie-breaker” rules identified involve dictatorial selections from tournament solutions. Given an agenda implementable tournament solution, it is possible to implement, via backward induction, the single-valued social choice rule which selects a particular voter’s most preferred candidate among those selected by the tournament solution.

These findings extend the results established in prior work. Using the fixed seeding rule on the amendment agenda, it was known that anonymous selections from the Banks Set could be implemented via dominance solvable voting (Moulin [1979]) and backward induction (Dutta and Sen [1993]). It was also known that the tie-breaker rule could be implemented (via backward induction) in the case where the tournament solution is the Condorcet Set (Herrero and Srivastava [1992]; Dutta and Sen [1993]).

Given the flexibility of the sufficient conditions identified in the main results, the fixed seeding and tie-breaker rules can actually implement selections from a much wider variety of tournament solutions. Of particular interest are the implementable refinements of the Uncovered Set discussed above (e.g. the iterations of the Uncovered Set, the Minimal Covering Set, the Banks Set and its iterations, the Tournament Equilibrium Set, the Bipartisan Set, and the Slater Set). For these tournament solutions, the associated fixed seeding and tie-breaker rules must also be Pareto efficient.

1.3 Additional Related Literature

Besides the related papers discussed above, it is also worth pointing out some additional related work on implementation, tournament solutions, and agenda voting:

Implementation

This paper is part of the broader literature on the implementation of majority voting rules. The scope for implementation varies significantly depending on the
solution concept and the permitted features of the mechanism. While none of the rules studied in this paper are Nash implementable\(^9\) (Özkal-Sanver and Sanver [2010]; Jackson [2001]), all become implementable once the solution concept is refined to undominated Nash equilibrium (Palfrey and Srivastava [1991]). When attention is restricted to bounded mechanisms however, this solution concept loses some of its power. Although certain rules, like the Uncovered Set, can be implemented, other rules, like the Condorcet Set, cannot (Jackson, Palfrey, and Srivastava [1994]). Considerably more flexible are the solution concepts of trembling-hand perfect equilibrium (Sjöström [1993]) and randomized subgame perfect equilibrium (Vartiainen [2007b]). Like agenda voting, they are capable of implementing every majority voting rule studied here (except the Copeland Set).\(^{10}\) With respect to subgame perfect implementation (Abreu and Sen [1990]; Moore and Repullo [1988]; Vartiainen [2007a]), the flexibility afforded by randomization should be emphasized. Many appealing rules cannot be implemented without randomization (Palfrey and Srivastava [1991]).

More specifically, this paper is part of a smaller literature related to implementation with particular voting mechanisms. Most related are the sequential mechanisms with stage games determined by veto voting (Armbruster and Boge [1983]; Felsenthal and Machover [1992]), the Kingmaker game (Dutta [1984]; Howard [1990]), bargaining (Howard [1992]), and weakest-link voting (Bag, Sabourian, and Winter [2009]). Like agenda voting, each of these implements selections from the Condorcet Set.

**Tournament Solutions**

This paper is also part of a vast social choice literature on tournament solutions which dates back to Condorcet [1785]. Laslier [1997] provides a comprehensive treatment of the main results (see also Brandt [2009] for a more recent survey). I mention only two issues that are particularly relevant.

Firstly, this paper is closely related to work on the axiomatic foundations of tournament solutions. Effectively, the sufficient conditions identified in this paper provide axiomatic foundations for a family of tournament solutions that can be implemented by agenda. Particularly relevant is the recent paper by Apesteguia, Ballester and Masatlioglu [2012] which characterizes the single-valued social choice rules induced by sophisticated voting on the simple and amendment agendas. Also related are pa-

\(^9\)This follows from the fact that every rule which satisfies COND violates Maskin monotonicity.

\(^{10}\)For both solutions concepts, a rule can be implemented if it selects the *Condorcet winner* when it exists and never selects a *Condorcet loser*. 

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papers providing characterizations of choice functions induced by backward induction (Xu and Zhou [2007]; Bossert and Sprumont [2013]) or sophisticated voting (Horan [2011]) on a game tree and choice correspondences induced by agenda implementable tournament solutions (Ehlers and Sprumont [2008]; Lombardi [2008, 2009]).

Secondly, this paper is also related to recent work in computational social choice (see e.g. Brandt, Conitzer, and Endriss [2012] for a recent survey). The most relevant work addresses issues related to the complexity (see e.g. Hudry [2009]) and approximate implementation (discussed above) of tournament solutions. Other related papers address the learnability of voting agendas (Procaccia et al. [2009]), the complexity of manipulating agenda elections (Russell and Walsh [2009]; Conitzer, Lang, and Xia [2009, 2011]; Vassilevska Williams [2010]; Stanton and Vassilevska Williams [2011a-c]), and the determination of agenda winners with incomplete information about voter preferences (Lang et al. [2007, 2012]; Xia and Conitzer [2011]).

**Agenda Voting**

Finally, this paper is part of an extensive literature in game theory related to voting agendas, which addresses questions ranging from agenda formation (Austin-Smith [1987]; Dutta, Jackson and Le Breton [2001a]; Duggan [2006]; Bernheim, Rangel, and Rayo [2006]; Penn [2008]; Bernheim and Slavov [2009]; Vartiainen [2012]) to strategic candidate behavior (Dutta, Jackson and Le Breton [2001b, 2002]) and incomplete information (Ordeshook and Palfrey [1988]) in situations where the agenda is fixed.

### 1.4 Layout of the Paper

The remainder of the paper is structured as follows. After addressing some preliminary matters in Section 2, I describe the main characterization results in Section 3. Section 4 applies these results to show that a wide variety of tournament solutions can be implemented by agenda. In Section 5, I address the issue of approximate implementation, first considering the approximation of the Copeland Set before addressing the question in more general terms. Finally, Section 6 examines the implications for implementation via dominance solvable voting schemes and backward induction.
2 Preliminaries

Let $X$ denote a finite set of choice alternatives. The population of agents is given by $N = \{1, \ldots, n\}$ where $n = |N|$ is odd. Let $\mathcal{L}(X)$ denote the collection of linear orders over the alternatives in $X$. An element $\vec{P} = (\succ_1, \ldots, \succ_n)$ of $\mathcal{L}^n(X)$ represents a profile of individual preference orders on $X$. For all profiles $\vec{P} \in \mathcal{L}^n(X)$, the majority relation $M(\vec{P})$ (or simply $M$ when the profile $\vec{P}$ is understood) is defined by $xM(\vec{P})y$ if and only if $x$ majority beats $y$ or, more formally,

$$\{|j \in N : x \succ_j y| > |j \in N : y \succ_j x|\}.$$

Because $n$ is odd, every majority relation $M$ defines a total\(^{11}\) and asymmetric relation, frequently called a tournament, on $X$. Let $\mathcal{M}(X)$ denote the collection of tournaments on $X$ and let $M|_Y$ denote the sub-tournament obtained by restricting $M$ to $Y \subseteq X$.

2.1 Majority Voting Rules

A social choice rule (SCR) on $X$ is a mapping $F : \mathcal{L}^n(X) \to 2X \setminus \emptyset$ which selects a non-empty subset of the outcomes in $X$ for all profiles $\vec{P} \in \mathcal{L}^n(X)$. To distinguish the case where $F$ is single-valued for every profile, a mapping $F : \mathcal{L}^n(X) \to X$ is called a social choice function. A binary social choice rule selects the same outcome(s) whenever the majority relations induced by the profiles $\vec{P}$ and $\vec{P}'$ coincide, namely $F(\vec{P}) = F(\vec{P}')$ whenever $M(\vec{P}) = M(\vec{P}')$. Effectively, a binary social choice rule (binary social choice function) is a mapping from the collection of majority relations $\mathcal{M}(X)$ to subsets of alternatives in $X$ (to alternatives in $X$).\(^{12}\)

A tournament solution is a binary social choice rule defined for every set of alternatives $X$ that satisfies two additional properties. The first, the Condorcet principle, requires that the maximal alternative (frequently called the Condorcet winner) is the only alternative chosen whenever it exists. The second property, neutrality, states that the set of chosen alternatives is unaffected by the labels attached to the alternatives. From a normative standpoint, both properties are appealing.

In order to state these properties more formally, some preliminary definitions are

\(^{11}\)Formally, $xMy$ or $yMx$ for all $x, y \in X$. Thus, $M$ is incomplete only because it is irreflexive.

\(^{12}\)Provided that the number of voters is large enough: if $n \geq c \cdot \frac{|X|}{\log |X|}$ for some constant $c$, every majority relation in $\mathcal{M}(X)$ is induced by some profile in $\mathcal{L}^n(X)$ (see Stearns [1959] and Erdős-Moser [1964], improving the result of McGarvey [1953]).
required. First, define $\max_P X \equiv \{ x \in X : xPy \text{ for all } y \in X \setminus \{ x \} \}$ to be the set of maximal alternatives in $X$ according to the binary relation $P$ (with $\min_P X$ defined analogously). Notice that when $P$ is a tournament, $\max_P X$ is either empty or single-valued. Next, define binary relations $P$ and $P'$ (over sets of alternatives $X$ and $X'$, respectively) to be isomorphic if there exists a bijection $\sigma : X \rightarrow X'$ such that $xPy$ if and only if $\sigma(x)P'\sigma(y)$ for all $x, y \in X$. With a slight abuse of notation, denote the isomorphism between the binary relations $P$ and $P'$ by $\sigma$ (so that $P' = \sigma P$). In the special case where $P = \sigma P$, the isomorphism $\sigma$ is called an automorphism.

**Definition 1** A tournament solution $S$ is a binary SCR which is defined for every $X$ and satisfies the following conditions for all tournaments $M \in \mathcal{M}(X)$:

- **Condorcet Principle** – if $\max_M X \neq \emptyset$, then $S(M) = \max_M X$; and
- **Neutrality** – for every bijection $\sigma : X \rightarrow X'$, $S(\sigma M) = \sigma S(M)$.

Informally, a tournament solution is a neutral mapping which satisfies the Condorcet principle and selects, for every tournament, a non-empty subset of the feasible alternatives. One solution $S$ weakly refines another solution $S'$ if, for every tournament, $S$ picks a subset of the alternatives selected by $S'$. Formally, $S \subseteq S'$ if $S(M) \subseteq S'(M)$ for every tournament $M$. The solution $S$ refines $S'$, denoted by $S \subseteq S'$, if $S$ weakly refines $S'$ (i.e. $S \subseteq S'$) and $S(M') \neq S'(M')$ for some tournament $M'$. In the sequel, $S$ always refers a tournament solution.

Arguably the most well-known known tournament solution is the Condorcet Set. Informally, this solution generalizes the notion of maximization to situations where there is no Condorcet winner. Formally, it can be defined as follows:

**Definition 2** The Condorcet Set (or Top Cycle) $TC(M)$ of a tournament $M$ on $X$ is the smallest subset of $X$ such that $xMx'$ for all $x \in TC(M)$ and $x' \in X \setminus TC(M)$.

Equivalently, the Condorcet Set $TC(M)$ of a tournament $M$ can be defined as the set of maximal alternatives in the transitive closure $cl(M)$ of $M$ so that

$$TC(M) = \max_{cl(M)} X = \{ x \in X : xM...My \text{ for all } y \in X \setminus \{ x \} \}.$$

The other solutions discussed in the introduction are defined in Section 4 below.

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13Formally, $S : \bigcup_X \mathcal{M}(X) \rightarrow \bigcup_X 2^X$ is a mapping such that $\emptyset \subset S(M) \subseteq X$ for any $M \in \mathcal{M}(X)$. 

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2.2 Implementation by Agenda

Formally, a voting agenda can be described as a labelled binary tree. A binary tree $B$ is a pair $(V, <)$ which consists of a finite set $V$ of nodes (or vertices) and a strict (but incomplete) transitive order $<$ on $V$. The order $<$ has a particular structure so that: all nodes have either zero or two successors; and, all nodes except one have a unique predecessor. The $<$-maximal vertices in $V$, denoted by $V_0$, are the leaves of the tree and the unique $<$-minimal vertex $v^*$ (with no predecessors) is the root.

In order to label the leaves $V_0$ of a binary tree $B$ with the alternatives in an index set $I = \{1, \ldots, i\}$ such that $i \leq |V_0|$, let $\ell : V_0 \to I$ define a surjection from the leaves to the elements of $I$. Together, the binary tree $B$ and the labelling $\ell$ define an agenda $A_i \equiv (B, \ell)$. The agenda $A_i$ can be tailored to any set $X$ of $i$ alternatives (i.e. $|X| = i$) by seeding the leaves. Formally, a seeding is a bijection $s : I \to X$ that identifies every alternative in $X$ with one or more leaves of $A_i$. Given $A_i$ and $s$, let $A^s_i$ denote the agenda seeded by the alternatives in $X$.

The agendas associated with two popular legislative procedures are easy to describe in terms of a list $L = (1, \ldots, i)$. For ease of presentation, let $L_j = (j, \ldots, i)$ denote the tail of the list $L$ starting from $j$ and let $(k, L_j)$ denote the list obtained by appending $k$ to the beginning of $L_j$.

The first amounts to sequential approval voting over the list of alternatives:

**Example 1 (Simple Agenda)** There are $i - 1$ stages of voting (at most). In the $j^{th}$ stage, the agents vote between $j$ and $L_{j+1}$. If $j$ wins, the voting ends and $j$ is selected. Otherwise, voting continues into the $(j + 1)^{st}$ stage.

![Figure 1](image-url) The simple agenda $Sim(L)$ on lists of $i = 4$ and $i = n$ alternatives.\(^{14}\)

\(^{14}\)Formally, the simple agenda can be defined recursively by $Sim(1, 2, \ldots, i) \equiv 1 \cdot Sim(2, \ldots, i)$.\(^{14}\)
Whereas the *simple agenda* is commonly employed by legislatures in the European legal tradition, the second type of agenda, called the *amendment agenda*, is more frequently used by legislatures in the common law tradition. Unlike the simple agenda (which involves considering alternatives one at a time), the amendment agenda amounts to sequential majority voting over *pairs* of alternatives from the list:

**Example 2 (Amendment Agenda)** There are exactly \( i - 1 \) stages of voting. The *provisional winner* \( pr_j \) is the lowest index alternative from the winning side of the \((j - 1)^{st}\) stage, with the exception that \( pr_1 = 1 \). In the \( j^{th} \) stage, the agents vote between \((pr_j, L_{j+2})\) and \((j + 1, L_{j+2})\). The outcome is the winner of the \( i^{th} \) stage.

By comparison with \( \text{Sim}(L) \), the amendment agenda \( \text{Am}(L) \) is more complicated:

![Figure 2](image)

**Figure 2** The amendment agenda \( \text{Am}(L) \) on lists of \( i = 4 \) and \( i = n \) alternatives.\(^\text{15}\)

To determine the *overall winner* \( v^*(A^*; M) \) for the majority relation \( M \) on the seeded agenda \( A^* \), one proceeds backward up the tree. The winner \( v(A^*; M) \) at a given leaf \( v \in V_0 \) is the alternative \( s(\ell(v)) \in X \) identified with \( v \) and the winner at a given non-leaf \( v \notin V_0 \) is determined by the majority relation between the winners at the left successor \( v_l \) of \( v \) and right successor \( v_r \) of \( v \). Formally:

\[
v(A^*; M) \equiv \begin{cases} 
  v_l(A^*; M) & \text{ if } v_l(A^*; M) M v_r(A^*; M) \\
  v_r(A^*; M) & \text{ otherwise}
\end{cases}
\]

\(^\text{15}\)The amendment agenda can be defined by \( \text{Am}(1, 2, ..., i) \equiv \text{Am}(1, 3, ..., i) \cdot \text{Am}(2, 3, ..., i) \).
When the voter profile $\vec{P}$ plays the complete information extensive-form game defined by $A^s$, the overall winner $v^*(A^s; M(\vec{P}))$ is the backward induction equilibrium outcome of the game $(A^s, \vec{P})$ (McKelvey and Niemi [1978]).$^{16}$ Equivalently, it is the outcome of the normal form game associated with $(A^s, \vec{P})$ which survives iterated deletion of weakly dominated strategies (Moulin [1979]).$^{17}$

It is clear that every seeded agenda $A^s$ on $X$ induces a binary social choice function on $M(X)$. Intuitively, a binary social choice function $F : L^n(X) \to X$ is implemented by the seeded agenda $A^s$ if, for every profile $\vec{P}$, the sophisticated voting outcome of the agenda game $(A^s, \vec{P})$ is $v^*(A^s; M)$. Formally:

**Definition 3** A binary social choice function $F : M(X) \to X$ is **implementable by seeded agenda** if there exists a seeded agenda $A^s$ on $X$ such that $F(M) = v^*(A^s; M)$ for all $M \in M(X)$.

By varying the seeding of the alternatives in $X$, one instead obtains a set-valued binary social choice rule. Intuitively, a tournament solution $S$ can be implemented by agenda if there exists a family of agendas $\mathcal{A} = \{A_i\}_{i=1}^\infty$ (containing exactly one agenda for every finite number of alternatives) such that, for all $M \in M(X)$ with $|X| = i$, every selected alternative $x \in S(M)$ is the overall winner for some seeding of the agenda $A_i$ and no unselected alternative $x \notin S(M)$ is the overall winner for any seeding of $A_i$. Let $S(X)$ denote the set of all seedings on $X$. Given a family of agendas $\mathcal{A}$, let $V_{\mathcal{A}}(M) \equiv \bigcup_{s \in S(X)} v^*(A^s; M)$ denote the alternatives in $X$ that are the sophisticated voting outcome for some seeding of the agenda $A_i$. Then:

**Definition 4** A tournament solution $S$ is **agenda implementable** if there exists a family of agendas $\mathcal{A}$ such that $S(M) = V_{\mathcal{A}}(M)$ for every tournament $M$.

There are a number of ways to interpret this notion of implementation. Most simply, the fact that $F$ is implementable indicates that there is an agenda $A$ where alternative $x$ is the winner on profile $\vec{P}$ for some seeding of $A$ if and only if $x \in F(\vec{P})$. This is broadly similar to the usual notion of implementation, namely that there is

---

$^{16}$This point is fairly subtle. Since the voting is simultaneous in every stage game, the game is technically an extensive form game of imperfect information. However, it can be transformed into a game of perfect information simply by assigning a voting order at each node. This does not affect the equilibrium outcome of the game (Duggan [2003]).

$^{17}$While the equilibrium outcome is unique for both solution concepts, it is worth pointing out that the equilibrium strategies are not generally unique (see Duggan [2003]; Hummel [2008]).
a game form $G$ where $x$ is an equilibrium of $G(\vec{P})$ if and only if $x \in F(\vec{P})$. Implementation by agenda may also be viewed in more probabilistic terms. In particular, $V_A(M)$ may be understood as the collection of alternatives that are implemented by randomizing over the seeding of the agenda. This has a similar flavor to implementation by randomized mechanism (Vartiainen [2007b]). In that context, $F$ is said to be implementable if there exists a game form $G$ such that, for every profile $\vec{P}$, the support of the mixed strategy equilibrium associated with $(G, \vec{P})$ is $F(\vec{P})$.

The essential feature of both interpretations is that part of the mechanism is left unspecified. One possibility is that this reflects institutional limitations that prevent the mechanism designer from controlling the seeding of the candidates. This is a natural assumption when, for example, the structure of the agenda is specified ex ante by the constitution. Another possibility is that the failure to specify the seeding is a deliberate omission by a mechanism designer whose primary goal is to get a sense of what candidates “might” be elected by agenda voting.

3 Conditions for Implementation

I first describe necessary and sufficient conditions for implementation of binary social choice functions by seeded agenda. Using these conditions, I then derive weak sufficient conditions for implementation of tournament solutions by agenda.

3.1 Binary Social Choice Functions

Theorem 1 establishes that a binary social choice function $F$ can be implemented by seeded agenda if, for every pair of majority relations $M$ and $M'$, there exists a seeded agenda which implements the desired outcomes $F(M)$ and $F(M')$ on these majority relations. Formally, a social choice function $F : \mathcal{M}(X) \to X$ is pairwise implementable for $M$ and $M'$ on $Y \subseteq X$ if there exists a seeded agenda $A^*$ on $Y$ such that $v^*(M; A^*) = F(M)$ and $v^*(M'; A^*) = F(M')$. To state the result:

**Theorem 1**  A binary social choice function $F : \mathcal{M}(X) \to X$ can be implemented by seeded agenda if and only if it is pairwise implementable on $X$ for all pairs of majority relations $M, M' \in \mathcal{M}(X)$.

