EC705 Placement Examination August/September 2006

Attempt all questions. Questions 1 and 2 each carries 10 points. Question 3 carries 15 points. Questions 4 and 5 each carries 20 points. Question 6 carries 25 points. The total in the test is 100.

Read the questions very carefully. You must explain your answers. You may draw graphs to help you visualize your work. Keep in mind, however, that a proof must be formal, and a graph almost always lacks the formality required by a proof.

1. Maximize $f(x, y, z) = x^2+y^2+5z^2$ subject to the constraints $x \geq 0$, $y \geq 0$, $z \geq 0$, $x-2y+z = 1$. Determine the maximum value of f and all the points $(x, y, z) \in\mathbb{R}^3$ which attain the contrained maximum. Justify your steps carefully.

2. Maximize $f(x, y, z) = x^2+y^2+5z^2$ subject to the constraints $x \geq 0$, $y \geq 0$, $z \geq 0$, $x+2y+z = 1$. Determine the maximum value of f and all the points $(x, y, z) \in\mathbb{R}^3$ which attain the contrained maximum. Justify your steps carefully.

3. Let $S \subset \mathbb{R}$. Let $f : S \to \mathbb{R}$ be a function. Give a definition of f being a continuous function. Let $I = \{1, 2, 3, ..., n\}$. Let $f : I \to \mathbb{R}$. According to your definition of continuity, is f a continuous function?

4. The notation $IN[x]$ denotes the integral part of the real number x. That is, $IN[x]$ is the largest integer smaller than or equal to x. For example, $IN[10.5] = 10$ (the largest integer smaller than 10.5 is 10). Let α, x and y be real numbers.

   - Give a condition on α for which a solution, in terms of x and y, to the following equations exists:

   $$\begin{aligned} IN[x] + y &= 1 \\ IN[x] - y &= \alpha \end{aligned}$$

   - If the solution exists, find all the solutions.
   - For a given α (say $\alpha = 1$), does the solution set, if it exists, form an affine set? Explain your answers. (A subset S of a linear space X is said to be an affine set if for every s and t in S, the combination $\theta s + (1 - \theta)t$ belongs to S, for any $\theta \in \mathbb{R}$.)

5. Let $f : X \to Y$ be a function, where X and Y are both convex and bounded subsets of $\mathbb{R}$. Let x_0 be a point in X. The upper contour set of x_0 is defined by $\{x \in X : f(x) \geq f(x_0)\}$.

   - A function f is said to be increasing if $x_1 \leq x_2$ implies $f(x_1) \leq f(x_2)$. Prove that if f is increasing, the upper contour set is convex.
   - A function f is said to be decreasing if $x_1 \leq x_2$ implies $f(x_1) \geq f(x_2)$. Prove that if f is decreasing, the upper contour set is convex.
   - Give an example of a function for which an upper contour set is not convex. A graph together with explanation will be sufficient.

- Give an example of a function for which an upper contour set consists of isolated points. A graph together with explanation will be sufficient.

6. Let $f : I \times I \to \mathbb{R}$, where $I \equiv [0, 1]$, the unit interval, and $f(x, y) = \max\{x, y\}$.

   - Solve the following problem: choose $(x, y) \in I \times I$ to maximize $f(x, y)$ subject to $x+y = \theta$, where θ is a parameter belonging to I (that is, $0 \leq \theta \leq 1$.). Call this problem Program L
   - Let $V(\theta) = f(x', y')$, where (x', y') is a solution of Program L. Is $V(\theta)$ differentiable? If it is differentiable, what is its derivative? If it is not differentiable, illustrate why it is not so for a particular value of θ.