Investigations in Number Theory
KHC MA 101
Spring 2012
Wednesday/Friday
9:00AM – 10:30 AM
Room: MCS 144

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Office Hours: Monday 10-11, Friday 4-5pm
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Course Description and Objectives

The goal of the course is to convey some notion of how mathematicians discover new mathematics. The only way to do this is to experience the frustration and excitement of figuring things out for oneself. Consequently, the course will begin with a few simple open-ended questions – questions that members of the class can begin to answer with little or no background, but which lead quickly into much deeper questions, some of which may extend beyond what mathematicians themselves can answer at the present time. The course will then proceed to develop ideas and tools for investigating these questions, so that the mathematics arises in response to the mathematical needs and inquiries of members of the class. New material will mostly be introduced via the homework problem sets, which will ask for examples, counterexamples, generalizations, conjectures, and bits of proof. The function of the lectures will be to compare the examples and solutions and hypotheses formulated by the various members of the class and to try to extract something from them which takes the class as a whole to the next level of understanding.

Mathematical Content
The course will cover the fundamental theorem of arithmetic, the abstract concept of a commutative ring and of unique factorization into irreducibles, and examples of commutative rings other than the rational integers in which unique factorization does or does not hold.

Required Texts

Attendance
Attendance in class is required

Academic Dishonesty
All KHC students are expected to maintain high standards of academic honesty and integrity. It is the responsibility of every student in the Kilachand Honors College to be aware of Boston
University’s Undergraduate Academic Conduct Code and to abide by its provisions. The University Honors College is committed to high academic and professional standards for its curriculum, faculty, and students. A description of policies and procedures to guide students in achieving their educational goals may be found at the following URL: 
http://www.bu.edu/academics/resources/academic-conduct-code/

Grading
- Mid-term Exam: 10%
- Final Exam: 20%
- Homework: 70%

Homework problem sets will be handed out in almost every class and collected at the end of the next class, and your homework grade will be based not only on what you hand in but also on your participation in the discussion of the homework during class.

Approximate Course Schedule

Week 1: Asking questions and bare-handed attempts to answer them.
  • What is the probability that a randomly chosen pair of positive integers is relatively prime?
  • When can an integer be expressed as a sum of two squares? Three squares? four squares? .... ?

Week 2: Examples of algebraic structure: number systems with addition, subtraction, and multiplication;
  • Comparing and contrasting various simple number systems.
  • The unit group. Zero-divisors.
  • Factorization. When are two factorizations “the same”? 

Week 3: Mathematical induction and the Well Ordering Principle
  • Scientific induction: discovering mathematical truths – examples and counterexamples.
  • Mathematical induction: reasoning about mathematics and establishing mathematical truths.

Week 4: Divisors and Irreducibility.
  • Divisors and common divisors – the Euclidean algorithm
    • The irreducible elements of a ring

Week 5: The Gaussian Integers.

Week 6: Polynomials over a ring.

Week 7: Other examples of commutative rings.

Week 8: Primes and irreducibles.
• Counting primes in \( \mathbb{Z}[i] \); in \( \mathbb{F}_p[x] \); in \( \mathbb{Z}[p^{-5}] \).

**Week 9**: The fundamental theorem of arithmetic and its generalizations.
• Prime factorization in \( \mathbb{Z}[i] \); in \( \mathbb{F}_p[x] \); in \( \mathbb{Z}[p^{-5}] \).
• Prime factorization in \( \mathbb{Z}[x] \).

**Week 10**: Multiplicative functions and the Chinese Remainder Theorem. **Week 11**: The Riemann zeta function.

**Week 12**: Sums of squares.
• In how many ways can a positive integer \( n \) be written as a sum of two squares?
• Dirichlet zeta functions.

**Week 13**: Density and probabilistic methods in \( \mathbb{Z} \).
• On average, in how many ways can a positive integer be written as a sum of two squares?
• What is the probability that a randomly chosen pair of positive integers is relatively prime?
• The value of \( \zeta(s) \) at \( s = 2 \).