The conditions for pairwise implementation depend on the global properties of the majority relations. For “distinct” majority relations $M$ and $M'$, any outcomes $x$
and \( x' \) in the Condorcet Sets of \( M \) and \( M' \) are pairwise implementable. For “comparable” majority relations, it is only possible to implement alternatives from the same “neighborhood.” Some definitions are required to formalize these concepts.

Define a **component** of a tournament \( M \) on \( X \) to be subset \( Y \subseteq X \) such that every alternative in \( Y \) bears the same relation to the alternatives in \( X \setminus Y \). Formally, \( y'Mx \) if and only if \( yMx \) for all \( x \in X \setminus Y \) and \( y, y' \in Y \).

A **decomposition** of a majority relation \( M \) on \( X \) is a partition of \( X \) into components. Clearly, the coarsest such decomposition is the degenerate partition \( \{X\} \). If \( M \) is a tournament such that \( TC(M) = X \), then \( M \) is said to be **strong**. For every strong tournament, the coarsest non-degenerate decomposition \( D(M) \) is unique (see e.g. Theorem 1.3.11 of Laslier [1997]). If \( M \) is a strong tournament and \( D(M) = X \) (so that the only non-degenerate decomposition is the trivial partition \( X \)), then \( M \) is said to be **simple**. Finally, a tournament that is not simple is said to be **composed**.

**Definition 5** For a tournament \( M \) on \( X \), the **global structure** \( \langle G(M), M_G \rangle \) is a pair consisting of the maximal non-degenerate decomposition \( G(M) \equiv D(M|_{TC(M)}) \) of \( M \) on the Condorcet Set \( TC(M) \) and the induced quotient relation \( M_G \equiv M/G(M) \). Moreover, any component \( g \in G(M) \) is called a **neighborhood**.

Majority tournaments \( M \) and \( M' \) are said to be **globally comparable** if they have the same global structure (so that \( \langle G(M), M_G \rangle = \langle G(M'), M_G' \rangle \)) and **globally distinct** otherwise. Based on the equivalence established in Theorem 1, the following **pairwise condition** characterizes the binary SCFs that can be implemented by seeded agenda:

**Proposition 1** Given a binary social choice function \( F : \mathcal{M}(X) \to X \), consider two majority relations \( M_1, M_2 \in \mathcal{M}(X) \) and the related outcomes \( F_{12} \equiv (F(M_1), F(M_2)) \).

**(I)** For globally distinct \( M_1 \) and \( M_2 \), \( F_{12} \) is pairwise implementable on \( X \) iff:

\[
F(M_j) \in TC(M_j) \quad \text{for } j = 1, 2.
\]

**(II)** For globally comparable \( M_1 \) and \( M_2 \), \( F_{12} \) is pairwise implementable on \( X \) iff:

(i) there is some neighborhood \( g \) such that \( F(M_j) \in g \) for \( j = 1, 2 \); and,

(ii) \( F_{12} \) is pairwise implementable on a subset \( g^* \) of the neighborhood \( g \).

---

\(^{18}\)To get a better intuition, notice that the Condorcet Set is a component for every tournament \( M \) on \( X \). In particular, it is the smallest component of \( M \) where \( X \setminus TC(M) \) is also a component of \( M \) such that \( cMx \) for some \( c \in TC(M) \) and \( x \in X \setminus TC(M) \).

\(^{19}\)Such tournaments are sometimes called strongly connected, irreducible, or cyclic.
For globally distinct majority relations, the conditions for pairwise implementation are relatively weak. For each majority relation, it is sufficient that the desired outcome belongs to the Condorcet Set. For globally comparable majority relations, the conditions for pairwise implementation are somewhat more restrictive. Specifically, the outcomes on both majority relations must belong to the same neighborhood $g$ of the Condorcet Set. Within the neighborhood $g$, the condition for pairwise implementation is recursive. Provided that the desired outcomes are pairwise implementable on a subset $g^* \subseteq g$, they are pairwise implementable on $X$.

The pairwise condition is closely related to the necessary conditions identified in prior work. McKelvey and Niemi [1978] observed that the overall winner on a seeded agenda must belong to the Condorcet Set (see Lemma 9 of Moulin [1986] for an elegant proof of this result):

**Condorcet Consistency** For all tournaments $M \in \mathcal{M}(X)$, $F(M) \in TC(M)$.

The second necessary condition, identified by Moulin [1986] (see Lemma 10 of his paper), relates to choice on tournaments that differ only on a *common component*. Formally, let $M_Y$ denote the tournament obtained by replacing the component $Y$ with an alternative $y^*$. If $Y$ is a component of $M$, let $M_Y$ denote the tournament on $(X \setminus Y) \cup \{y^*\}$ with $M_Y$ defined by $xM_XX'$ if (i) $xMx'$, (ii) $x' = y^*$ and $xMy$ for $y \in Y$, or (iii) $x = y^*$ and $yMx'$ for $y \in Y$. Moulin observed that changes to the majority relation $M$ on $Y$ only have an effect when the overall winner is in $Y$:

**Adjacency** For all pairs of tournaments $M, M' \in \mathcal{M}(X)$ with a common component $Y$ such that $M_Y = M'_Y$:

(i) $F(M) \notin Y$ implies $F(M') = F(M)$; and,

(ii) $F(M) \in Y$ implies $F(M') \in Y$.

Proposition 1(I) establishes that Condorcet Consistency is necessary and sufficient for pairwise implementation on globally distinct majority relations. In turn, Proposition 1(II) shows that a recursive version of Adjacency (which also requires Condorcet Consistency) is necessary and sufficient for pairwise implementation on globally comparable majority relations. While it is stated differently in their paper (see the Appendix), Srivastava and Trick [1996] also establish the sufficiency of this condition when the Condorcet Sets of $M$ and $M'$ are distinct (see Theorem 2 of their paper). It is straightforward to see that their result is a corollary of Proposition 1.
Proposition 1 was first established by Srivastava and Trick [1996]. Theorem 1 follows directly from this result, although the proof involves novel algebraic methods and does not rely on the explicit construction of an implementing mechanism.

3.2 Tournament Solutions

The necessary conditions discussed in the last section have natural analogs for social choice rules that are not single-valued (as first noted by Laslier [1997]). Extended to correspondences, Condorcet Consistency (COND) strengthens the Condorcet principle by requiring that selected alternatives belong to the Condorcet Set whenever there is no Condorcet winner. Formally:

\textbf{COND} \quad \text{For all tournaments } M, S(M) \subseteq TC(M).

As it applies to correspondences, Moulin’s Adjacency condition is more commonly known as Weak Composition Consistency (WCOM). Given a pair of tournaments \( M \) and \( M' \) that differ only on a common component \( Y \), this IIA-like condition requires that differences between the alternatives chosen on \( M \) and \( M' \) be limited to \( Y \):

\textbf{WCOM} \quad \text{For all } X \text{ and pairs of tournaments } M, M' \in \mathcal{M}(X) \text{ with a common component } Y \text{ such that } M_Y = M'_Y:\n
\begin{enumerate}
  \item \( x \in S(M) \setminus Y \) implies \( x \in S(M') \setminus Y \); and,
  \item \( y \in S(M) \cap Y \) implies \( y' \in S(M') \) for some \( y' \in Y \).
\end{enumerate}

It is straightforward to show that WCOM and COND are necessary to implement a tournament solution by agenda. However, the following simple example illustrates that these conditions are not sufficient to ensure implementation by agenda:

\textbf{Example 3} \quad \text{Consider the majority tournament } M \text{ on } X = \{w, x, y, z\} \text{ defined by:}

\begin{tikzpicture}
  \node (w) at (0,0) {w};
  \node (z) at (1,0) {z};
  \node (x) at (0,1) {x};
  \node (y) at (1,1) {y};
  \draw (w) -- (x);
  \draw (x) -- (y);
  \draw (y) -- (z);
  \draw (z) -- (w);
  \draw (x) -- (w);
  \draw (z) -- (y);
\end{tikzpicture}

\text{Now, suppose that } S \text{ is a tournament solution such that } z \notin S(M) \text{ and } w \in S(M).
It is easy to complete $S$ into a mapping $S : \mathcal{M}(X) \to 2^X$ that satisfies WCOM, COND, and neutrality.\footnote{In particular, COND and neutrality jointly pin down $S(M')$ for all $M'$ on $X$ such that $TC(M') \subseteq X$. Given neutrality, the only indeterminacy in the mapping is whether $x, y \in S(M)$. And, it turns out this is not germane to whether $S$ satisfies any of the three requirements listed.} However, $z \not\in S(M)$ and $w \in S(M)$ cannot both arise for a tournament solution which is implementable by agenda. To see this, first consider the tournament $M'$ that coincides with $M$ except that $wM'z$. Observe that $\{w, z\}$ is a common component of $M$ and $M'$. By neutrality, $w \notin S(M')$ and $z \in S(M')$. Next, suppose that $S$ is implemented by some agenda $A$ (on four alternatives). As a consequence of WCOM, $v^*(A^*; M') = z$ for every seeding $s$ such that $v^*(A^*; M) = w$.

In turn, these choices lead to a contradiction. To grasp the basic idea, fix a seeding $s$ of $A$ and observe that there must be some instance of $w$ (call it $w^*$) that never faces $y$ or $z$ on $A^*$. Otherwise, $w$ cannot be chosen from $A^*$ for $M$. In other words, there is a path $p$ in $A^*$ from the leaf seeded with $w^*$ to the root $v^*$ such that, for $M$, $w^*$ faces only instances of $x$ or $w$ along $p$. In fact, the same must be true for $M'$. The insight is that reversing the majority preference between $z$ and $w$ cannot affect the alternatives that $w^*$ faces along the path $p$. Consequently, $v^*(A^*; M') = w$ which is a contradiction (since the assumptions entail $w \notin S(M')$).

One way to resolve the problem posed by this example to strengthen COND by insisting that every alternative selected from a component $Y$ of $M$ belong to the Condorcet Set of the sub-tournament $M|_Y$ on $Y$. In Example 3, this rules out the possibility that $w \in S(M)$ (and, likewise, that $z \in S(M')$). Formally, this \textit{Strong Condorcet Consistency} (SCOND) requirement can be stated as follows:

**SCOND**  \textit{For all tournaments $M$ with a component $Y$, $S(M) \cap Y \subseteq TC(M|_Y)$.}

The main result of the paper establishes that this requirement, combined with WCOM, is sufficient to ensure that a tournament solution can be implemented by agenda:

**Theorem 2 (Main Characterization)** If $S$ is agenda implementable, then it satisfies Weak Composition Consistency and Condorcet Consistency. Conversely, if $S$ satisfies Weak Composition Consistency and Strong Condorcet Consistency, then it is agenda implementable.

The proof of sufficiency leverages Theorem 1 by exploiting the natural correspondence between tournament isomorphisms and agenda seedings. To get the basic
idea, consider a seeded agenda \( A^s \) and a tournament \( M \). Applied to the seeding \( s \), any permutation \( \sigma : X \to X \) induces a new seeded agenda \( A^{\sigma s} \). Clearly, the winner associated with this re-seeding may be determined from the winner for the isomorphic tournament \( \sigma^{-1} M \) on the original seeding. In particular, it must be that \( v^*(A^{\sigma s}; M) = \sigma v^*(A^s; \sigma^{-1} M) \). Given a tournament \( M \), the winners associated with all seedings of \( A \) can then be determined directly from the winners for isomorphic tournaments on a single seeding \( s \). In other words, it possible to reconstruct \( V_A(M) \) simply by keeping track of \( v(A^s; \cdot) \). The proof of Theorem 2 exploits this insight.

While the gap in Theorem 2 appears quite modest, at least one tournament solution, namely the Condorcet Set, falls between the necessary and sufficient conditions. While this solution satisfies \( \text{COND} \), it does not satisfy \( \text{SCOND} \). Intuitively, the issue is that this solution is “not selective enough” within components. Illustrating this in terms of Example 3, \( w \in TC(M) \) even though \( w \notin TC(M|_{\{w,z\}}) \). However, Miller [1977] shows that the Condorcet Set can be implemented by the simple agenda.

A conceptually different way to resolve the problem in Example 3 is to strengthen \( \text{WCOM} \). In particular, one might insist that the alternatives selected from the component \( \{w, z\} \) on \( M \) coincide with the alternatives selected from the sub-tournament \( M|_{\{w,z\}} \). Since \( zMw \), it follows that \( z \) is the Condorcet winner on \( \{w, z\} \) so that \( S(M|_{\{w,z\}}) = z \) and, hence, \( w \notin S(M) \). (Likewise, \( z \notin S(M') \).)

To formalize, let the composition \( M \equiv \Pi(M^*; M_1, \ldots, M_i) \) of a summary tournament \( M^* \) on \( I = \{1, \ldots, i\} \) with tournaments \( M_j \) on \( X_j \) be defined as a tournament on \( X \equiv \bigcup_{j \in I} X_j \) with \( M \) defined by:

\[
xMy \text{ if } \begin{cases} 
  xM_jy & \text{ for } x, y \in X_j \\
  jM^*k & \text{ for } x \in X_j, y \in X_k \text{ and } j \neq k
\end{cases}
\]

For every composed tournament \( M = \Pi(M^*; M_1, \ldots, M_i) \), each of the sub-tournaments \( M_j \) is a component of \( M \). Whereas \( \text{WCOM} \) imposes some choice regularity across composed tournaments with the same summaries and alternatives in the \( M_j \) components, \( \text{Composition Consistency} \) (COM) imposes the stronger requirement that the selections from composed tournaments are determined recursively. In particular:

\footnote{However, the proof can be adapted to show that the Condorcet Set is agenda implementable. In fact, the proof can be adapted to show that other solutions are implementable as well—like the \text{Condorcet Non-Losers} of each component in the coarsest non-degenerate decomposition of \( TC(M) \). See footnote 25 and Section 5 for the appropriate definitions.}
COM

For all composed tournaments \( M = \Pi(M^*; M_1, \ldots, M_i) \):

\[
S(M) = \bigcup_{j \in S(M^*)} S(M_j).
\]

Like solutions which satisfy WCOM and SCOND, those which satisfy COM and COND can also be implemented by agenda. This is a consequence of the following:

**Lemma 1** (I) If \( S \) satisfies Composition Consistency, it satisfies Weak Composition Consistency. (II) If \( S \) satisfies Composition Consistency and Condorcet Consistency, it satisfies Strong Condorcet Consistency.

By Lemma 1, COM and COND are stronger than SCOND and WCOM. Thus:

**Corollary 1 (Characterization with COM)** If \( S \) satisfies Composition Consistency and Condorcet Consistency, then it is agenda implementable.

While less general than the sufficient conditions in Theorem 2, this characterization is more practical. It is generally known whether established tournament solutions satisfy COM and COND. By comparison, SCOND and, to a lesser extent, WCOM have not been studied as extensively.\(^{23}\) Accordingly, Corollary 1 provides a more straightforward way to show that a known solution can be implemented.

There is a third set of sufficient conditions, nested between these two characterizations, which helps clarify the gap between necessity and sufficiency. This characterization imposes Sen’s \( \alpha \) [1971] in situations where the sub-menu of interest is a component of the majority relation. Formally, this Component \( \alpha \) (COM-\( \alpha \)) requirement can be stated as follows:

**COM-\( \alpha \)** For all tournaments \( M \) with a component \( Y \), \( S(M) \cap Y \subseteq S(M|_Y) \).

It is straightforward to see that COM-\( \alpha \) and COND together imply SCOND.\(^{24}\) Thus:

**Corollary 2 (Characterization with COM-\( \alpha \))** If \( S \) satisfies the necessary conditions and Component \( \alpha \), then it is agenda implementable.

\(^{23}\)To my knowledge, SCOND has not been studied at all before this paper.

\(^{24}\)COM is clearly stronger than COM-\( \alpha \). To see this, note that COM implies \( S(M) \cap Y = S(M|_Y) \).
This result shows that, by adding a fairly innocuous requirement to the necessary conditions identified in prior work, one obtains conditions which are sufficient for implementation. In a sense, this has a similar flavor to Maskin’s [1999] characterization of Nash implementation with No Veto Power. The result helps clarify the intuition that the gap between the necessary and sufficient conditions essentially boils down to a mild form of recursivity.

As a final point, it is worth noting that many popular solutions satisfy WCOM or COM (as discussed at greater length in Section 4). While these properties are usually justified on the normative basis that small changes in the majority preference “should not” have a dramatic impact on which alternatives are selected, the results in this section provide a distinctly positive justification (not unlike the justification for strategyproofness). In particular, they establish that either property, in combination with a suitably strengthened version of the Condorcet property, is sufficient to ensure that a tournament solution can be implemented by agenda.

\section{Implementable Tournament Solutions}

In this section, I apply the sufficient conditions from Section 3 to show that many tournament solutions discussed in the literature can be implemented by agenda. I then show how to construct additional agenda implementable solutions from solutions that satisfy either set of sufficient conditions. For the unfamiliar reader, I first review the definitions of the tournament solutions discussed in this section.

\subsection{Seven Popular Tournament Solutions}

Besides the Condorcet Set, arguably the best known tournament solution is the Uncovered Set. It is based on an idea of Landau [1953] that was later fleshed out by Miller [1977, 1980] and Fishburn [1977]:

\textbf{Definition 6} The Uncovered Set $UC(M)$ of a tournament $M$ on $X$ is the subset of alternatives that majority defeat every other alternative in at most two steps:

$$UC(M) \equiv \max_{M \cup M^2} X = \{ x \in X : xMy \text{ or } xzM My \text{ for all } y \in X \}.$$
Given a tournament $M$ on $X$, a set $Y \subseteq X$ is a covering set for $M$ if it is externally UC-stable in the sense that $x \not\in UC(M|_{Y \cup \{x\}})$ for all $x \in X \setminus Y$. Let $\mathcal{C}(M)$ denote the collection of covering sets for $M$. It is easy to see that the Uncovered Set $UC(M)$ is a covering set for every tournament $M$. As such, the collection $\mathcal{C}(M)$ is non-empty. What is more, $\mathcal{C}(M)$ includes a covering set that refines all other $Y \in \mathcal{C}(M)$. Using this insight, Dutta [1988] proposed the following refinement of the Uncovered Set:

**Definition 7** The Minimal Covering Set $MC(M)$ of $M$ is the minimal member of $\mathcal{C}(M)$ with respect to set inclusion. More formally, $MC(M) \equiv \min_{\subseteq} \mathcal{C}(M)$.

Another solution that refines the Uncovered Set, originally characterized by Banks [1985], exploits the notion of maximal transitive chains. Formally, a set $Y \subseteq X$ is a transitive chain of a tournament $M$ on $X$ if $M|_{Y}$ is transitive. A transitive chain is said to be maximal if there is no $x \in X \setminus Y$ such that $Y \cup \{x\}$ is a transitive chain. Let $\mathcal{T}(M)$ denote the collection of maximal transitive chains of $M$. Then:

**Definition 8** The Banks Set $BA(M)$ of $M$ is the set of alternatives in $X$ that are at the top of some maximal transitive chain. Formally:

$$BA(M) \equiv \{x \in X : x = \max_Y M \text{ for some } Y \in \mathcal{T}(M)\}.$$ 

The next two solutions were motivated by analogies with game theory. Schwartz [1990] developed a tournament solution, called the Tournament Equilibrium Set, that is related to cooperative game theory. The solution is based on the idea of retentiveness. A non-empty $Y \subseteq X$ is said to be S-retentive for $M$ if $S(\{x \in X : xMy\}) \subseteq Y$ for all $y \in Y$ such that $\{x \in X : xMy\} \neq \emptyset$. Let $\mathcal{R}_S(M)$ denote the collection of S-retentive sets for $M$. Given a binary relation $P$ on a set $Z$, define

$$\min^*_P Z \equiv \{z \in Z : zPy \text{ for no } y \in Z \setminus \{z\}\}$$

to be the (weakly) $P$-minimal alternatives in $Z$. With this convention in place:

**Definition 9** The Tournament Equilibrium Set $TEQ(M)$ of $M$ consists of the alternatives in the minimal $TEQ$-retentive subsets of $X$. In other words:

$$TEQ(M) \equiv \bigcup \min^*_\subseteq \mathcal{R}_{TEQ}(M).$$
Laffond, Laslier, and Le Breton [1993] proposed a tournament solution, called the *Bipartisan Set*, that is based on non-cooperative game theory. Given a tournament $M$ on $X = \{x_1, \ldots, x_i\}$, the *tournament matrix* $M = (m_{jk})$ associated with $M$ is an $i \times i$ matrix with $jk$-entry given by:

$$m_{jk} \equiv \begin{cases} 
1 & \text{if } x_j M x_k \\
0 & \text{if } j = k \\
-1 & \text{otherwise}
\end{cases}$$

For every tournament $M$, the associated tournament matrix $M$ determines a two-player zero-sum game. The Bipartisan Set of $M$ is determined by the unique Nash equilibrium of this game. Specifically:

**Definition 10** The Bipartisan Set $BP(M)$ of $M$ consists of the alternatives in $X$ that are part of the support of the unique Nash equilibrium of the game $M$.

The last two solutions share something in common with scoring rules (like those proposed by Borda and Simpson). Slater [1961] proposed to select the winners of the linear orders closest to the majority tournament. To formalize this idea, let

$$\Delta(M, >) \equiv |\{(x, y) \in X^2 : xMy \text{ and } y > x\}|$$

denote the number of differences between a tournament $M$ and a linear order $>$ on $X$. Let $SL(M) \equiv \arg\min_{> \in L(X)} \Delta(M, >)$ denote the linear orders, called Slater orders, that are closest to $M$. Then:

**Definition 11** The Slater Set $SL(M)$ of $M$ is the subset of alternatives in $X$, called Slater winners, that are maximal for some Slater order. Formally:

$$SL(M) \equiv \{x \in X : x = \max_{> \in SL(M)} X \text{ for some } >M \in SL(M)\}.$$
Definition 12 The **Copeland Set** $CO(M)$ of $M$ consists of the alternatives in $X$, called Copeland winners, with maximal Copeland score. In other words:

$$CO(M) \equiv \arg \max_{x \in X} co(x, M).$$

While these are certainly the most widely discussed tournament solutions, they are by no means the only ones that have been proposed. Some additional tournament solutions are discussed later in the paper. For a more exhaustive list, consult Laslier’s monograph [1997] or Brandt’s thesis [2009].

### 4.2 Implementing the Popular Solutions

Most of the popular tournament solutions are known to satisfy COM and COND:

**Remark 1** The Uncovered Set, the Minimal Covering Set, the Banks Set, the Tournament Equilibrium Set, and the Bipartisan Set all satisfy Composition Consistency and Condorcet Consistency.

Given Corollary 1, these tournament solutions can be implemented by agenda:

**Proposition 2** The following tournament solutions are agenda implementable:

(i) the Uncovered Set;
(ii) the Minimal Covering Set;
(iii) the Banks Set;
(iv) the Tournament Equilibrium Set; and,
(v) the Bipartisan Set.

Of these tournament solutions, the Banks Set is the only one previously known to be implementable by agenda—something which is a consequence of that solution’s unusual history. While most tournament solutions are developed by imposing appealing properties that define a social choice rule, the Banks Set was developed in response to academic interest in the amendment agenda (see e.g. Farquharson [1957/1969]; Miller [1977, 1980]; Moulin [1979]; Shepsle and Weingast [1982, 1984]). Only later did Banks [1985] identify the set which bears his name as the tournament solution implemented by the amendment agenda.
Unlike the solutions covered by Proposition 2, the Condorcet Set, the Slater Set, and the Copeland Set all violate COM.\footnote{It is worth noting that each of these three tournament solutions nonetheless satisfies COND. Some tournament solutions, like the Condorcet Non-Losers as defined by $CNL(M) \equiv X \setminus \min_M X$ (see Brandt [2009]), violate both properties.} As discussed in Section 3, the Condorcet Set falls in the gap between the necessary and sufficient conditions of Theorem 2. While it satisfies WCOM and is implemented by the simple agenda, it violates SCOND. In contrast, the Slater Set satisfies both WCOM and SCOND:

**Remark 2** The Slater Set satisfies Weak Composition Consistency and Strong Condorcet Consistency.

Given Theorem 2, the Slater Set can be implemented by agenda:

**Proposition 3** The Slater Set is agenda implementable.

Unlike the other tournament solutions discussed above, the Copeland Set cannot be implemented by agenda. This follows from the fact that it violates WCOM (see e.g. Corollary 8.5.3 of Laslier [1997] adapting an observation due to Moulin [1986]). In the next section, I show that it is nonetheless possible to implement the Copeland winners approximately.

Before moving on, I pause to make two comments about the results above:

(1) In light of Propositions 2 and 3, it might be argued that the sufficient conditions identified in Section 3 are general enough. Since it would appear that almost every tournament solution discussed in the literature which satisfies WCOM also satisfies SCOND, little may stand to be gained by identifying a necessary and sufficient condition that simultaneously weakens SCOND and strengthens COND.

(2) Propositions 2 and 3 should be viewed as existence results. While they establish that a variety of tournament solutions can be implemented by agenda, they provide no indication about the structure of the implementing agendas. To date, the only tournament solutions with known implementing agendas are the Condorcet Set and the Banks Set. An appealing feature of the agendas known to implement these solutions is their recursive structure. Ideally, it would be nice to identify recursive implementing agendas for the solutions covered by Propositions 2 and 3 as well.
Even without constructing implementing agendas, one can nonetheless establish lower bounds on their size. For some tournament solutions, the complexity of the “search problem” (the problem of finding some candidate in $S(M)$ for a given tournament $M$) provides a reasonable lower bound. If the search problem for an agenda implementable tournament solution is NP-hard, the number of nodes in every implementing agenda must be exponential in $|X|$ unless $P = NP$ and the search problem is NP-complete. Since the search problem for the Slater Set is NP-hard (Hudry [2010]), for instance, it would seem safe to assume that any agenda which implements this solution must be at least exponential.

4.3 Implementing Additional Solutions

One appealing feature of tournament solutions that satisfy the sufficient conditions of Theorem 2 or Corollary 1 is that they can be used to construct additional tournament solutions that are implementable.

Given tournament solutions $S$ and $S'$, perhaps the most natural way to construct a new social choice rule is to take their union or intersection tournament by tournament (so that $S \cup S'(M) \equiv S(M) \cup S'(M)$ and $S \cap S'(M) \equiv S(M) \cap S'(M)$ for every tournament $M$). A third possibility is to compose the two solutions by applying $S$ to the alternatives in $S'(M)$ to obtain $S \cdot S'(M) \equiv S(M|_{S'(M)})$ for every $M$.

The Minimal Covering Set suggests a fourth possibility, namely to consider the minimal externally $S$-stable sets of $M$. Given a tournament $M$ on $X$, let $\mathcal{E}_S(M)$ denote the collection of subsets $Y \subseteq X$ that are externally $S$-stable in the sense that $x \notin S(M|_{Y \cup \{x\}})$ for all $x \in X \setminus Y$ and $Y \in \mathcal{E}_S(M)$. Define $\widehat{S}(M) \equiv \bigcup \min_{\subseteq} \mathcal{E}_S(M)$ to be the union of the minimal externally $S$-stable sets.

It is straightforward to see that all four set operations preserve neutrality and the Condorcet principle. As such, $S \cup S'$, $S \cdot S'$, $\widehat{S}$, and $S \cap S'$ (provided that it is non-empty for every $M$) define tournament solutions. Moreover, each of these operations preserves some of the properties discussed in Section 3:

**Lemma 2** Consider tournament solutions $S$ and $S'$. Then:

(I) If $S$ and $S'$ satisfy Weak Composition Consistency (respectively, Strong Condorcet Consistency), then $S \cup S'$ satisfies Weak Composition Consistency (respectively, Strong Condorcet Consistency).
(II) If $S$ and $S'$ satisfy Composition Consistency (respectively, Condorcet Consistency), then: (i) $S \cdot S'$ satisfies Composition Consistency (respectively, Condorcet Consistency); and, (ii) $S \cap S'$ satisfies Composition Consistency (respectively, Condorcet Consistency) provided $S(M) \cap S'(M) \neq \emptyset$ for all $M$.

(III) If $S$ satisfies Composition Consistency and Condorcet Consistency, then $\widehat{S}$ satisfies Composition Consistency and Condorcet Consistency.

Given Theorem 2 and Corollary 1, Lemma 2 implies the following:

Corollary 3 (Construction of Implementable Solutions) (I) If $S$ and $S'$ satisfy Weak Composition Consistency and Strong Condorcet Consistency, then $S \cup S'$ is agenda implementable. (II) If $S$ and $S'$ satisfy Composition Consistency and Condorcet Consistency: (i) $S \cdot S'$ and $\widehat{S}$ are agenda implementable; and, (ii) $S \cap S'$ is agenda implementable provided that $S(M) \cap S'(M) \neq \emptyset$ for every tournament $M$.

Corollary 3 has a variety of practical implications. As discussed at greater length in Section 5, it guarantees that there are agenda implementable approximations for a wide range of tournament solutions. Moreover, it provides a natural way to construct a variety of agenda implementable tournament solutions. To illustrate:

(1) Since $MC = \widehat{UC}$ by definition and the Uncovered Set satisfies COM and COND, it follows directly from Corollary 3 that the Minimal Covering Set can be implemented by agenda voting. In other words, there is no need to check that this tournament solution satisfies the sufficient conditions directly. Similarly, Corollary 3 also establishes that $BA$ can be implemented by agenda. This tournament solution, known as Minimal Extending Set, was only recently proposed by Brandt [2011].

(2) Since $BA(M) \cap MC(M) \neq \emptyset$ for every tournament $M$ (by Proposition 7.1.7 of Laslier [1997]) and both of these solutions satisfy COM and COND, Corollary 3 establishes that their intersection, $BA \cap MC$, can be implemented by agenda. This serves to illustrate that, in some cases, a natural common refinement of two agenda implementable solutions is itself implementable. Incidentally, Corollary 3 also establishes that the union and composition of these solutions can be implemented.

(3) Some tournament solutions are not idempotent in the sense that there are tournaments $M$ such that $S \cdot S(M) \neq S(M)$. When this is the case, $S \cdot S$ refines $S$. Two prominent examples are the Uncovered Set and the Banks Set (see e.g.
Theorems 5.1.7 and 7.1.3 of Laslier [1997]). Given that both of these solutions satisfy COM and COND, Corollary 3 establishes that the $k$-iterations $UC^k$ and $BA^k$ can be implemented by agenda for all $k \in \mathbb{N}$. These solutions, known as the $k$-Uncovered Set and the $k$-Banks Set, are refinements of the underlying solution concepts.

(4) It is sometimes possible to combine composition and intersection to obtain even tighter refinements which are agenda implementable. One natural common refinement of the $k$-Uncovered Sets is their intersection. Formally, this solution, known as the Iterated Uncovered Set $UC^\infty$, is defined by $UC^\infty(M) \equiv \bigcap_{k \in \mathbb{N}} UC^k(M)$ for every tournament $M$.\footnote{While the definition of $UC^\infty$ may appear to involve a countable intersection, the finiteness of $M$ ensures that there are only finitely many distinct $UC^k(M)$ for any tournament $M$. Consequently, the definition of $UC^\infty$ only involves a finite intersection.} In the same fashion, one can define the Iterated Banks Set $BA^\infty$ (which is even finer than $UC^\infty$). Since the underlying solutions satisfy COM and COND, Corollary 3 establishes that $UC^\infty$ and $BA^\infty$ can be implemented by agenda. This observation tightens the result of Coughlan and Le Breton [1999]. While they construct a family of agendas $A$ such that $V_A \subseteq BA^\infty$, they cannot rule out the possibility that $V_A \neq BA^\infty$.

5 Approximate Implementation

In this section, I first discuss a way to implement the Copeland winners approximately before providing a general framework for approximating tournament solutions which are not agenda implementable.

5.1 Approximating the Copeland Winners

Although the Copeland Set is not agenda implementable, there exists an implementable solution, called the Composition Copeland Set, for which every selected alternative has a relatively high Copeland score. The idea is to modify the Copeland Set, by adding certain alternatives and removing others, so that it satisfies WCOM and SCOND. The subtlety is to avoid adding alternatives with low Copeland scores.

To formalize this approach, some definitions are required. Given a tournament $M$ with component $Y$, the component Copeland score $co(Y, M)$ is the Copeland score of $y^*$ on the tournament $M_Y$ (where, recall, $Y$ is replaced with a single alternative
y*). Given a strong tournament \( M \), let \( D^*(M) \) denote the \( X_j \in D(M) \) for which \( |X_j| + 2 \cdot \text{co}(X_j, M) \) is maximal:

\[
D^*(M) \equiv \arg \max_{X_j \in D(M)} (|X_j| + 2 \cdot \text{co}(X_j, M)).
\]

Intuitively, the collection \( D^*(M) \) consists of those components in \( D(M) \) which contain alternatives with the highest Copeland scores in the worst case. The Composition Copeland Set is defined recursively in terms of these components:

**Definition 13** For every tournament \( M \), the **Composition Copeland Set** \( CO^*(M) \) is defined by:

\[
CO^*(M) \equiv \begin{cases} 
\bigcup_{X_i \in D^*(M)} CO^*(M_i) & \text{if } M \text{ is strong} \\
CO^*(M|_{TC(M)}) & \text{otherwise}
\end{cases}
\]

From the definition, the Composition Copeland Set satisfies neutrality and the Condorcet principle and, hence, defines a tournament solution. Since it also satisfies WCOM and SCOND, the Composition Copeland Set can be implemented by agenda:

**Remark 3** The Composition Copeland Set satisfies Weak Composition Consistency and Strong Condorcet Consistency. Consequently, it is agenda implementable.

For simple tournaments, the Composition Copeland Set selects the Copeland winners. If \( M \) is simple, then \( D(M) = X \) so that \( D^*(M) = CO(M) \) and, consequently, \( CO^*(M) = CO(M) \). For other tournaments, the relationship with the Copeland Set is less straightforward. In particular, the Composition Copeland Set may include alternatives (from large components) which are not Copeland winners or exclude alternatives (from small components) which are Copeland winners.\(^{27}\) Even so, the scores of the selected alternatives track the Copeland winner(s) fairly closely:

**Proposition 4 (Approximation of the Copeland Set)** For every tournament \( M \), the score of an alternative in the Composition Copeland Set is at least two-thirds that

\(^{27}\)Corollary to Lemma 10 of Moulin [1986] establishes that no seeded agenda \( A^* \) on \( X \) selects from the Copeland Set \( CO(M) \) for every tournament \( M \). It follows that every agenda implementable solution must select Copeland non-winners for some tournament.
of the Copeland winner(s) – i.e. for every \( M, x \in CO^*(M) \), and \( w \in CO(M) \):

\[
\frac{co(x, M)}{co(w, M)} > \frac{2}{3}
\]

To put this result into perspective, consider a seeded agenda \( A^s \) on \( X \) and let \( co_{A^s}(M) \) denote the Copeland score of the chosen alternative \( v^*(A^s; M) \). Denote the collection of seeded agendas on \( X \) by \( A_X \) and the greatest lower bound of \( co_{A^s}(M)/co(w, M) \) for tournaments \( M \) on \( X \) by

\[
co^-(A_X) \equiv \max_{A^s \in A_X} \min_{M \in M(X)} \frac{co_{A^s}(M)}{\max_{w \in X} co(w, M)}.
\]

By Remark 3 (and Remark 7 of the Appendix), Proposition 4 implies \( co^-(A_X) > 2/3 \). This is a significant improvement on the lower bounds of \( \Omega(\log(|X|)/|X|) \) (Fischer, Procaccia, and Samorodnitsky (FPS) [2011]) and \( \Omega(1/\sqrt{|X|}) \) (Iglesias, Ince, and Loh [2012]) established in prior work.\(^{28}\) It even improves on the “probabilistic” lower bound of \( 1/2 - O(1/|X|) \) that FPS obtain by randomizing over the seedings of agendas related to the simple agenda.

Interestingly, FPS use the necessary conditions in Theorem 2 to establish a deterministic upper bound of \( co^-(A_X) \leq 3/4 + O(1/|X|) \) for all \( X \). Combined with Proposition 4, their result establishes fairly narrow bounds for \( co^-(A_X) \). In particular:

**Corollary 4** It is possible to implement only alternatives whose Copeland score is at least \( 2/3 \) that of the Copeland winner(s). However, for every \( X \), it is impossible to achieve a ratio better than \( 3/4 + O(1/|X|) \).

To conclude, it is worth commenting on the relationship between the Composition Copeland Set and some other tournament solutions. By following the same kind of reasoning as Landau [1953], it is easy to see that this solution, like the Copeland Set, refines the Uncovered Set (see Lemma 8 of the Appendix).

Unlike the Composition Copeland Set, the Uncovered Set may contain alternatives whose relative score is arbitrarily small. To see this, consider the linear order \( > \) on \( \{x_1, \ldots, x_i\} \) (with \( i \geq 3 \)) given by the index order and define \( M \) to be the same as \( > \) except that \( x_i M x_k \) for all \( k \neq i - 1 \). Since \( x_i \) beats every alternative except \( x_{i-1} \) and

\[^{28}\text{Formally, “}f(n) \text{ is } \Omega(g(n))\text{” means } f(n) \geq c \cdot g(n) \text{ for some positive } c \text{ and } n \text{ sufficiently large.}\]
$x_{i-1}$ beats $x_i$, $x_{i-1} \in UC(M)$. Since $co(x_{i-1}, M) = 1$ and $co(x_1, M) = i - 2$ however, $co(x_{i-1}, M)/co(x_1, M)$ gets arbitrarily small as the number of alternatives increases.

5.2 A General Method of Approximation

A conceptually different way to approximate a non-implementable solution $S$ by agenda is to identify implementable solutions which are close to $S$ in a set-theoretic sense. A natural approach is to focus on implementable solutions which nest $S$. This approach was first proposed by Litvakov [1981] as a method to approximate choice correspondences by correspondences satisfying particular properties.29

Formalizing this approach, denote by $S(Q)$ the class of tournament solutions that satisfies some property (or properties) $Q$. Then, a solution $S \notin S(Q)$ has a lower approximation (upper approximation) in $S(Q)$ if it has a greatest lower bound (least upper bound) in $S(Q)$. In other words:

**Definition 14** Consider a tournament solution $S \notin S(Q)$. Then:

$S$ is an $S(Q)$-lower approximation of $S$ if (i) $S^- \in S(Q)$, (ii) $S^- \subseteq S$, and (iii) $S_Q \subsetneq S^-$ for all $S_Q \in S(Q) \setminus S^-$ such that $S_Q \subseteq S$; and

$S$ is an $S(Q)$-upper approximation of $S$ if (i) $S^+ \in S(Q)$, (ii) $S \subseteq S^+$, and (iii) $S^+ \subsetneq S_Q$ for all $S_Q \in S(Q) \setminus S^+$ such that $S \subseteq S_Q$.

Laffond, Lainé, and Laslier [1996] (see also Laslier [1997]) study upper approximations, which they call composition-consistent hulls, for the class of tournament solutions satisfying COM. The main result of this section establishes that a variety of non-implementable tournament solutions have upper approximations which satisfy COM and COND. In order for a tournament solution $S$ to have an upper approximation, $S$ must refine (one of) the coarsest tournament solution(s) which satisfies COM and COND. In light of this observation, consider the correspondence, called the Maximal Set, which selects recursively from the Condorcet Set of every component in the coarsest non-degenerate decomposition:

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29 Since it was published in Soviet engineering journal, this article can be difficult to obtain. However, the results are also reported in Aizerman [1985] (Theorems 24 and 25) as well as Aizerman and Aleskerov [1995] (Theorems 5.12-5.14).
Definition 15 For every tournament $M$, the Maximal Set $\text{Max}(M)$ is defined by:

$$\text{Max}(M) \equiv \begin{cases} 
\bigcup_{X_i \in D(M)} \text{Max}(M_i) & \text{if } M \text{ is strong} \\
\text{Max}(M|_{TC(M)}) & \text{otherwise}
\end{cases}$$

From the definition, the Maximal Set satisfies neutrality and the Condorcet principle and, hence, defines a tournament solution. As shown by the following remark, the Maximal Set is also the coarsest tournament solution satisfying COM and COND.

Remark 4
(i) The Maximal Set satisfies Composition Consistency and Condorcet Consistency. Consequently, it is agenda implementable. (ii) If $S$ satisfies Composition Consistency and Condorcet Consistency, then $S \subseteq \text{Max}$. And, (iii) the Maximal Set is distinct from the Uncovered Set.

Since the Maximal Set is the coarsest tournament solution satisfying COM and COND, the last part of Remark 4 shows that the Uncovered Set refines the Maximal Set. Since the Maximal Set satisfies COND, it weakly refines the Condorcet Set. It is easy to see that there are tournaments $M$ such that $\text{Max}(M) \neq TC(M)$ so that the Maximal Set is a refinement of the Condorcet Set. Having said this, it is worth noting that the two solutions coincide for simple tournaments. If $M$ is simple, $D(M) = X$ so that, by definition, $\text{Max}(M) = TC(M)$.

Given the sufficient conditions, Lemma 2 and Remark 4 establish the following:

Theorem 3 (Approximation) Consider a tournament solution $S$. Then:

(I) $S$ has an upper approximation among the agenda implementable solutions that satisfy Composition Consistency and Condorcet Consistency if and only if $S \subseteq \text{Max}$.

(II) $S$ has a lower approximation among the agenda implementable solutions that satisfy Weak Composition Consistency and Strong Condorcet Consistency if and only if there exists a tournament solution $S' \subseteq S$ that satisfies Weak Composition Consistency and Strong Condorcet Consistency.

To illustrate the limitations of this result, consider the task of approximating the Copeland Set. As shown by Laffond, Lainé, and Laslier [1996] (Proposition 12), the Uncovered Set is the upper approximation of the Copeland Set among the solutions which satisfy COM. Since the Uncovered Set satisfies COND, $CO^+ = UC$. As noted
in the previous subsection, there are tournaments where the Uncovered Set contains alternatives whose relative score is arbitrarily small. In this sense, $CO^+$ provides only a poor approximation of the Copeland Set. What is worse, Theorem 3 does not ensure the existence of a lower approximation. Moulin [1986] observed (in the Corollary to Lemma 10) that no seeded agenda $A^*$ on $X$ selects from the Copeland Set for every tournament $M$ on $X$. From the necessary conditions of Theorem 2, it follows that no solution $S \subseteq CO$ satisfies WCOM and COND. As such, the Copeland Set has no lower approximation that satisfies WCOM and SCOND.

While Theorem 3 is intended as a tool to approximate non-implementable tournament solutions, it also provides approximations for agenda implementable solutions. It is straightforward to see that the Condorcet Set, for instance, has a lower approximation (because $UC \subseteq TC$ and $UC$ satisfies WCOM and SCOND) but no upper approximation (because $Max \not\subseteq TC$). Not surprisingly, the lower approximation is the Maximal Set (see Remark 8 of the Appendix).

The same reasoning establishes the more general point that the Maximal Set is the lower approximation of any tournament solution $S$ that it refines. In other words, the Maximal Set is the largest tournament solution satisfying WCOM and SCOND:

**Corollary 5** If $S$ satisfies Weak Composition Consistency and Strong Condorcet Consistency, then $S \subseteq Max$.

In combination with Remark 4(i), this establishes that there is a well-defined upper bound on tournament solutions that satisfy the sufficient conditions identified in Section 3. In particular, these conditions cannot be used to show that a tournament solution is implementable unless the solution refines the Maximal Set.

## 6 Related Solution Concepts

The sufficient conditions identified in Section 3 have implications for implementation via backward induction and dominance solvable voting.

Implementation via backward induction involves sequential mechanisms which are similar to seeded agendas. The only differences are the fact that a single agent (rather than the collection of all agents) is appointed to “vote” at each node and any number of subgames (not just two subgames) may be attached to a non-terminal
node.\(^{30}\) Formally, a *sequential mechanism* is a finite game tree \(\Gamma\) where: (i) every non-terminal node is seeded with an agent \(j \in N\); and, (ii) every terminal node is seeded with an alternative \(x \in X\). Given a profile \(\vec{P}\), the *backward induction solution* (or equilibrium outcome) of the perfect information extensive-form game \((\Gamma, \vec{P})\) is denoted by \(BI(\Gamma, \vec{P}) \in X.\(^{31}\)

**Definition 16** A social choice function \(F : \mathcal{L}^n(X) \to X\) is *implementable via backward induction* if there exists a sequential mechanism \(\Gamma\) such that \(F(\vec{P}) = BI(\Gamma, \vec{P})\) for all \(\vec{P} \in \mathcal{L}^n(X)\).

Dominance solvable voting is a more general solution concept associated with normal form games.\(^{32}\) Let \((S, \pi)\) denote an \(n\)-player normal form game where the pure strategy set of agent \(j \in N\) is given by \(S_j\), the set of all pure strategy combinations by \(S = \Pi_{j \in N} S_j\), and the payoff function by \(\pi : S \to X\). Given a profile \(\vec{P}\), the perfect information normal form game \((S, \pi, \vec{P})\) is said to be *dominance solvable* if, after eliminating all of the weakly dominated strategies in each stage of elimination, the remaining strategies are payoff-equivalent. If \(S' \subseteq S\) denotes the strategies remaining after exhaustive elimination, then \(\{\pi(s') : s' \in S'\}\) must be a singleton. A game form \((S, \pi)\) is said to be *\(d\)-solvable* if \((S, \pi, \vec{P})\) is dominance solvable for all profiles \(\vec{P} \in \mathcal{L}^n(X)\). Given a \(d\)-solvable game form \((S, \pi)\), denote the *iterated weak dominance solution* for profile \(\vec{P}\) by \(DS(S, \pi, \vec{P}) \in X\).

**Definition 17** A social choice function \(F : \mathcal{L}^n(X) \to X\) is *implementable via dominance solvable voting* if there exists a \(d\)-solvable game form \((S, \pi)\) such that \(F(\vec{P}) = DS(S, \pi, \vec{P})\) for all \(\vec{P} \in \mathcal{L}^n(X)\).

Arguably the most desirable properties for social choice rules are anonymity, neutrality, and Pareto efficiency. Whereas neutrality requires the social choice rule to be independent of the labels attached to the alternatives, anonymity requires the rule to be independent of the labels attached to the agents. To formalize, some definitions are required. Given a profile \(\vec{P}\) and a bijections \(\sigma : X \to X\), let \(\sigma \vec{P}\) denote the profile where every agent’s preferences are permuted according to \(\sigma\). Given a bijection \(\tau : N \to N\), let \(\tau \vec{P}\) denote the profile where the agents are permuted according to \(\tau\).

\(^{30}\)Having said this, it is without loss of generality to restrict attention to binary game trees.

\(^{31}\)The fact that \(BI(\Gamma, \vec{P})\) is a singleton follows from strict preferences and perfect information.

\(^{32}\)Moulin [1979] shows that every rule implemented via backward induction can be implemented by dominance solvable voting. Golberg and Gurvich [1986] give examples of dominance solvable voting schemes that cannot be implemented via backward induction.
Definition 18 A social choice rule \( F : \mathcal{L}^n(X) \to 2^X \) is said to be:

**Anonymous** if \( F(\tau \vec{P}) = F(\vec{P}) \) for every \( \vec{P} \) and permutation \( \tau : N \to N \);

**Neutral** if \( F(\sigma \vec{P}) = \sigma F(\vec{P}) \) for every \( \vec{P} \) and permutation \( \sigma : X \to X \); and,

**Pareto Efficient** if there exists some \( j(\vec{P}, f, x) \in N \) such that \( f \succ_j(\vec{P}, f, x) \) \( x \) for every \( \vec{P}, f \in F(\vec{P}) \), and \( x \in X \setminus \{ f \} \).

An important distinction from implementation by agenda is that dominance solvable voting and backward induction involve social choice functions. Despite this difference, there is a close connection between implementation by agenda and these two solution concepts. In particular, the sophisticated voting outcome on \( A^* \) is the iterated weak dominance solution of the strategic form game associated with \( A^* \) (Moulin [1979]) or, equivalently, the backward induction solution of the sequential mechanism obtained by assigning a voting order at each node (Duggan [2003]).

Definition 19 Given a tournament solution \( S \) implemented by the agenda \( A \) on \( X \), let \( S_{A^*} : \mathcal{L}^n(X) \to X \) denote the social choice function which selects the winner of \( A^* \) for all profiles \( \vec{P} \in \mathcal{L}^n(X) \), i.e.

\[
S_{A^*}(\vec{P}) \equiv v(A^*; M(\vec{P})).
\]

Because the voting on \( A^* \) is by majority, the sophisticated voting outcomes on \( A^* \) must be anonymous. Provided that \( S \) (weakly) refines the Uncovered Set, the sophisticated outcomes must also be Pareto efficient (Miller [1980]):

Remark 5 Given a tournament solution \( S \) implemented by the agenda \( A \) on \( X \):

(i) \( S_{A^*} \) can be implemented via backward induction and dominance solvable voting.

(ii) \( S_{A^*} \) is anonymous and, if \( S \subseteq UC \), then \( S_{A^*} \) is also Pareto efficient.

While it is anonymous, the sophisticated outcome associated with the seeded agenda \( A^* \) is not neutral. Intuitively, the reason is that \( A^* \) must induce a sufficient variety of outcomes on candidate-isomorphic profiles (e.g. profiles \( \vec{P} \) and \( \sigma \vec{P} \)) to ensure that every candidate in \( S(M(\vec{P})) \) is selected for some seeding of \( A \). A natural way to obtain a neutral rule from tournament solutions implementable by agenda is to select one agent to act as the “tie-breaker” among the winners for different seedings:

\[33\] Alternatively, it is the solution of a sequential Kingmaker mechanism (Dutta and Sen [1993]).
**Definition 20** Given a tournament solution $S$ on $X$, let $S_j : L^n(X) \rightarrow X$ denote the social choice function which selects the most preferred alternative of agent $j \in N$ for all profiles $\vec{P} \in L^n(X)$, i.e.

$$S_j(\vec{P}) \equiv \max_{\succeq_j} S(M(\vec{P})).$$

If $S$ can be implemented by the agenda $A$, then $S_j$ can be implemented by having agent $j \in N$ select from among the winners on the seedings of $A$ (i.e. $A^1, ..., A^s, ..., A^{|X|!}$). To see this, consider the following game:

Since every fixed seeding rule $S_{A^s}$ can be implemented via backward induction (and, consequently, via dominance solvable voting), the tie-breaker rule for $S$ can also be implemented with these solution concepts.

**Remark 6** Given a tournament solution $S$ implemented by the agenda $A$ on $X$:

(i) $S_j$ can be implemented via backward induction and dominance solvable voting.

(ii) $S_j$ is neutral and, if $S \subseteq UC$, then $S_j$ is also Pareto efficient.

Given Theorem 2, Remarks 5 and 6 immediately imply the following:

**Theorem 4 (Neutral/Anonymous Rules I)** If $S$ satisfies Weak Composition Consistency and Strong Condorcet Consistency, then for all sets of alternatives $X$:

(I) Every $S_{A^s} : L^n(X) \rightarrow X$ defines an anonymous SCF which is implementable via backward induction. Moreover, $S_{A^s}(\vec{P}) \in S(M(\vec{P}))$ for all $\vec{P} \in L^n(X)$.

(II) Every $S_j : L^n(X) \rightarrow X$ defines a neutral SCF which is implementable via backward induction. Moreover, $S_j(\vec{P}) \in S(M(\vec{P}))$ for all $\vec{P} \in L^n(X)$.

(III) Provided that $S \subseteq UC$, each of the $S_{A^s}$ and $S_j$ is also Pareto efficient.

The results of Section 4 show that it is possible to construct rules that are either anonymous and Pareto efficient or neutral and Pareto efficient from a wide range
of tournament solutions including the Uncovered Set and its iterations, the Minimal Covering Set, the Banks Set and its iterations, the Tournament Equilibrium Set, the Bipartisan Set, and the Slater Set.

By no means do these two families exhaust the appealing rules that can be constructed from agenda implementable tournament solutions. In fact, it is possible to recombine any two social choice rules that can be implemented via backward induction by “adding” together the implementing mechanisms as in the tie-breaker rule (Golberg and Gurvich [1986]). If \( F \) and \( F' \) are implemented by the sequential mechanisms \( \Gamma \) and \( \Gamma' \), the following mechanism can also be implemented:

![Diagram of mechanisms](attachment:mechanism_diagram.png)

While this “adding” operation does preserve Pareto efficiency, it need not preserve anonymity or neutrality. To ensure that the resulting social choice function also satisfies anonymity or neutrality, one must take a different approach.

Basically, the idea is to have the agents decide collectively on the appointment of the tie-breaker for a given tournament solution \( S \). One natural mechanism involves sequential veto (Mueller [1978]; Moulin [1979, 1980]). Given a collection of agents \( N = \{1, \ldots, n\} \), the permutation \( \tau : N \to N \) defines an order \( \tau^{-1}(1), \ldots, \tau^{-1}(n-1) \) for veto. In any stage \( 1 \leq j < n \), the agent \( \tau^{-1}(j) \) is afforded the opportunity to veto one of the \( n + 1 - j \) remaining agents. Ultimately, the alternative selected is the alternative \( S_{\tau^{-1}(j)}(M) \) chosen according to the tie-breaker rule by the agent \( j^* \) who remains “unvetoed” in the last stage of the veto game.

Another natural mechanism involves sequential approval (Moulin [1980]). In that case, \( \tau : N \to N \) defines an order \( \tau^{-1}(1), \ldots, \tau^{-1}(n-1) \) for approval. In any stage \( 1 \leq j < n \), the agents vote on the approval of agent \( \tau^{-1}(j) \). If approved unanimously, this agent is appointed as the tie-breaker and \( S_{\tau^{-1}(j)}(M) \) is selected. Otherwise, agent \( \tau^{-1}(j) \) is eliminated and the agents vote on the approval of agent \( \tau^{-1}(j+1) \).

Fixing an agenda implementable tournament solution \( S \) and a permutation \( \tau \), the sequential veto mechanism defines a social choice function \( S^\tau : \mathcal{L}^n(X) \to X \) that can be implemented via backward induction. Similarly, the sequential approval
mechanism defines an implementable social choice function $S^A_\tau : \mathcal{L}^n(X) \to X$. It is straightforward to show that the sequential veto mechanism $S^V_\tau$ is neutral while the sequential approval mechanism $S^A_\tau$ is anonymous. Thus:

**Corollary 6 (Neutral/Anonymous Rules II)** If $S$ satisfies Weak Composition Consistency and Strong Condorcet Consistency, then for all sets of alternatives $X$:

(I) Every $S^A_\tau : \mathcal{L}^n(X) \to X$ defines an anonymous SCF which is implementable via backward induction. Moreover, $S^A_\tau(\vec{P}) \in S(M(\vec{P}))$ for all $\vec{P} \in \mathcal{L}^n(X)$.

(II) Every $S^V_\tau : \mathcal{L}^n(X) \to X$ defines a neutral SCF which is implementable via backward induction. Moreover, $S^V_\tau(\vec{P}) \in S(M(\vec{P}))$ for all $\vec{P} \in \mathcal{L}^n(X)$.

(III) Provided that $S \subseteq UC$, each of the $S^A_\tau$ and $S^V_\tau$ is also Pareto efficient.

It is not difficult to generate a variety of other appealing rules in the same fashion.

### 7 Conclusions and Open Problems

In this paper, I study the implementation of majority voting rules. By identifying weak sufficient conditions, I establish that a rich variety of tournament solutions can be implemented by sophisticated agenda voting. I also show that it is possible to combine tournament solutions which satisfy the sufficient conditions to obtain additional solutions that are implemented by agenda. For solutions which cannot be implemented, I provide a method to implement approximately. Finally, I examine the implications of these results for implementation with the less restrictive solution concepts of dominance solvable voting and backward induction. While these results settle some long-standing questions, they also raise a variety of questions:

1. **Implementing Agendas**: The paper shows that a wide variety of tournament solutions can be implemented by agenda. Except for the Condorcet and Banks Sets, the structure of the implementing agendas remains an open question. For any of these solutions, it would be ideal to identify a family of implementing agendas which (like the simple and amendment agendas) can be defined recursively.

2. **Complexity of Agendas**: The search problem gives a lower bound on the size of the agendas which implement a tournament solution. If the search problem is

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[^34]: I follow the order in which the questions arise in the paper (not their order of importance).
NP-hard, it seems reasonable to assume that the implementing agendas must be exponential. However, the search problem for many solutions is polynomial (see Hudry [2009]). In that case, the search problem gives a polynomial lower bound on the size of the agendas, a bound which seems out of reach for some tournament solutions. While the search problem of the Banks Set is polynomial (Hudry [2004]), for example, it seems unlikely that it can be implemented with polynomial-sized agendas (given that the amendment agenda is exponential). Consequently, it would be helpful to have some way to obtain tighter lower bounds.

(3) New Solutions: The paper introduces two new tournament solutions, the Composition Copeland Set and the Maximal Set, which are based on modifying established solutions so that they satisfy WCOM or COM. In recent work, Brandt [2009, 2011] studies new classes of tournament solutions which are the minimal stable sets and minimal retentive sets of other solutions. Given the central importance of WCOM and COM for implementation, it may be useful to carry out a similar exercise for these properties. As noted in Section 5, this work has been started by Laffond, Lainé, and Laslier [1996] and Laslier [1997].

(4) Tighter Bounds for Copeland: While the lower bound in Proposition 4 is a marked improvement over the bounds given in earlier work, there may yet be room for improvement. For one, I have not been able to show that the bound is tight for the Composition Copeland Set. Calculating by hand, I have considered a range of tournaments and have yet to find one where the score ratio is less than 3/4.

(5) Better Approximations: Given the discussed limitations of Theorem 3, it may be worth investigating another approach to approximation. For a non-implementable solution $S$, one might try to extend Slater’s logic by identifying a solution $S'$ that satisfies WCOM and SCOND while minimizing the distance to $S$ under a suitable metric. A similar question was previously suggested by Laslier [1997] (Chapter 9).
8 Appendix

Theorem 1 is proved in 8.1 while Theorem 2 is proved in 8.2. The other proofs are given afterward.

8.1 Proof of Theorem 1 (and Proposition 1)

8.1.1 Proof of Proposition 1

Proposition 1(I) follows from a result of Srivastava and Trick [1996]. Some definitions are required to state their result. A subset PS ⊆ X is said to be prime if there exists no non-trivial partition PS = {PS_i}_{i=1}^k of PS such that: (i) PS is a decomposition of M and M' on PS; and, (ii) the quotient relations induced by PS agree so that M/PS = M'/PS.

Srivastava and Trick’s Theorem F(M) and F(M') are pairwise implementable on a subset of X if and only if there exists a prime set PS ⊆ X s.t. F(M) ∈ TC(M|PS) and F(M') ∈ TC(M'|PS).

Proposition 1(I) is a consequence of the following lemma.

Lemma 3 If \langle G(M), M_G \rangle \neq \langle G(M'), M'_G \rangle, then TC(M) ∪ TC(M') is a prime set.

Proof. For parsimony, I abbreviate TC(M) to TC and TC(M') to TC'. Suppose TC ∪ TC' is not a prime set and let S define a partition of TC ∪ TC' that satisfies conditions (i) and (ii) above. There are four possibilities: (a) TC ∩ TC' = ∅; (b) TC \ TC' \neq ∅, TC' \ TC \neq ∅, and TC ∩ TC' \neq ∅; (c) TC = TC'; and (d) TC \ TC' is symmetric to (d).

(a) Pick x ∈ TC and x' ∈ TC'. By definition of the Condorcet Set, xMx' and x'Mx. Suppose x ∈ s and x' ∈ s' for distinct components of S. Then, xMx' and x'Mx imply s(M/S)s' and s'(M'/S)s. This contradiction establishes TC ∪ TC' ⊆ s for some component s of S. Since TC ∩ TC' = ∅, S = \{s\} so that S does not satisfy condition (i). This is the desired contradiction.

(b) The same argument given in (a) establishes that TC\TC'∪TC'\TC ⊆ s for some component of S. Since TC ∩ TC' \neq ∅, pick some \bar{x} ∈ TC ∩ TC' from a different component \bar{s} \neq s of S. (This is without loss of generality: if no such component exists, S does not satisfy condition (i) and a contradiction obtains.) Since TC' \ TC is non-empty, then there is an x' ∈ TC' \ TC such that \bar{x}Mx' (from the definition of the Condorcet Set TC). Then, \bar{s}(M/S)s so that \bar{s}(M'/S)s. In turn, this implies \bar{x}Mx' for all \bar{x} ∈ TC' \ s and all x' ∈ TC' ∩ s. So, \{TC' \ s, TC' ∩ s\} is a decomposition of TC' which contradicts the assumption that TC' is a Condorcet Set.

(c) By the minimality of G(M), S refines G(M). Likewise, S refines G(M'). Since M'/S = M/S, it follows that G(M') is also a decomposition of M on TC. By definition of G(M), G(M) is coarser than G(M'). The same argument shows that G(M') is coarser than G(M'). This establishes that G(M) = G(M') and M_G = M'_G. But, this contradicts \langle G(M), M_G \rangle \neq \langle G(M'), M'_G \rangle.

(d) Pick x− ∈ TC and x' ∈ TC' \ TC such that x'Mx-. (By assumption, TC' \ TC \neq ∅.) Since x-Mx', it follows that x− and x' must be in the same component s of S. Letting

\[ TC^- = \{x ∈ TC : x'Mx\ \text{for some } x' ∈ TC' \setminus TC\} \]
the preceding observation establishes $TC^- \subseteq s$. Now consider

$$\text{TC}^+ \equiv \text{TC} \setminus \text{TC}^- = \{x \in \text{TC} : x M' x' \text{ for all } x' \in \text{TC}' \setminus \text{TC}\}.$$ 

There are two possibilities: $\text{TC}^+ \subseteq s$, or there is some $x^+ \in \text{TC}^+$ such that $x^+ \notin s$. In the first case, $x^+ M' x'$ for all $x' \in \text{TC}' \setminus \text{TC}$ implies $s(M'/S)s'$ for any other component $s'$ of $S$. (If there is no other component, $S$ does not satisfy condition (i) and a contradiction obtains.) So, $\{s, \text{TC}' \setminus s\}$ is a decomposition of $\text{TC}$ which contradicts the assumption that $\text{TC}$ is a Condorcet Set. In the second case, pick $x^+ \in \text{TC}^+ \setminus s$ and let $s^+$ be the corresponding component of $S$. Since $x^+ M' x'$ for all $x' \in \text{TC}' \setminus \text{TC}$, it follows that $s^+(M'/S)s$ so that $s^+(M/S)s$. As such, $x^+ M x$ for all $x \in \text{TC} \cap s$ and $x^+ \in \text{TC} \setminus s$ which establishes that $\{\text{TC} \setminus s, \text{TC} \cap s\}$ is a decomposition of $\text{TC}$ and contradicts the assumption that $\text{TC}$ is a Condorcet Set. ■

**Proof of Proposition 1(I).** ($\Leftrightarrow$) By Lemma 3, $\text{TC} \cup \text{TC}'$ is a prime set. By the Theorem of Srivastava and Trick, all $x \in \text{TC} \equiv \text{TC}(M|_{\text{TC} \cup \text{TC}'})$ and $x' \in \text{TC}' \equiv \text{TC}(M'|_{\text{TC} \cup \text{TC}'})$ are pairwise implementable for $\text{TC} \cup \text{TC}' \subseteq X$. To complete the proof, fix a pair $x \in \text{TC}$ and $x' \in \text{TC}'$ and a seeded agenda $A^*$ on $\text{TC} \cup \text{TC}'$ that pairwise implements $(x, x')$. Next, construct a seeded agenda whose left branch at the root corresponds with $A^*$ and whose right branch is a seeded agenda on $X \setminus (\text{TC} \cup \text{TC}')$. (When $X \setminus (\text{TC} \cup \text{TC}') = \emptyset$, the right branch can be omitted.) By construction, the desired outcome emerges from the left branch for each of the two majority relations and defeats whatever emerges from the right. ($\Rightarrow$) If $x$ and $x'$ are pairwise implementable, $x \in \text{TC}$ and $x' \in \text{TC}'$ (by Lemma 9 of Moulin [1986]). ■

Proposition 1(II) is a consequence of the following lemma:

**Lemma 4** Given a collection of globally comparable majority relations $M^{c} \subseteq M(X)$ with maximal non-degenerate decomposition $G(M)$, the binary SCF $F^{c} : M^{c} \rightarrow X$ is implementable by seeded agenda if and only if $F^{c}$ is implementable by seeded agenda on some $g^{*} \subseteq g \in G(M)$.

**Proof.** For parsimony, let $\text{TC} = \text{TC}(M)$ and $G(M) = \{g_{1}, \ldots, g_{k}\}$. First, fix an element $x \in g_{i}$ and suppose that $g_{i} M_{G_{i+1}}$ for $i < k$ and $g_{k} M_{G_{i+1}}$ (otherwise, the components can be relabeled so that this is true). Construct a simple agenda $A(x) = \text{Sim}(L_{x})$ using $L_{x} = (x, g_{k}, g_{1} \setminus g\{x\}, X \setminus \text{TC})$. (As in the proof of Proposition 1(i), the bottom branch may be omitted when $X \setminus \text{TC} = \emptyset$.) To the branch labeled $x$, append the item $x$. To the branches labelled by $g_{i}$ (respectively, $g_{1} \setminus \{x\}$ and $X \setminus \text{TC}$), append a seeded agenda $A_{i}$ containing all outcomes in $g_{i}$ (resp. $g_{1} \setminus \{x\}$ or $X \setminus \text{TC}$). By construction, $A(x)$ implements $x$ on $M^{c}$ (see e.g. Lemma 8.3.3 of Laslier [1997]). Moreover, it can be associated with the trivial agenda $a(x) = x$ that implements $x$ on $\{x\} \subseteq g_{i}$.

Let $\mathcal{A}_{1} = \{A(x) : x \in \text{TC}\}$ and let $\mathcal{A}_{1}(g^{*}) = \{a(x) : x \in g^{*}\}$ define the collection of seeded agendas $a(x)$ for $x \in g^{*} \subseteq g \in G(M)$. By construction, every $F^{c}$ implemented by seeded agenda on $X$ can be obtained by concatenating agendas in $\mathcal{A}_{1}$. Since $F^{c}(M) \in \text{TC}$ for all $M \in M^{c}$ (by Lemma 9 of Moulin [1986]), one can ignore agendas $A(x)$ where $x \notin \text{TC}$. Likewise, every $F^{c}$ implemented by seeded agenda on $g^{*} \subseteq g$ can be obtained by concatenating agendas in $\mathcal{A}_{1}(g^{*}) = \{x : x \in g^{*}\}$.
Define $A_n = \{A_{n-1} + A_k : A_{n-1} \in A_{n-1} \text{ and } A_k \in A_k \text{ for } k < n\}$ to be the seeded agendas obtained by concatenating a seeded agenda in $A_{n-1}$ with a “smaller” seeded agenda in $A_k$. Moreover, let $V_n = \{F^c : F^c = v^*(A_n; \cdot) \text{ for some } A_n \in A_n\}$ denote the Condorcet SCFs implemented by some $A_n \in A_n$. Define $A_n(g^*)$ and $V_n(g^*)$ in a similar fashion.

Using strong induction, I establish that $F^c \in V_n$ if and only if $F^c \in V_n(g^*)$ for some $g^* \subseteq g \in G(M)$. The claim is trivial for the base case $n = 1$. So, suppose that it holds for $n \leq N$. To establish the claim for $n = N + 1$:

$(\Rightarrow)$ Consider any $F^c = v^*(A_N + A_k; \cdot) \in V_{N+1}$. By the induction step, $v^*(A_N; \cdot) = v^*(a_N(g^1); \cdot)$ for some $a_N(g^1)$ on $g^1 \subseteq g_1$ and $v^*(A_k; \cdot) = v^*(a_k(g^2); \cdot)$ for some $a_k(g^2)$ on $g^2 \subseteq g_2$. There are two cases: (i) $g_1 \neq g_2$; and, (ii) $g_1 = g_2$.

(i) Without loss of generality, suppose $g_1(M_G)g_2$. Then, $F^c = v^*(A_N + A_k; \cdot) = v^*(A_N; \cdot) = v^*(a_N(g^1); \cdot)$ so that $a_N(g^1)$ implements $F^c$ on $g^1 \subseteq g_1 \in G(M)$ (by the induction step).

(ii) In this case,

$$F^c(M) = v^*(A_N + A_k; M) = \max_M \{v^*(A_N; M), v^*(A_k; M)\}$$

$$= \max_M \{v^*(a_N(g^1); M), v^*(a_k(g^1); M)\} = v^*(a_N(g^1) + a_k(g^2); M)$$

for all $M \in M^c$ so that $a_N(g^1) + a_k(g^2)$ implements $F^c$ on $g^1 \cup g^2 \subseteq g_1 \in G(M)$.

$(\Leftarrow)$ Suppose that $F^c = v^*(a_N(g^1) + a_k(g^2); \cdot) \in V_{N+1}(g^*)$ for $a_N(g^1)$ on $g^1 \subseteq g^* \subseteq g$ and $a_k(g^2)$ on $g^2 \subseteq g^* \subseteq g$. By the induction step, $v^*(a_N(g^1); \cdot) = v^*(A_N; \cdot) \in V_N$ and $v^*(a_k(g^2); \cdot) = v^*(A_k; \cdot) \in V_N$. Following the same reasoning as case (ii) above, $F^c(M) = \ldots = v^*(A_N + A_k; M)$ for all $M \in M^c$ so that $A_N + A_k$ implements $F^c$. ■

**Proof of Proposition 1(II).** Given Lemma 4, let $M^c = \{M, M'\}$. ■

### 8.1.2 Proof of Theorem 1

The proofs of these results rely on algebraic methods. Some preliminary definitions are required.

**Preliminaries**

Given a majority tournament $M$ on $X$, let the **tournament algebra $X$** be defined by a pair $(X, +)$ consisting of $X$ and a binary operation $+$ such that $x + y = x$ if and only if $xMy$ or $x = y$.\(^{35}\) Tournament algebras can be extended to products. Given a collection $\{X_i\}_{i=1}^m$ of tournament algebras, the **product algebra $\Pi_{i=1}^m X_i$** is defined by $(\Pi_{i=1}^m X_i, +)$ where $+$ applies the operations $+$ component-wise so that $x + y \equiv (x_i + y_i)_{i=1}^m$. The projection of $x \equiv (x_i)_{i=1}^m \in \Pi_{i=1}^m X_i$ onto any collection $J \subseteq \{1, ..., m\}$ of the component algebras is $\pi_J(x) = \Pi_{i \in J} x_i$. A **subdirect product** of $(\Pi_{i=1}^m X_i, +)$ is a sub-algebra $Y \equiv (Y, +)$ of $\Pi_{i=1}^m X_i$ (i.e. $Y \subseteq \Pi_{i=1}^m X_i$ and $Y$ is closed under the binary operation $+$) such that $Y \equiv \{\pi_{\{i\}}(y) : y \in Y\} = X_i$ for every component $Y_i$. The subdirect product $Y$ is said to be **weakly indecomposable** if there is no bi-partition $(J, K)$ of the $m$ components such that $Y = \pi_J(Y) \times \pi_K(Y)$ (up to re-ordering of the components).

\(^{35}\)More generally, an algebra $X$ is a set $X$ that is closed under a collection of $n$-ary operations.
A tournament algebra \((X, +)\) is strong if \(TC(M) = X\) where \(M\) is the majority relation induced by the binary operation \(+\) (so that \(xMy\) if and only if \(x + y = x\) and \(x \neq y\)). A congruence \(\beta\) on \(Y \equiv (Y, +)\) is an equivalence relation on \(Y\) such that \((x + y)\beta (x' + y')\) if and only if \(x\beta x'\) and \(y\beta y'\). The largest congruence on \(Y\) is the complete relation \(1_Y = Y \times Y\) while the smallest is the trivial relation \(\text{Id}_Y = \{(y, y) : y \in Y\}\). Given a congruence \(\beta\) on \(Y\), the quotient algebra \(Y/\beta\) is \((Y/\beta, +_\beta)\) where \(Y/\beta\) is the partition of \(Y\) induced by \(\beta\) and \(+_\beta\) is the binary operation \(y +_\beta y' / \beta \equiv \{Z \in Y/\beta : y + y' \in Z\}\). Finally, \(Y\) is simple if its only congruences are \(1_Y\) and \(\text{Id}_Y\).

**Proofs**

The proofs of these results rely on a theorem in universal algebra established by Maroti [2002] (combining Lemmas 5.10, 5.13, and 5.14 of his Ph.D. dissertation). To state Maroti’s theorem:

**Maroti’s Theorem** If \(Y\) is a weakly indecomposable subdirect product of \(m\) strong tournament algebras, then \(Y\) has a largest congruence \(\beta \neq Y \times Y\) and \(Y/\beta\) is a simple tournament algebra.

They also rely on the following claim:

**Claim 1** (I) The natural numbers \(h\) and \(h + 1\) are co-prime. (II) If \(a\) and \(b\) are co-prime, then every pair of congruence relations of the form \(x = k \mod a\) and \(x = l \mod b\) has a solution.

**Proof.** (I) Two naturals \(a\) and \(b\) are co-prime if and only if \(\gcd(a, b) = 1\). By Bézout’s identity, it suffices to find two integers \(x\) and \(y\) such that \(ax + by = 1\). If \(a = h\) and \(b = h + 1\), \(x \equiv -1\) and \(y \equiv 1\) gives the desired result. (II) This follows from the Chinese remainder theorem.

To simplify the presentation below, consider the following definitions. Let \(\mathcal{M}(X) = \{M_i\}_{i \in I}\) denote the collection of states (i.e., majority relations) on \(X\). For parsimony, I abbreviate \(TC(M_i)\) to \(TC_i\). If there are \(n\) alternatives, denote the domain by \(X_n\) so that \(\mathcal{M}(n)\) defines the collection of states on \(X_n\). Let \(\mathcal{M}_d^n(X) = \{M_j\}_{j \in J}\) denote a collection of \(J \subseteq I\) globally distinct states in \(\mathcal{M}(X)\) so that \(\mathcal{M}_d(n)\) denotes a maximal collection of globally distinct states in \(\mathcal{M}(n)\). Let \(\mathcal{M}_j^n(n)\) denote the maximal collection (or class) of states that are globally comparable to \(M_j \in \mathcal{M}_d(n)\) and let \(K(j) \subseteq I\) denote the set of indices associated with \(\mathcal{M}_j^n(n)\). Finally, let \(\mathcal{M}(n) = \{\mathcal{M}_j^n(n)\}_{j \in J}\) denote the partition dividing \(\mathcal{M}(n)\) into classes of globally comparable states.

One can identify every binary SCF \(F: \mathcal{M}(X) \rightarrow X\) with a vector \(\vec{x} \equiv (x_i)_{i \in I} \in \Pi_{i \in I} X_i\). Using this approach, let \(\mathcal{F}(n) = \{\vec{x} \in \Pi_{i \in I} TC_i : \vec{x} \text{ is implementable}\}\) denote the collection of binary SCFs on \(X_n\) that can be implemented by seeded agenda. Let \(\mathcal{F}_d^n(X) = \{\pi_j(\vec{x}) \in \Pi_{j \in J} TC_j : \vec{x} \in \mathcal{F}(X)\}\) denote the collection of binary SCFs on \(\mathcal{M}_d^n(X)\) that can be implemented by seeded agenda. And, let \(\mathcal{F}_j^n(n) = \{\pi_{K(j)}(\vec{x}) \in \Pi_{k \in K(j)} TC_k : \vec{x} \in \mathcal{F}(n)\}\) denote the collection of binary SCFs on \(\mathcal{M}_j^n(n)\) that can be implemented by seeded agenda.

**Proposition 5** Given any collection of globally distinct majority relations \(\mathcal{M}_d\), the binary SCF \(F^d: \mathcal{M}_d \rightarrow X\) is implementable by seeded agenda if and only if \(F(M) \in TC(M)\) for all \(M \in \mathcal{M}_d\).
Proof. ($\Rightarrow$) If $F^d : M^d \to X$ is implementable, it is also pairwise implementable for every pair of states in $M^d$. From Proposition 1(I), $F^d(M) \in TC$ for all $M \in M^d$.

($\Leftarrow$) Let $M^d_i = \{M_i\}_{i \in I}$ and suppose that $|TC_i| > 1$. To establish the result, I show $F^d_j(X) = \Pi_{i=1}^n TC_i$. The proof is by induction on the number of globally distinct states $|I|$. Proposition 1(I) establishes the base case $|I| = 2$. Suppose that the result holds for $|I| = N$. To complete the induction, I show the result for $|I| = N + 1$. To simplify the notation, let $\overline{X} \equiv \Pi_{i=1}^{n+1} TC_i$ and $Y \equiv F_{N+1}(T^d)$ so that $\overline{X}_j = \pi_j(X)$ and $Y_j = \pi_j(Y)$ define the projections onto the states in $J$.

To see that $Y = \overline{X}$, suppose otherwise.

First, note that $Y$ is a subdirect product of $\overline{X}$. By the induction hypothesis, $F^d_j((n)) = \Pi_{i \in J(n)} TC_i$ for every collection $J(n)$ of $n \leq N$ states. Accordingly, $\pi_i(F^d_j(n)) = TC_i$. Second, each component algebra of $Y$ is strong because $Y_i = TC_i$. Finally, $Y$ is weakly indecomposable. To see this, suppose $Y = \pi_j(Y) \times \pi_k(Y)$. By the induction step, $\pi_j(Y) = \Pi_{i \in J TC_j}$ and $\pi_k(Y) = \Pi_{k \in K TC_k}$ so that $Y = \Pi_{i \in J} TC_j \times \Pi_{k \in K} TC_k = \overline{X}$. But this contradicts the assumption that $Y \neq \overline{X}$ and establishes $Y$ is weakly indecomposable.

As such, the conditions of Maroti’s theorem are satisfied. Applying his theorem, $Y$ has a unique largest congruence $\beta \neq Y \times Y$ and $Y/\beta$ is a simple tournament algebra. There are two cases:

(i) $|X_j| = |X_k| = h + 1 > 1$ for all $j, k \leq N + 1$; and,

(ii) there are distinct states $j$ and $k$ such that $|X_j| \neq |X_k|$.

(i) Pick any two states $j$ and $k$ and consider any distinct $a, b \in Y$. Label the alternatives in $X_j$ so that the sequence $\{x_j^l\}_{l=0}^{h+1}$ defines a complete cycle $x^0_jM_j...M_jx^l_jM_j...M_jx^{h+1}_j = x^0_j$ in $X_j$. And, label the alternatives in $X_k$ so that $\{x^m_k\}_{m=0}^{h+1}$ defines a complete “reverse cycle” $x^0_k = x^{h+1}_kM_k...M_kx^m_kM_k...M_kx^0_k$ in $X_k$. By the base case, there is a $x^{(l,m)}_{-jk} \in \Pi_{i \in I \setminus \{j,k\}} X_i$ s.t. $x^{(l,m)}_{-jk} \equiv \langle x_j^l \times x^m_k \times x^{(l,m)}_{-jk} \rangle \in Y$. Without loss of generality, let $a \equiv x^{(0,0)}$. By construction, $x^{(l,m)}$ and $x^{(l+1,m+1)}$ are unranked by $\Pi_{i=1}^{n+1} M_i$. Since $Y/\beta$ is a tournament, $(x^{(l,m)}, x^{(l+1,m+1)}) \in \beta$ for $l \leq h$ and $m \leq h$ so that $(a, x^{(l+1,m+1)}) \in \beta$.

By Theorem 7 of Harary and Moser [1966], there exists an $h$-length cycle $C_j \subseteq X_j$ containing $b_j$. Let $l^*$ be the lowest index $l$ such that $x_j^l \in C_j$ and let $x^* = x^{(l^*,l^*)}$. So, it is possible to label the elements of $C_j$ so that the sequence $\{x_j^l\}_{l=0}^h$ defines a complete cycle $x^0_j = x^h_jM_j...M_jx^l_jM_j...M_jx^h_j = x^0_j$ in $C_j$. Because $h$ and $h + 1$ are co-prime, $(x^{(l,m)}, x^{(l',m')}) \in \beta$ for every $l, l' \leq h$ and $m, m' \leq h + 1$ (by Claim 1). In particular, $(x^*, b) \in \beta$. Since $(a, x^*) \in \beta$ (by the first argument), it follows that $(a, b) \in \beta$ so that $\beta = Y \times Y$.

(ii) Fix components $j$ and $k$ such that $|X_j| = h' > h = |X_k|$ and consider any distinct $a, b \in Y$. By the same approach as in the previous case, define a complete cycle on $X_j$ and a complete reverse cycle on $X_k$ such that $a$ corresponds to the first element in each sequence. By Theorem 7 of Harary and Moser [1966], there exists an $(h + 1)$-length cycle $C_j \subseteq X_j$ that contains $b_j$. Let $l^*$ be the lowest index $l$ such that $x_j^l \in C_j$ and let $x^* = x^{(l^*,l^*)}$. By the same argument given in the previous case, $(a, x^*) \in \beta$ and $(x^*, b) \in \beta$ so that $(a, b) \in \beta$ so that $\beta = Y \times Y$.

In both cases, $Y \neq X$ implies $\beta = Y \times Y$. But this contradicts the assumption that $\beta \neq Y \times Y$. Thus, $Y = \overline{X}$. Given a collection of distinct states $M^d$, it then follows that $F^d$ is implementable
Lemma 5. Given a complete collection of globally comparable states \( \mathcal{M}^c \) with maximal non-degenerate decomposition \( G(M) \), the binary SCF \( F^c : \mathcal{M}^c \to X \) is pairwise implementable for all \( M, M' \in \mathcal{M}^c \) if and only if \( F^c \) is pairwise implementable for all \( M, M' \in \mathcal{M}^c \) on a subset \( g^* \) of some \( g \in G(M) \).

Proof. Let \( \mathcal{P}W(n) = \{ \bar{x} \in \Pi_{i \in I} TC_i : \bar{x} \) satisfies the pairwise condition on \( \mathcal{M}(n) \} \) represent the collection of binary SCFs that are pairwise implementable on \( X_n \). Consider the similarity class \( \mathcal{M}^c(n) = \{ M_k \}_{k \in K} \) with global structure \( G(M) = \{ g_i \}_{i \in L} \). Let \( \mathcal{P}W^c(n) = \{ \pi_K(\bar{x}) \in \Pi_{k \in K} TC_k : \bar{x} \in \mathcal{P}W(n) \} \) represent the choice functions that are pairwise implementable on \( \mathcal{M}^c(n) \). First note

\[
\mathcal{P}W^c(n) = \bigcup_{i \in L} \mathcal{P}W_i^c(n)
\]

where \( \mathcal{P}W_i^c(n) = \{ \pi_K(\bar{x}) \in \Pi_{k \in K} TC_k : \bar{x} \in \mathcal{P}W^c(n) \cap \Pi_{k \in K} g_k \} \) is the sub-collection of \( \mathcal{P}W^c(n) \) selecting from \( g_i \in G(X_n) \). To see this, fix adjacent states \( M \) and \( M' \) in \( \mathcal{M}^c(n) \) such that \( F^c(M) = x \in g_i \) and \( F^c(M') = x' \). By assumption, \( (F^c(M), F^c(M')) \) is pairwise implementable. From Proposition 1(II), \( x \in g_i \) implies \( x' \in g_i \). By the same argument, \( F^c(M'') \in g_i \) for all \( M'' \in \mathcal{M}^c(n) \).

Let \( \mathcal{P}W_i^c(n)|_{g^*} \) define the sub-collection of \( \mathcal{P}W_i^c(n) \) that is pairwise implementable on \( g^* \subseteq g_i \). And, let \( \mathcal{P}W_i^c(n)[g^*] \) define the sub-collection of \( \mathcal{P}W_i^c(n) \) with range \( g^* \subseteq g_i \) (so \( \bigcup_{k \in K} \{ F^c(M_k) \} = g^* \) for all \( F^c \in \mathcal{P}W_i^c(n)[g^*] \)). By construction, \( \mathcal{P}W_i^c(n) = \bigcup_{g^* \subseteq g_i} \mathcal{P}W_i^c(n)[g^*] \). To establish the desired result, it suffices to prove \( \mathcal{P}W_i^c(n)|_{g^*} = \mathcal{P}W_i^c(n)[g^*] \) for all \( g^* \subseteq g_i \). Using this identity:

\[
\mathcal{P}W^c(n) = \bigcup_{i \in L} \bigcup_{g^* \subseteq g_i} \mathcal{P}W_i^c(n)|_{g^*}
\]

as required. To show \( \mathcal{P}W_i^c(n)|_{g^*} = \mathcal{P}W_i^c(n)[g^*] \) for all \( g^* \subseteq g_i \), first consider the following:

Claim If \( F^c \in \mathcal{P}W_i^c(n) \), \( M, M' \in \mathcal{M}^c(n) \), and \( TC(M|_{g_i}) = \{ F^c(M') \} \), then \( F^c(M) = F^c(M') \).

Proof. Consider any state \( M \in \mathcal{M}^c(n) \) such that \( xMx' \) for all \( x' \in g_i \setminus \{ x \} \) and \( M' \in \mathcal{M}^c(n) \) such that \( F^c(M') = x \in g_i \). Without loss of generality, suppose \( M \neq M' \) and \( F^c(M) = x' \). (If \( M = M' \), then \( F^c(M) = x \) holds trivially.) By assumption, \( x' \) and \( x \) are pairwise implementable on \( M \) and \( M' \) for some \( g^* \subseteq g_i \). By the Theorem of Srivastava and Trick, there exists a prime set \( \{ x, x' \} \subseteq PS \subseteq g_i \) such that \( x' \in TC(M|_{PS}) \) and \( x \in TC(M'|_{PS}) \). By the assumption about \( M \), \( TC(M|_{PS}) = \{ x \} \). Thus, \( F^c(M) = x \). Combined with the assumption that \( F^c(M) = x' \), it follows that \( x' = x \) as required.

The result follows by showing \( \mathcal{P}W_i^c(n)[g^*] = \mathcal{P}W_i^c(n)|_{g^*} \). The fact that \( F^c(M) = x \) for any \( M \in \mathcal{M}^c(n) \) such that \( xMx' \) for all \( x' \in g^* \setminus \{ x \} \) implies \( \mathcal{P}W_i^c(n)[g^*] \subseteq \mathcal{P}W_i^c(n)|_{g^*} \). To establish \( \mathcal{P}W_i^c(n)|_{g^*} \subseteq \mathcal{P}W_i^c(n)[g^*] \), there are two cases to consider: (i) \( g^* = g_i \); and, (ii) \( g^* \not\subseteq g_i \).
(i) For $M, M'$ that are globally distinct on $g_i$, it is sufficient to show that $F^c(M) \in TC(M|g_i)$ and $F^c(M') \in TC(M'|g_i)$. To see this, consider $F^c \in \mathcal{PW}_1(n)[g_i]$ and fix some $M$ such that $|TC(M|g_i)| > 1$ and some $x' \in TC(M|g_i)$. (The fact that $F^c(M) \in TC(M|g_i)$ for every $M$ such that $|TC(M|g_i)| = 1$ follows from the Claim above and the assumption that $F^c \in \mathcal{PW}_1(n)[g_i]$.) Consider an $M'$ such that $M'|x \setminus g_i = M|x \setminus g_i$, $M'|x \setminus g_i = M_i|g_i\setminus \{x'\}$, and $x'M'x$ for all $x \in g_i \setminus \{x'\}$. (More plainly, $M'$ differs from $M$ only by putting $x'$ at the top of $g_i$.) Since $F^c \in \mathcal{PW}_1(n)[g_i]$, $x'$ is chosen for some $M'' \in \mathcal{M}^c(n)$. (More plainly, $M'$ differs from $M$ only by putting $x'$ at the top of $g_i$.) By the Claim above, it follows that $F^c(M') = x'$. By construction, $\{x'\} \subseteq PS \subseteq TC(M|g_i)$ for any non-trivial prime set $PS$ on $M$ and $M'$. By the Theorem of Srivastava and Trick, it follows that $F^c(M') \in TC(M|g_i)$. This shows $F^c(M) \in TC(M|g_i)$ for all $M \in \mathcal{M}^c(n)$.

Next, consider $M, M'$ that are globally comparable on $g_i$ with $G(M|g_i) = G(M'|g_i) = \{g_i^j\}_{j \in I}$. Without loss of generality, suppose $F^c(M) \in g_i^j$. It is sufficient to show that $F^c(M)$ and $F^c(M')$ are pairwise implementable for some $g \subseteq g_i$. From the Theorem of Srivastava and Trick, $F^c(M)$ and $F^c(M')$ are pairwise implementable for some prime set $PS$ such that $F^c(M) \in PS$. By definition, it must be that $PS \subseteq g_i^j$ for any prime set such that $F^c(M) \in PS$. This establishes the desired result.

(ii) Pick $F^c \in \mathcal{PW}_1(n)[g^*]$ for some $g^* \subseteq g_i$. Fix an $M \in \mathcal{M}^c$ and consider the state $M^{i\sigma^*}$ defined by $M^{i\sigma^*}|x \setminus g^* = M|x \setminus g^*$, $M^{i\sigma^*}|g^* = M|g^*$, and $x'M^{i\sigma^*}x$ for all $x \in X \setminus g^*$ and $x \in g^*$. (More plainly, $M^{i\sigma^*}$ differs from $M$ only by putting $g^*$ at the bottom.) By construction, any non-trivial prime set $PS$ on $M$ and $M^{i\sigma^*}$ must contain some $x' \in X \setminus g^*$. Since $F^c(M)$ and $F^c(M^{i\sigma^*})$ are pairwise implementable, $F^c(M) = F^c(M^{i\sigma^*})$. Otherwise, $TC(M^{i\sigma^*}|_{PS}) \subseteq X \setminus g^*$ so that $F^c(M^{i\sigma^*}) \in X \setminus g^*$ which contradicts the assumption that $F^c \in \mathcal{PW}_1(n)[g^*]$. This establishes that $F^c(M) = F^c(M')$ for all $M, M' \in \mathcal{M}^c(n)$ such that $M|g^* = M'|g^*$.

To see that $F^c(M) \in TC(M|g^*)$ for all $M \in \mathcal{M}^c(n)$, fix an $\overline{M}$ such that $x\overline{M}x'$ for all $x \in g^*$ and $x' \in g_i \setminus g^*$. (More plainly, $\overline{M}$ differs from $M$ only by putting $g^*$ at the top of $g_i$.) By the same reasoning as in (i) above, $F^c(\overline{M}) \in TC(\overline{M}|g^*)$. Since $F^c(M) = F^c(\overline{M})$ for all $M, \overline{M} \in \mathcal{M}^c(n)$ such that $M|g^* = \overline{M}|g^*$, then $F^c(M) \in TC(M|g^*)$ for all $M \in \mathcal{M}^c(n)$.

To complete the proof, fix any $M \in \mathcal{M}^c$ and consider $M^{i\sigma^*}$ defined by $M^{i\sigma^*}|x \setminus g^* = M|x \setminus g^*$, $M^{i\sigma^*}|g^* = M|g^*$, and $x'M^{i\sigma^*}x'$ for all $x \in g^*$ and $x' \in X \setminus g^*$. (More plainly, $M^{i\sigma^*}$ differs from $M$ only by putting $g^*$ at the top.) Now consider $M'$ globally comparable to $M$ on $g^*$. By construction, $M^{i\sigma^*}|g^* = M|g^*$ and $M^{i\sigma^*}|g^* = M|g^*$ so that $F^c(M) = F^c(M^{i\sigma^*})$ and $F^c(M') = F^c(M^{i\sigma^*})$. Moreover, $M^{i\sigma^*}$ and $M^{i\sigma^*}$ are globally comparable on $g^*$. Without loss of generality, suppose that $G(M|g^*) = G(M'|g^*) = \{g_i^j\}_{j \in I} = \{g_i^j\}_{j \in I}$ and $F^c(M) \in g_i^j$. Following the same reasoning as in (i) above, $F^c(M^{i\sigma^*})$ and $F^c(M^{i\sigma^*})$ are pairwise implementable for some prime set $PS \subseteq g_i^j$, which establishes the desired result.

**Proposition 6** Given a collection $\mathcal{M}^c$ consisting of every majority relation with a particular global structure (i.e. a complete collection of globally comparable majority relations), a binary social choice function $F^c : \mathcal{M}^c \to X$ is implementable by seeded agenda if and only if it is pairwise implementable on $X$ for every pair of majority relations $M, M' \in \mathcal{M}^c$.

**Proof of Proposition 6 and Theorem 1.** ($\Rightarrow$) If $F : \mathcal{M}(X) \to X$ (respectively $F^c : \mathcal{M}^c \to X$) is implementable, it is implementable for every pair of states in $\mathcal{M}(X)$ (respectively $\mathcal{M}^c$).
As in Lemma 5, let \( \mathcal{PW}(n) \) represent the choice functions that satisfy the pairwise condition on \( \mathcal{M}(n) \) and let \( \mathcal{PW}_j(n) \) represent the choice functions that satisfy the pairwise condition on the similarity class \( \mathcal{M}_j(n) = \{ M_k \}_{k \in K(j)} \). Finally, let \( J(n) = |\mathcal{M}(n)| \) represent the number of similarity classes in \( \mathcal{M}(n) \). For Proposition 6, I show (I) \( V_j(n) = \mathcal{PW}_j(n) \) for all \( j \in J(n) \). For Theorem 1, I show (II) \( V(n) = \Pi_{j \in J} V_j(n) \) for all \( n \). Results (I) and (II) establish \( V(n) = \Pi_{j \in J} \mathcal{PW}_j(n) \). Since \( \mathcal{PW}(n) = \Pi_{j \in J} \mathcal{PW}_j(n) \) by Proposition 1, it follows that \( V(n) = \mathcal{PW}(n) \). The proof is by strong induction on the size of the domain \( n \) and the number of similarity classes \( J(n) \).

For \( n \in \{1, 2, 3\} \), it is easy to see that (I) and (II) hold. (For \( n = 2 \), there are 2 globally distinct states each consisting of a linear order. For \( n = 3 \), there are 8 states and 5 similarity classes (3 classes that consist of two linear orders each and 2 classes consisting of one cycle). For all \( m < n \), next suppose that \( V(m) = \Pi_{j \in J(m)} V_j(m) \) and \( V_j(m) = \mathcal{PW}_j(m) \) for all \( j \in J(m) \). To complete the induction, it suffices to show that (I) and (II) hold for \( n \).

(I) Consider any non-trivial class similarity \( \mathcal{M}_j(n) \in \mathcal{M}(n) \) (so that \( |\mathcal{M}_j(n)| > 1 \) or, equivalently, \( |G(M)| > 1 \) for all \( M \in \mathcal{M}_j(n) \)). Without loss of generality, suppose \( G(M) = \{ g^l \}_{l \in L(j)} \) so that \( |g^l| < n \). By Lemma 4:

\[
\forall j \in J, \forall g \in g^l, \forall l \in L(j), \exists V_{j_l}(n) \text{ such that } \mathcal{PW}_j(n) = \bigcup_{l \in L(j)} \bigcup_{g \in g^l} \mathcal{PW}_{j_l}(n) \mid g^l
\]

where \( V_{j_l}(n) \mid g^l \) is the collection of binary SCFs that are implementable on \( g^l \subseteq g^l \). Lemma 5 above establishes that:

\[
\forall j \in J, \forall g \in g^l, \forall l \in L(j), \exists \mathcal{PW}_{j_l}(n) \text{ such that } \mathcal{PW}_j(n) = \bigcup_{l \in L(j)} \bigcup_{g \in g^l} \mathcal{PW}_{j_l}(n) \mid g^l
\]

By induction assumptions (I) and (II), \( V_{j_l}(n) \mid g^l = \mathcal{PW}_{j_l}(n) \mid g^l \) for all \( g \subseteq g^l \). Consequently, \( V_j(n) = \mathcal{PW}_j(n) \) which establishes the desired result.

(II) First, let \( J^*(n) = \{ j \in J(n) : |V_j| > 1 \} \). Given \( V_j(n) = \mathcal{PW}_j(n) \) for every \( j \in J^*(n) \), the result follows by induction on \( J \). For ease of notation, let \( \pi_j(C(n)) = \pi_j \). To establish the base case \( J = \{1, 2\} \), suppose \( \pi_{1,2} \neq \pi_1 \times \pi_2 \). Note that \( \pi_{1,2} \) is a subdirect product of \( \pi_1 \times \pi_2 \). For \( M_j \in \mathcal{M}_j(n) \), there exists a seeded agenda \( \mathcal{A}(x) \) that implements any outcome in \( x \in TC_j \). (The construction is similar to that given in Lemma 4.) This observation also establishes that the sub-algebra on each state is strong. Finally, the assumption that \( \pi_{1,2} \neq \pi_1 \times \pi_2 \) implies that \( \pi_{1,3} \) is weakly indecomposable. To see this, suppose that there are two disjoint collections \( \mathcal{M}_P = \{ M_p : p \in P \} \) and \( \mathcal{M}_Q = \{ M_q : q \in Q \} \) such that \( \mathcal{M}_P \cup \mathcal{M}_Q = \mathcal{M}_1(n) \cup \mathcal{M}_2(n) \) and \( \pi_{1,2} = \pi_P(V(n)) \times \pi_Q(V(n)) \). Now, consider any \( M_1, M_1' \in \mathcal{M}_1(n) \) and suppose that \( M_1 \in \mathcal{M}_P \) and \( M_1' \in \mathcal{M}_Q \). It follows that it is possible to pairwise implement \( x \in g \) and \( x' \notin g' \) for all \( \neq g' \). This contradicts Proposition 1 and establishes \( \mathcal{M}_1(n) \subseteq \mathcal{M}_P \) or \( \mathcal{M}_1(n) \subseteq \mathcal{M}_Q \). A similar argument shows \( \mathcal{M}_2(n) \subseteq \mathcal{M}_P \) or \( \mathcal{M}_2(n) \subseteq \mathcal{M}_Q \). Since the collections \( \mathcal{M}_P \) and \( \mathcal{M}_Q \) are non-trivial, then \( \mathcal{M}_1(n) = \mathcal{M}_P \) and \( \mathcal{M}_2(n) = \mathcal{M}_Q \) without loss of generality. But, this contradicts the assumption that \( \pi_{1,2} \neq \pi_1 \times \pi_2 \) and establishes that \( \pi_{1,2} \) is weakly indecomposable.

Accordingly, the theorem of Maroti applies. Let \( \beta \) define the largest congruence of \( Y \) such that \( \beta \neq \pi_{1,2} \times \pi_{1,2} \). By Proposition 1, it is possible to pairwise implement \((x_1, x_2)\) and \((x'_1, x'_2)\) on \( M_1 \in \mathcal{M}_1(n) \) and \( M_2 \in \mathcal{M}_2(n) \) so that \( x_1 M_1 x'_1 \) and \( x'_2 M_2 x_2 \). Using the same approach as
in Proposition 5, it follows that \( \beta = \pi_{\{1,2\}} \times \pi_{\{1,2\}} \). But, this contradicts the assumption that \( \beta \neq \pi_{\{1,2\}} \times \pi_{\{1,2\}} \) and establishes that \( \pi_{\{1,2\}} = \pi_{1} \times \pi_{2} \) in the base case \( J = \{1,2\} \).

Now, assume that the result holds for \( |J| = j \). In order to complete the induction, it suffices to show that the result holds for \( |J| = j + 1 \). Following the same line of argument as in the base case (and Proposition 5), the result \( \pi_{J} = \prod_{j \in J} \pi_{j} \) can be established by contradiction. This proves \( \pi_{J^*}(V(n)) = \prod_{j \in J^*} V^j_f(n) \). It then follows that \( V(n) = \prod_{j \in J^*} V^j_f(n) \).

\[ \Box \]

8.2 Proof of Theorem 2

The proof of Theorem 2 leverages Theorem 1. It also depends on the following results:

Remark 7: Consider a tournament \( M \), a seeded agenda \( A^s \), and a tournament isomorphism \( \sigma \) such that \( M' = \sigma M \). Then, \( v^*(A^{\sigma s}; M') = \sigma v^*(A^s; M) \), where \( \sigma s \) is the induced agenda obtained by permuting the labels of the alternatives in \( X \) according to \( \sigma \).

Lemma 6: A tournament solution \( S_X : \mathcal{M}(X) \to 2^X \) is agenda implementable if and only if there exists a binary SCR \( F : \mathcal{M}(X) \to X \) such that:

(i) \( F \) is implementable; and, (ii) \( S_X(M) = \{ \sigma F(M') : (\sigma, M') \text{ s.t. } M = \sigma M' \} \).

Proof. To simplify the presentation, define \( \Sigma F(M) = \{ \sigma F(M') : (\sigma, M') \text{ s.t. } M = \sigma M' \} \).

(If) Suppose that \( F \) satisfies conditions (i) and (ii). Then, there exists a seeded agenda \( A^s \) such that \( S_X(M) = \Sigma F(M) \) for all \( M \). To see \( S_X \) can be implemented by agenda, fix a tournament \( M \).

First, consider a non-trivial permutation \( \sigma \) over \( X \). Observe that \( \sigma \) induces a new seeding \( \sigma s \neq s \) as well as an isomorphic tournament \( \sigma^{-1} M \) (obtained by permuting the labels of the alternatives in \( X \) according to \( \sigma^{-1} \)). For \( \sigma s \), the chosen alternative on \( M \) is given by \( v^*(A^{\sigma s}; M) = \sigma v^*(A^s; \sigma^{-1} M) = \sigma F(\sigma^{-1} M) \) so that \( v^*(A^{\sigma s}; M) \in \Sigma F(M) = S_X(M) \) and, hence, \( V_A(M) \subseteq S_X(M) \).

Next, consider a non-trivial tournament isomorphism \( \sigma \) such that \( M = \sigma M' \). Observe that \( \sigma \) induces a new seeding \( \sigma s \neq s \) (where \( \sigma s \) is obtained by permuting the labels of the alternatives in \( X \) according to the tournament isomorphism \( \sigma \)). Since \( \Sigma F(M) = S_X(M) \) by assumption, \( \sigma F(M') \in S_X(M) \). Now, observe that \( \sigma F(M') = \sigma v^*(A^s; M') = \sigma v^*(A^s; \sigma^{-1} M) = v^*(A^{\sigma s}; M) \) so that \( \sigma F(M') \in V_A(M) \) and, hence, \( S_X(M) \subseteq V_A(M) \).

(Only if) Suppose that \( S_X \) can be implemented by agenda. Then, there exists an agenda \( A \) such that \( S_X(M) = \bigcup_{s \in \mathcal{S}(X)} v^*(A^s; M) \) for every tournament \( M \). Fix a seeding \( s \) of \( A \) and define \( F \) by \( F(M) = v^*(A^s; M) \) for all \( M \in \mathcal{M}(X) \). By construction, \( F \) is implementable.

To see that \( F \) satisfies condition (ii), fix a tournament \( M \). First, consider a non-trivial permutation \( \sigma \) over \( X \). By the same reasoning as the “if” direction, \( v^*(A^{\sigma s}; M) = \sigma F(\sigma^{-1} M) \) so that \( S_X(M) \subseteq \Sigma F(M) \). Next, consider a non-trivial tournament isomorphism \( \sigma \) such that \( M = \sigma M' \). By the same reasoning as the “if” direction, \( \sigma F(M') = v^*(A^{\sigma s}; M) \) so that \( \Sigma F(M) \subseteq S_X(M) \).

Given a tournament \( M \) on \( X \) with \( |X| = i \), the group of tournament isomorphisms is \( S_i \) (i.e., the group of permutations on \( X \)). Denote the subgroup of tournament automorphisms by \( Aut(M) \). The quotient group \( S_i/Aut(M) \) is the group of isomorphisms on \( M \) that are not automorphic.
Given a global structure $G = \langle \{g_k\}_{k=1}^m, M_G \rangle$ with $\bigcup_{k=1}^m g_k = X$, one can likewise define the group of isomorphisms $I(G)$, the subgroup of automorphisms $\text{Aut}(G)$, and the quotient group $I(G)/\text{Aut}(G)$. Global structures $G$ and $G'$ are isomorphic if (i) the induced quotient rankings of $M$ and $M'$ are isomorphic (i.e. $M_G = \sigma M_G$), and (ii) $|g_j| = |g'_{\sigma(j)}|$ for every neighborhood. Global structures $G$ and $G'$ are automorphic if, in addition, $g_j = g'_{\sigma(j)}$ for every neighborhood. As such, $I(G)/\text{Aut}(G)$ consists of mappings that preserve the overall structure of the induced quotient ranking while reshuffling (some of) the alternatives in the neighborhoods. The proof of Theorem 2 depends on the fact that the quotient group $I(G)/\text{Aut}(G)$ is sufficiently large. Intuitively, this ensures that there are sufficiently many distinct global structures isomorphic to $G$ to implement the alternatives from each component of $G$.

Lemma 7 (i) For every simple tournament $M$ on $X$ with $|X| = i \geq 4$, $|S_i/\text{Aut}(M)| \geq i$.

(ii) For every global structure $G$ on $X$ with $m$ components and $|X| \geq 4$, $|I(G)/\text{Aut}(G)| \geq m$.

Proof. (i) First observe that $|S_i/\text{Aut}(M)| = |S_i|/|\text{Aut}(M)| = i!/|\text{Aut}(M)|$. Since $3! = 6 \geq 3\sqrt{3} = (\sqrt{3})^3$ and $i > \sqrt{3}$ for $i \geq 4$, it follows that $(i - 1)! \geq (\sqrt{3})^{i-1}$ for $i \geq 4$. By Theorem 1 of Dixon [1967], $|\text{Aut}(M)| \leq (\sqrt{3})^{i-1}$. Hence, $|S_i|/i = (i - 1)! \geq |\text{Aut}(M)|$ for $i \geq 4$ which establishes the desired result.

(ii) For every global structure $G$, $m \geq 5$, $m = 3$, or $m = 1$. Consider each case in turn:

- For $m \geq 5$, the desired result is a consequence of part (I). For each component $g_k$ of $G$, fix an alternative $x_k$. Consider the subclass $I^*(G)$ (resp. $\text{Aut}^*(G)$) of isomorphisms (resp. automorphisms) that permute only the alternatives in $X^* = \{x_k\}_{k=1}^m$. By part (I), it follows that $|I^*(G)/\text{Aut}^*(G)| \geq m$ which, in turn, establishes the desired result.

- For $m = 3$ and $|X| \geq 4$, there must be one component (call it $g_3$) with $|g_3| \geq 2$. For components $g_1$ and $g_2$, fix alternatives $x_1$ and $x_2$. For component $g_3$, fix alternatives $x_{31}$ and $x_{32}$. Now, consider permutations $\sigma_1 = (x_1x_{31})$ and $\sigma_2 = (x_2x_{32})$. Observe that no two global structures in $\{G, \sigma_1G, \sigma_2G\}$ are automorphic.

- For $m = 1$, $I(G)/\text{Aut}(G)$ consists of the identity mapping which establishes the result.

Proof of Theorem 2. (Only if) See Theorems 8.5.1-2 of Laslier [1997].

(If) Consider a tournament solution $S$ that satisfies WCOM and SCOND. It suffices to show that $S_X : \mathcal{M}(X) \to 2^X$ can be implemented by agenda for all finite $X$. The proof is by strong induction on the size of $X$. For $|X| \leq 3$, the combination of WCOM and SCOND are equivalent to the requirement that $S_X(M) = TC(M)$ for all $M \in \mathcal{M}(X)$. Since $TC$ can be implemented by the simple agenda (see e.g. Lemma 8.3.3 of Laslier [1997]), the claim is true for $|X| \leq 3$.

Suppose the claim is true for $|X| \leq i$. To establish the claim for $|X| = i + 1$, the goal is to construct a binary social choice function $F : \mathcal{M}(X) \to X$ such that (i) $F$ is implementable and (ii) $S_X(M) \equiv \{\sigma F(M') : (\sigma, M') \text{ s.t. } M = \sigma M'\}$ for all $M \in \mathcal{M}(X)$. Lemma 6 then ensures that $S_X$ can be implemented by agenda, which is the desired result.

First, fix a global structure $G = \langle \{g_k\}_{k=1}^m, M_G \rangle$ and consider the subcollection of components $g_k \in G$ s.t. $S_X(M) \cap g_k^* \neq \emptyset$ for some tournament $M$ with global structure $G$. Call this subcollection
\(G^*\). Observe that \(S^*_k \equiv S^*_X \cap g^*_k\) defines a tournament solution on \(g^*_k\) and, moreover, \(S^*_k\) satisfies WCOM and SCOND.

To see that \(S^*_k\) defines a binary SCR on \(g^*_k\), suppose that \(x \in S(M) \cap g^*_k\) for some \(M\) with global structure \(G\). By definition, \(g^*_k\) is a component of every tournament with global structure \(G\). By WCOM, it follows that \(S^*_k(M') \neq \emptyset\) for any other \(M'\) with global structure \(G\). Moreover, since \(S\) satisfies neutrality, WCOM (which is stronger than the Condorcet property), and SCOND, it is straightforward to see that \(S^*_k\) satisfies these properties as well.

Thus, \(S^*_k\) is a tournament solution on \(g^*_k\) that satisfies WCOM and SCOND. By the induction hypothesis, it follows that \(S^*_k\) can be implemented by agenda. By Lemma 6, there is a binary social choice rule \(F^*_k : M(g^*_k) \rightarrow g^*_k\) such that (i) \(F^*_k\) is implemented by a seeded agenda on \(g^*_k\) and, moreover, \(F^*_k\) satisfies the identity in condition (ii) of Lemma 6.

To extend this construction into a binary social choice rule \(F : M(X) \rightarrow X\), it is sufficient to apply Lemma 7(II). Let \(\Gamma(X)\) denote a collection of global structures on \(X\) s.t. (a) \(G, \hat{G} \in \Gamma(X)\) are isomorphic, and (b) for every global structure \(G'\) on \(X\), there is a \(G \in \Gamma(X)\) isomorphic to \(G'\).

For any global structure \(G = \langle \{g^*_k\}_{k=1}^m, M_G \rangle \in \Gamma(X)\), Lemma 7 establishes that there are at least \(m\) distinct global structures on \(X\) that are isomorphic to \(G\). Using this result, define a partition \(G^*\) of the global structures isomorphic to \(G\) that divides them into \(|G^*| \leq m\) classes. Denote a generic class in \(G^*\) by \(G^*_j\). For any \(G' = \langle \{g^*_k\}_{k=1}^m, M_G \rangle \in G^*_j\), let \(g^*_k\) denote the component isomorphic to \(g_k\) (i.e. the \(k^{th}\) component of the global structure \(G\)). As established above, for all \(G' \in G^*_j\), one can implement a binary social choice function \(F_{G'}\) on \(g^*_k\) that satisfies requirements (i) and (ii).

Carrying out this construction for every \(G \in \Gamma(X)\) defines a binary social choice function \(F_{G'}\) for every global structure \(G'\) on \(X\). To extend these into a social choice rule \(F : M(X) \rightarrow X\), define \(F(M) = F_{G}(M|_g)\) if \(M\) has global structure \(G\) (and the domain of \(F_{G}\) is \(M|_g\)).

To complete the proof, it suffices to show that \(F\) satisfies the desired properties (i) and (iii).

To see that \(F\) is implementable, fix \(M, M' \in M(X)\). By the quoted theorem of Horan [2012], \(F\) is implementable if \(F(M)\) and \(F(M')\) are pairwise implementable. By the quoted proposition of Horan [2012], there are two cases to consider. If \(M\) and \(M'\) are globally distinct, it is clear (by the construction of \(F\) and the fact that \(S_X\) satisfies SCOND) that \(F(M) \in TC(M)\) and \(F(M') \in TC(M')\) as required. If \(M\) and \(M'\) have global structure \(G\), the construction of \(F_{G}\) likewise ensures that \(F(M), F(M') \in g\) for some \(g \in G\). By the induction hypothesis, it follows that \(F(M)\) and \(F(M')\) are pairwise implementable on \(g\).

To see that \(S_X(M) = \Sigma F(M) \equiv \{\sigma F(M') : (\sigma, M') \text{ s.t. } M = \sigma M'\} \) for all \(M \in M(X)\), fix a tournament \(M\) with global structure \(G\) and subcollection \(G^*\) of components that intersect with \(S_X(M)\). Fix some \(G^*_j\). By construction, there exists a tournament \(M' \in G^*_j\) such that \(M' = \sigma M\) and \(S^*_k(M'|_{\sigma^{-1}g_k}) = \Sigma F^*_k(M'|_{\sigma^{-1}g_k})\). From this identity and the definition of \(F\), it follows that \(S_X(M) \cap g^*_k = [\Sigma F(M')] \cap g^*_k\). Since this holds for all \(g^*_k \in G^*\), it follows that \(S_X(M) = \Sigma F(M)\). 

**8.3 Proof of the other Results in Sections 3 and 4**

**Proof of Lemma 1.** (I) Suppose that \(S\) satisfies COM. Consider two tournaments \(M\) and \(M'\) on \(X\) with a common component \(Y\) s.t. \(M_Y = M'_Y\). If \(X \setminus Y = \{x_1, ..., x_i\}\), then \(M =
Without loss of generality, it follows that $y \in Y$. Consequently, $S(M) \setminus Y = S(M') \setminus Y$. Next, suppose $S(M) \cap Y \neq \emptyset$ so that $y \in S(M)$ for some $y \in Y$. By COM, $y^s \in S(M)$. By a second application of COM, $S(M') \cap Y \neq \emptyset$ (since $S(M') \cap Y \neq \emptyset$).

(II) Suppose that $S$ satisfies COM and SCOND. Consider a tournament $M$ on $X$ with a component $Y$. If $X \setminus Y = \{x_1, \ldots, x_i\}$, then $M = \Pi(M'_Y : M'_Y, \{x_1, \ldots, x_i\})$. Suppose that $y \in S(M)$ for some $y \in Y$. By COM, $y^s \in S(M_Y)$. By definition, $y^s \in S(M)$ as required.


For SCOND: It is well known that each of these tournament solutions satisfies SCOND. See e.g. Theorem 5.1.7(viii) of Laslier [1997] for (i). Since $UC(M) \subseteq TC(M)$, the set inclusions $BA(M) \subseteq MC(M)$ (Proposition 7.1.8), $MC(M) \subseteq UC(M)$ (Proposition 5.3.2), $BP(M) \subseteq MC(M)$ (Theorem 6.3.3), and $TEQ(M) \subseteq BA(M)$ (Proposition 7.2.2) establish (ii)-(v).36 ■

Proof of Remark 2. For WCOM: Established by Propositions 3.4.8 of Laslier [1997].37

For SCOND: Consider a tournament $M$ on $X$ with a component $Y$. Suppose that $y \in SL(M)$ for some $y \in Y$. By definition, $y$ is at the top of some Slater order $\succ$ for $M$. By Proposition 3.4.3(iv) of Laslier, $y \succ |Y|$ is a Slater order for $M'|Y$. So, $y \in SL(M'|Y)$. Since $SL(M') \subseteq TC(M')$ (for every tournament $M'$) by Theorem 3.1.2(viii) of Laslier, $y \in TC(M'|Y)$. Thus, $SL(M) \cap Y \subseteq TC(M'|Y)$ as required. ■

Proof of Lemma 2. (I) For WCOM: Consider two tournaments $M$ and $M'$ on $X$ with a common component $Y$ s.t. $M_Y = M'_Y$. First, suppose $x \in S(M) \cup S'(M)$ for some $x \in X \setminus Y$. Without loss of generality, it follows that $x \in S(M)$. Since $S$ satisfies WCOM, it follows that $x \in S(M')$. Consequently, $x \in S(M') \cup S'(M')$ which establishes that $[S(M) \cup S'(M)] \setminus Y = [S(M') \cup S'(M')] \setminus Y$ as required. Next, suppose that $[S(M) \cup S'(M)] \cap Y \neq \emptyset$ so that $y \in S(M) \cup S'(M)$ for some $y \in Y$. Without loss of generality, it follows that $y \in S(M) \cap Y$. Since $S$ satisfies SCOND, it follows that $S(M') \cap Y \neq \emptyset$. Consequently, $[S(M') \cup S'(M')] \cap Y \neq \emptyset$ as required.

For SCOND: Consider a tournaments $M$ on $X$ with a component $Y$ s.t. such that $y \in S(M) \cup S'(M)$ for some $y \in Y$. Without loss of generality, it follows that $y \in S(M)$. Since $S$ satisfies SCOND, it follows that $y \in TC(M'_Y)$ so that $[S(M) \cup S'(M)] \cap Y \subseteq TC(M'_Y)$ as required.

(II) For COM: Propositions 3-4 of Laffond, Lainé, and Laslier [1996] establish (i)-(ii). For SCOND: For (i), suppose that $x \in S \cdot S'(M)$. By definition, $x \in S'(M)$. Since $S'$ satisfies SCOND, it follows that $x \in TC(M)$. Consequently, $S \cdot S'(M) \subseteq TC(M)$. For (ii), suppose that $x \in S(M) \cap S'(M)$. By definition, $x \in S(M)$ (and also $x \in S'(M)$). Since $S$ satisfies SCOND, it follows that $x \in TC(M)$. Consequently, $S(M) \cap S'(M) \subseteq TC(M)$ as required.

(III) For COM: See Proposition 6 of Brandt [2011]. For SCOND: The proof is by induction on the size of $X$. It is straightforward to check the claim for $|X| \leq 2$. Suppose the claim is true for $|X| \leq i$. To establish the claim for $|X| = i + 1$, there are two cases to consider. If $M$ is strong, it follows immediately that $\tilde{S}(M) \subseteq X = TC(M)$ as required. So suppose that $M$ is not strong.

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36 Another reference for the set inclusions described here is Laffond, Laslier, and Le Breton [1995].

37 Another reference for the results quoted in this proof is Banks, Bordes, and Le Breton [1991].
(and, hence, composed). In that case, it follows that \( M = \Pi(M^*; M_1, M_2) \) where \( X_1 = \text{TC}(M) \) and \( X_2 = X \setminus \text{TC}(M) \). Since \( \hat{S} \) satisfies COM, it follows that \( \hat{S}(M) = \bigcup_{i \in \hat{S}(M^*)} \hat{S}(M_i) \). From the base case of the induction, it follows that \( \hat{S}(M^*) \subseteq \text{TC}(M^*) = \{1\} \). Since \( \hat{S}(M^*) \neq \emptyset \), it follows that \( \hat{S}(M) = \hat{S}(M_1) \subseteq \text{TC}(M) \) as required. □

8.4 Proof of the Results in Section 5

Proof of Remark 3. Given that the Composition Copeland Set satisfies WCOM and SCOND, Theorem 2 implies that \( CO^* \) is agenda implementable. The following establish that \( CO^* \) satisfies WCOM and SCOND.

For WCOM: Consider tournaments \( M \) and \( M' \) on \( X \) with a common component \( Y \) s.t. \( M_Y = M'_Y \). Suppose \( x \in CO^*(M) \) for some \( x \in X \setminus Y \). (Similar reasoning establishes that \( CO^*(M) \cap Y \neq \emptyset \) implies \( CO^*(M') \cap Y \neq \emptyset \).) By definition of the Composition Copeland Set, \( x \in \text{TC}(M) \).

There are two cases:

**Case 1:** If \( \text{TC}(M) \cap Y = \emptyset \), it follows that \( M|_{\text{TC}(M)} = M'|_{\text{TC}(M')} \). By definition of the Composition Copeland Set, \( CO^*(M) = CO^*(M') \) so that \( x \in CO^*(M') \).

**Case 2:** If \( \text{TC}(M) \cap Y \neq \emptyset \), it must be that \( Y \subseteq \text{TC}(M) \). (The fact that \( x \in \text{TC}(M) \) rules out the other possibility that \( \text{TC}(M) \subseteq Y \).) There are two sub-cases to consider: (i) \( \text{TC}(M) \) is simple and (i) \( \text{TC}(M) \) is not simple. In sub-case (i), it follows that \( Y \) is a singleton and, hence, \( M = M' \) so that \( x \in CO^*(M') \). In sub-case (ii), the definition of the coarsest non-degenerate decomposition \( D(M) = \{X_1, \ldots, X_i\} \) and the fact that \( x \not\in Y \) allow for only two possibilities: either (a) \( x \in X_j \) and \( Y \subseteq X_k \) for some \( k \neq j \), or (b) \( x \in X_j \) and \( Y \subseteq X_j \). For (a), the definition of the Composition Copeland Set implies \( x \in CO^*(M_j) \). Since \( D(M') = D(M) \), it follows that \( x \in CO^*(M'_j) \) and, hence, \( x \in CO^*(M') \). For (b), it must be that \( x \in \text{TC}(M_j) \). But, this leads back to the situation of cases 1 and 2 (except on \( X_j \) instead of \( X \)).

Since \( X \) is finite, the process terminates in cases 1, 2(i), or 2(ii)(a), so that \( x \in CO^*(M') \).

For SCOND: Consider a tournaments \( M \) on \( X \) with a component \( Y \) s.t. \( y \in CO^*(M) \) for some \( y \in Y \). By definition of the Composition Copeland Set, \( y \in \text{TC}(M) \). There are two cases:

**Case 1:** If there exists a \( y' \in Y \) s.t. \( y' \not\in \text{TC}(M) \), it follows that \( y \in \text{TC}(M|_Y) \).

**Case 2:** If \( Y \subseteq \text{TC}(M) \), the definition of the coarsest non-degenerate decomposition implies that: either (i) \( Y = \text{TC}(M) \) or, (ii) \( Y \subseteq X_j \) for some \( X_j \in D(M) \). In sub-case (i), \( \text{TC}(M|_Y) = \text{TC}(M) \) so that \( y \in \text{TC}(M|_Y) \) follows immediately. In sub-case (ii), it must be that \( y \in \text{TC}(M_j) \). But, this leads back to the situation of cases 1 and 2 (except on \( X_j \) instead of \( X \)).

Since \( X \) is finite, the process terminates in cases 1 or 2(i), so that \( y \in \text{TC}(M|_Y) \). □

**Lemma 8** For every tournament \( M \), \( CO^*(M) \subseteq UC(M) \).

**Proof.** Fix a component in \( X_j \in D^*(M) \) and suppose that there is some component \( X_k \) such that \( kM^*j \) and \( kM^*i \) for every component \( X_i \) such that \( jM^*i \). Then, \( \text{co}(X_k, M) \geq |X_j| + \text{co}(X_j, M) \) so that \( 2 \cdot \text{co}(X_k, M) > |X_j| + 2 \cdot \text{co}(X_j, M) \) which contradicts the assumption that \( X_j \in D^*(M) \). This shows that \( j \in UC(M^*) \). By recursively repeating the argument on \( D^*(M_j) \), the result follows. □
Lemma 9  For all tournaments $M$ on $X$ and all $x \in CO^*(M)$, $co(x, M) \geq (|X| - 1)/2$.

Proof. The proof that is by strong induction on the size of $X$. It is straightforward to check the claim for $|X| \leq 3$. Suppose the claim is true for $|X| \leq i$.

To establish the claim for $|X| = i + 1$, suppose that $M$ is strong with $D(M) = \{X_1, ..., X_i\}$. Otherwise, $CO^*(M) = CO^*(M|_{TC(M)})$ so that $co(x, M) = co(M|_{TC(M)}) + (|X| - |TC(M)|)$. By the induction hypothesis, $co(x, M|_{TC(M)}) \geq (|TC(M)| - 1)/2$ so that $co(x, M) \geq (|TC(M)| - 1)/2 + |X| - |TC(M)| > (|X| - 1)/2$.

Suppose that $x \in X_k$. From the definition of the Composition Copeland Set, $|X_k| + 2 \cdot co(X_k, M)$ is maximal. Consequently, $(|X_k| - 1)/2 + co(X_k, M) \geq (|X_j| - 1)/2 + co(X_j, M)$ for all $x_j \in D(M)$. If $(|X_k| - 1)/2 + co(X_k, M) < (|X| - 1)/2$, it follows that

$$\sum_{y \in X_j} co(y, M) = \sum_{y \in X_j} co(y, M) + |x_j| \cdot co(X_j, M) = |X_j| \cdot \left(\frac{|X_j| - 1}{2} + co(X_k, M)\right) < |X_j| \cdot (|X| - 1)/2$$

for every component $X_j$ so that $\sum_{y \in X} co(y, M) < |X| \cdot (|X| - 1)/2$. But, this contradicts the fact that $\sum_{y \in X} co(y, M) = |X| \cdot (|X| - 1)/2$. So, $(|X_k| - 1)/2 + co(X_k, M) \geq (|X| - 1)/2$. By the induction hypothesis, $co(x, X_k) \geq (|X| - 1)/2$ which establishes that $co(x, M) \geq (|X| - 1)/2$ as required. ■

Proof of Proposition 4. Consider a tournament $M$ on $X$. Consider any $x \in CO^*(M)$ and $w \in CO(M)$. The proof that $co(x, M)/co(w, M) > 2/3$ is by strong induction on the size of $X$. The claim is trivially true for $|X| \leq 3$ since $CO^*(M) = CO(M)$ for every tournament $M$ on three or fewer alternatives.

So, suppose that the claim is true for $|X| \leq i$. To establish the claim for $|X| = i + 1$, let $\{X_1, ..., X_i\}$ denote $D(M|_{TC(M)})$ and let $M_j$ denote the sub-tournament $M|_{X_j}$ on $X_j$. Without loss of generality, there are two cases: (i) $x, w \in X_j$; and, (ii) $x \in X_k$, $w \in X_j$ for $X_k \neq X_j$.

In the first case, $co(x, M) = co(x, M_j) + co(X_j, M)$ and $co(w, M) = co(w, M_j) + co(X_j, M)$. Since $w$ is a Copeland winner on $M$, it must also be a Copeland winner on $M_j$. Likewise, since $x$ is a Composition Copeland winner on $M$, it must also be a Composition Copeland winner on $M_j$. By the induction hypothesis, it then follows that $co(x, M_j)/co(w, M_j) > 2/3$. Consequently:

$$\frac{co(x, M)}{co(w, M)} = \frac{co(x, M_j) + co(X_j, M)}{co(w, M_j) + co(X_j, M)} > \frac{2}{3}$$

In the second case, suppose that $co(x, M)/co(w, M) \leq 2/3$. Since $co(x, M) = co(x, M_k) + co(X_k, M) + co(X_j, M)$, $co(w, M) = co(w, M_j) + co(X_j, M)$, $co(x, M_j) \leq |X_j| - 1$, and $co(x, M_k) \geq (|X_k| - 1)/2$ (by Lemma 9 and the fact that $x$ is a Composition Copeland winner on $M_k$), it follows that:

$$\frac{(|X_k| - 1)/2 + co(X_k, M)}{(|X| - 1) + co(X_j, M)} \leq \frac{co(x, M_k) + co(X_k, M)}{co(w, M_j) + co(X_j, M)} = \frac{co(x, M)}{co(w, M)} \leq \frac{2}{3}$$

By definition of the Composition Copeland Set, $|X_k| + 2co(X_k, M) \geq |X_j| + 2co(X_j, M)$. Combined with the previous inequality, this gives $(|X_k| - 1)/2 + co(X_k, M) \leq |X_j| - 1$.

But, this is a contradiction. By assumption, there is some component $X_l \in D(M|_{TC(M)})$ that dominates $X_j$ (i.e. $x_lM_{x_j}$ for all $x_l \in X_l$ and $x_j \in X_j$). But $co(X_k, M) < |X_j|$ implies that...
Construction, the definition of the maximal set implies that Max hypothesis applies. Consequently, since \( |t_j| = |X_j| \) for all \( j \in \{1, \ldots, k\} \), the induction hypothesis applies. By definition of the Maximal Set, it then follows that \( \text{Max}(M) = \bigcup_{j=1}^{k} \text{Max}(M_j) \) and establishes that \( \text{Max}(M) \) is the trivial partition of \( X \) for \( |X| \geq 2 \).

Proof of Remark 4. (i) For COM: Consider a tournament \( M \) on \( X \). The proof that the Maximal Set satisfies COM is by strong induction on the size of \( X \). For \( |X| \leq 3 \), the definition of the Maximal Set implies \( \text{Max}(M) = UC(M) \) so that COM is trivially satisfied.

Now, suppose that Maximal Set satisfies COM for \( |X| \leq i \). To establish the claim for \( |X| = i+1 \), suppose that \( M = \Pi(M^*_1; M_1, \ldots, M_k) \) on \( X \) where the sub-tournaments \( M_j \) are on \( X_j \). Without loss of generality, the only possibility is \( \text{TC}(M) = \bigcup_{j=1}^{k} X_j \) for \( k \leq k \).

If \( M \) is not strong \( (\text{TC}(M) \subset X) \), \( \text{Max}(M) = \text{Max}(M|_{\text{TC}(M)}) \) by definition of the Maximal Set. Since \( \text{TC}(M) \leq i \), the induction hypothesis applies. By definition of the Maximal Set, it then follows that \( \text{Max}(M|_{\text{TC}(M)}) = \bigcup_{j=1}^{k} \text{Max}(M_j) \). Finally, the definition of the Maximal Set also gives

\[
\text{Max}(M^*) = \text{Max}(M|_{\text{TC}(M^*)}) = \{1, \ldots, j^*\}.
\]

Combining these identities gives \( \text{Max}(M) = \bigcup_{j \in \text{Max}(M^*)} \text{Max}(M_j) \) as required.

If \( M \) is strong, there are two possibilities. When \( M \) is simple, it follows that \( k = i + 1 \) or \( k = 1 \). In the first case, the identity \( \text{Max}(M) = \bigcup_{j \in \text{Max}(M^*)} \text{Max}(M_j) \) is a consequence of the fact that \( \text{Max}(x_j) = x_j \) for every singleton tournament. In the second case, \( \text{Max}(M) = \text{Max}(M_1) = \bigcup_{j \in \text{Max}(M^*)} \text{Max}(M_j) \).

When \( M \) is not simple, there are two possibilities. If \( \{X_1, \ldots, X_k\} \) is the trivial partition of \( X \), then \( \text{Max}(M) = \bigcup_{j \in \text{Max}(M^*)} \text{Max}(M_j) \) follows \( \text{Max}(x_j) = x_j \) for every singleton tournament.

Otherwise, suppose without loss of generality that \( |X_1| \geq 2 \). Observe that the definition of \( D(M) = \{X_1', \ldots, X_k'\} \) implies that the partition \( \{X_1, \ldots, X_k\} \) is weakly finer than \( D(M) \). From the definition of the Maximal Set, \( \text{Max}(M) = \bigcup_{j \in D(M)} \text{Max}(M_j) \). Since \( |X_j'| \leq i \) for all \( j \leq l \), the induction hypothesis applies to each of the sub-tournaments \( M_j' \) on \( X_j' \). Thus, \( \text{Max}(M_j') = \bigcup_{h \in \text{Max}(M_j^*)} \text{Max}(M_{jh}) \) where \( M_j' = \Pi(M_j^*; M_{j1}, \ldots, M_{jk}) \) is the composed tournament on the components of \( \{X_1, \ldots, X_k\} \) contained in \( X_j' \) and \( M_j^* \) is the summary tournament associated with this decomposition. Now, consider the composed tournament \( M^* = \Pi(M^{**}; M_1^*, \ldots, M_k^*) \) whose sub-tournaments are the summaries \( M_j^* \) and whose summary \( \text{Max}(M) \) is the summary tournament on \( D(M) \). Since \( |X_1| \geq 2 \) by assumption, \( M^* \) is a tournament on \( i \) or fewer alternatives and the induction hypothesis applies. Consequently, \( \text{Max}(M^*) = \bigcup_{j \in \text{Max}(M^*)} \text{Max}(M_j^*) \). Since \( M^{**} \) is simple by construction, the definition of the maximal set implies that \( \text{Max}(M^{**}) = D(M) \). Combining the various observations from this paragraph establishes that

\[
\bigcup_{X_j' \in D(M)} \text{Max}(M_j') = \bigcup_{X_j' \in D(M)} \bigcup_{h \in \text{Max}(M_j^*)} \text{Max}(M_{jh}) = \bigcup_{j \in \text{Max}(M^*)} \bigcup_{h \in \text{Max}(M_j^*)} \text{Max}(M_{jh}) = \bigcup_{j \in \text{Max}(M^*)} \text{Max}(M_j)
\]

so that \( \text{Max}(M) = \bigcup_{X_j' \in D(M)} \text{Max}(M_j') = \ldots = \bigcup_{j \in \text{Max}(M^*)} \text{Max}(M_j) \) as required.

For COND: Consider a tournament \( M \) on \( X \) and suppose that \( x \in \text{Max}(M) \). If \( M \) is cyclic, then \( TC(M) = X \) so that \( x \in TC(M) \). Otherwise, the definition of the Maximal Set implies
Max(M) = Max(M_{TC(M)}). Thus, \( x \in Max(M_{TC(M)}) \) so that \( x \in TC(M) \).

(ii) Fix a tournament solution \( S \) that satisfies COM and COND. Consider a tournament \( M \) on \( X \). The proof that \( S(M) \subseteq Max(M) \) is by strong induction on the size of \( X \). For \( |X| \leq 3 \), the combination of WCOM and SCOND are equivalent to the requirement that the selected alternatives coincide with the Condorcet Set \( TC(M) \). Thus, \( S(M) = TC(M) = Max(M) \) for \( |X| \leq 3 \) which establishes the base case.

Now, suppose that \( S(M) \subseteq Max(M) \) for \( |X| \leq i \). To establish \( S(M) \subseteq Max(M) \) for \( |X| = i + 1 \), there are two cases to consider. If \( M \) is simple, the definition of the Maximal Set establishes that \( Max(M) = X \). Since \( S(M) \subseteq X \) by definition, it follows that \( S(M) \subseteq Max(M) \) as required. If \( M \) is not simple, write \( M \) as a composed tournament \( M = \Pi(M^*; M_1, ..., M_k) \) on \( X \) where the sub-tournaments \( M_j \) are on \( X_j \). Since \( S \) and the Maximal Set satisfy COM (by assumption and Max by part (I) above), \( S(M) = \bigcup_{j \in S(M^*)} S(M_j) \) and \( Max(M) = \bigcup_{j \in Max(M^*)} Max(M_j) \). Since \( M \) is not simple, \( k \leq i \) and \( |X_j| \leq i \) for all \( j \leq k \). Thus, the induction hypothesis applies to each of the \( M_j \) and \( M^* \). Thus, \( S(M_j) \subseteq Max(M_j) \) (for all \( j \leq k \)) and \( S(M^*) \subseteq Max(M^*) \). Consequently,

\[
S(M) = \bigcup_{j \in S(M^*)} S(M_j) \subseteq \bigcup_{j \in Max(M^*)} S(M_j) \subseteq \bigcup_{j \in Max(M^*)} Max(M_j) = Max(M)
\]

so that \( S(M) \subseteq Max(M) \) as required.

(iii) Consider the tournament \( M \) on \( \{x_1, ..., x_5\} \) defined by

\[
x_1Mx_2Mx_3Mx_1, x_3Mx_4Mx_5Mx_3, x_4Mx_1, x_4Mx_2, x_5Mx_1, \text{ and } x_5Mx_2.
\]

Straightforward computation establishes that \( UC(M) = \{x_3, x_4, x_5\} \). Since \( M \) is simple however, \( Max(M) = X \) by definition. Thus, \( UC(M) \subseteq Max(M) \).

Proof of Theorem 3. (I) (Only if) This follows from Remark 4(iii). (If) Since \( S \subseteq Max \) by assumption, Remark 4(i) establishes that \( S \) has an upper bound satisfying COM and COND. From Corollary 3(II)(ii), it follows that \( S \) has a least upper bound that satisfies COM and COND. In other words, \( S \) has an upper approximation that satisfies COM and COND.

(II) (Only if) This is immediate. (If) By assumption, \( S \) has a lower bound satisfying WCOM and SCOND. From Corollary 3(I), it follows that \( S \) has a greatest lower bound that satisfies WCOM and SCOND. In other words, \( S \) has a lower approximation that satisfies WCOM and SCOND.

Remark 8 \( TC^- = Max \).

Proof. Clearly, \( Max \subseteq TC^- \) since \( Max \) satisfies WCOM and SCOND and \( Max \subseteq TC \). So, consider a tournament \( M \) and some \( x \in TC^-(M) \). By way of contradiction, suppose that \( x \notin Max(M) \). By definition of the Maximal Set, there is some component \( Y \) (within a component of a component ... etc. of \( D(M) \)) such that \( x \notin TC(M_{|Y}) \). But, this contradicts the fact that \( TC^- \) satisfies SCOND. So, \( x \in Max(M) \) which implies \( TC^- \subseteq Max \). Since \( Max \subseteq TC^- \), \( TC^- = Max \).
**Proof of Corollary 5.** Suppose \( S \) satisfies WCOM and SCOND. Consider a tournament \( M \) and some \( x \in S(M) \). The proof that \( x \in \text{Max}(M) \) is identical to Remark 8 with \( S \) in place of \( TC^- \). ■

### 8.5 Proof of the Results in Section 6

**Proof of Remark 5.** (i) This follows from Theorem 2 and the results cited in the text.

(ii) Both points follow directly from the argument stated in the text. ■

**Proof of Remark 6.** (i) Given the game form in the text, this follows from Remark 5(i).

(ii) The point about Pareto efficiency follows from the argument stated in the text. To show that all tie-breaker rules must be neutral, fix a tournament solution \( S \) on \( X \), an agent \( j \in N \), and a permutation \( \sigma : X \to X \). Since \( \sigma M(\vec{P}) = M(\sigma \vec{P}) \), \( \sigma \) induces an isomorphism on \( M \). Because \( S \) is neutral (as a tournament solution), \( S(\sigma M(\vec{P})) = \sigma S(M(\vec{P})) \). Since \( S \) is implemented by some agenda \( A \) (by assumption), \( x^\sigma \in S(\sigma M(\vec{P})) \) is the winner on \( A^\sigma \) for any \( x \in S(M(\vec{P})) \) that is the winner on \( A^\sigma \). Since \( S_j(\vec{P}) = \max_{\succ_j} S(M(\vec{P})) \), then \( \max_{\sigma(\succ_j)} S(M(\sigma \vec{P})) = \sigma S_j(M(\vec{P})) \) as required.

**Proof of Corollary 6.** (I) It is straightforward to see that: (i) the sequential approval game \( \Gamma^A(S,\tau, \vec{P}) \) defines a unique backward induction equilibrium outcome \( S^A(\vec{P}) \) for every profile \( \vec{P} \); and, moreover, (ii) \( S^A \) is anonymous.

(II) Fix a tournament solution \( S \) on \( X \), a permutation \( \tau : N \to N \), and a profile \( \vec{P} \). Let \( S^n(M(\vec{P})) \equiv \{ x \in S(M(\vec{P})) : x = S_j(M(\vec{P})) \text{ for some } j \in N \} \) denote the multi-set of alternatives (i.e. the same alternative may appear more than once) that are tie-breaker outcomes for some \( j \in N \). Following the argument given in Moulin [1980, 1984], any backward induction equilibrium of the sequential veto game \( \Gamma^V(S,\tau, \vec{P}) \) is given by

\[
S^1(M(\vec{P})) \equiv S^n(M(\vec{P})) \setminus \{ x_{\tau^{-1}(1)}, \ldots, x_{\tau^{-1}(n-1)} \}
\]

where \( x_{\tau^{-1}(j)} \in \arg \min_{\tau^{-1}(j)} S^{j+1}(M(\vec{P})) \) and \( S^j(M(\vec{P})) \equiv S^{j+1}(M(\vec{P})) \setminus \{ x_{\tau^{-1}(j+1)} \} \) for \( 1 \leq j < n \). It is straightforward to see that: (i) the game \( \Gamma^V(S,\tau, \vec{P}) \) defines a unique backward induction equilibrium outcome \( S^V(\vec{P}) \) for every profile \( \vec{P} \); and, moreover, (ii) \( S^V \) is neutral.

(III) This follows from the argument stated in the text. ■

### References


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