1. Introduction and Background

Multiple-wavelength measurements of light transport in tissue can yield information about scattering and absorption properties that are pertinent to the physiological state of the tissue. The scattering properties of tissue are related to tissue morphology, and the absorption properties are related to the tissue biochemistry. Several chromophores in tissue have absorption bands in the near UV, visible, and near IR, including hemoglobin, nicotinamide adenine dinucleotide, melanin, and bilirubin. A variety of techniques are being developed to measure scattering properties of tissue and to detect and measure concentrations of chromophores.\(^1\)\(^-\)\(^6\) Frequently these techniques are based on the diffusion approximation and are single-wavelength techniques, although a few researchers have made broader wavelength measurements.\(^3\),\(^7\)

Measurement over a broad wavelength range offers potential advantages over single-wavelength measurements. When one is working in the diffusion approximation, where contributions to the signal that are due to scattering and absorption can be separated, expected advantages of broad wavelength measurements are: (1) measurements of concentrations of several absorbers can be performed simultaneously with improved accuracy, and (2) knowledge of the wavelength dependence of scattering will aid in the determination of structural features. In geometries where the diffusion approximation is not valid (e.g., small separations of the source and detector), a method of separating the contributions to the signal that are due to scattering and absorption has not yet been developed. Nonetheless, recent \textit{in vivo} clinical studies involving a statistically significant number of measurements have demonstrated that the measurement of elastic scatter or fluorescence over broad wavelength ranges has the potential to distinguish cancerous and noncancerous tissue.\(^8\)\(-\)\(^11\) These results could be improved on with detailed answers to the following questions: (1) What is the sensitivity of the measurements to specific features of the tissue? (2) How can the measurement procedures and equipment be adjusted to enhance sensitivity to specific features? As an approach to addressing the above two questions, in this paper we examine the wavelength dependence of scattering by employing both Mie theory calculations and experimental measurements of polystyrene spheres.\(^12\)

Although the exact nature of the scattering centers in tissue has not been determined, it is possible to
place some limits on scatterer sizes and refractive indices. Scattering is expected to occur from cells, organelles, and proteins. Mammalian cells vary in size from approximately 3 to 20 \( \mu m \) in diameter.\(^{13}\) Organelles such as mitochondria and nuclei have dimensions in the range \( 1-5 \mu m \). Subnuclear components such as nucleoli and regions of condensed and extended chromatin may have dimensions as large as 1 \( \mu m \). Proteins typically have dimensions of the order of fractions of a micrometer. For example, a hemoglobin molecule is roughly 64 \AA\ in diameter. Therefore it is expected that scattering centers can vary in size from very small particles such as proteins, which have dimensions measured in angstroms, to cells 20 \( \mu m \) in diameter. There is also a range of indices of refraction in tissue. Extracellular fluid has a real refractive index between 1.348 and 1.351.\(^{14}\) Refractive indices of the cytoplasm and nucleus of Chinese hamster cells have been measured to be 1.3703 and 1.392, respectively.\(^{15}\) Materials of higher index such as lipids (1.46)\(^{16}\) and proteins (1.51)\(^{17}\) are also present in tissue.

The tissue parameters that determine light transport are the boundary conditions and, depending on the measurement geometry, a subset of the following parameters: the inverse mean free path for scattering \( \mu_s \), the inverse mean free path for absorption \( \mu_a \), the probability distribution of scattering angles \( P(\theta) \), and the reduced scattering coefficient \( \mu_s' \). When source-to-detector separations are large (such that the diffusion approximation is valid), only \( \mu_s' \) and \( \mu_a \) are needed to describe light transport. When source-to-detector separations are small, \( \mu_s P(\theta) \), and \( \mu_a \) can all be important factors determining light transport. In this paper we examine both regimes.

In Section 2, the expected wavelength dependence of the reduced scattering coefficient in tissue as a function of particle size is discussed. In Subsections 3.A and 3.B broad wavelength measurements of absorption and scattering coefficients with a cw system are presented for Intralipid and suspensions of polystyrene spheres. Using the wavelength dependence of the measured reduced scattering coefficient, the size of the scattering particles in Intralipid is estimated. In Subsection 3.C measurements of polystyrene spheres at small fiber separations are presented.

### 2. Theoretical Predictions

If simplifying assumptions are made that the scattering centers in tissue are homogeneous dielectric spheres and that the electromagnetic field incident on each sphere is a plane wave, Mie theory can be used to predict the wavelength dependence of the scattering. For a sphere of radius \( r \) and index of refraction \( n_s \) in a medium with index \( n_m \), scattering is determined by the size parameter \( x = 2\pi r n_m / \lambda \) and the ratio of indices of refraction \( m = n_s/n_m \). Mie theory can be used to calculate \( P(\theta) \) and the efficiency factors for scattering and absorption \( Q_s \) and \( Q_a \), respectively.\(^{19}\) The efficiency factors for individual scattering events are related to the macroscopic scattering coefficients \( \mu_s', \mu_a' \), and \( \mu_a \) by the following:

\[
\mu_s = Q_s N_s A_s, \tag{1}
\]

\[
\mu_a' = Q_a N_s A_s (1 - g), \tag{2}
\]

\[
\mu_a = Q_a N_s A_s. \tag{3}
\]

\( N_s \) is the number density of spheres, \( g = \langle \cos \theta \rangle \) is the expectation of the cosine of the photon scattering angle, and \( A_s = \pi r^2 \). For our calculations we assume that the light incident on the spheres is unpolarized.

Both \( Q_s \) and \( P(\theta) \) are complicated and sometimes rapidly varying functions of wavelength. The wavelength dependence of \( Q_s \), which is related to \( \mu_s \) by Eq. (1), varies greatly with both the size and refractive index of the scatterer as demonstrated in Fig. 1.

Figure 2 is a surface plot of \( \log(P(\theta)) \) for a wide range of wavelengths for a 0.5- \( \mu m \) sphere (\( n_s = 1.46 \) and \( n_m = 1.33 \)). There are oscillations in \( P(\theta) \) as a function of both wavelength and angle. Note that \( P(180^\circ) \) shows more structure as a function of wavelength than \( P(0^\circ) \). This structure depends on the scatterer.
Therefore measurements in geometries where sensitivity to high-angle scattering is enhanced are expected to be more useful for differentiating average structural features of the scatterers. Other authors have reached similar conclusions.\textsuperscript{20,21}

Considering the complicated dependencies of both $m_s$ and $P(u)$ on wavelength one might expect $m_s \propto \lambda^{-0.37}$ to be a rapidly varying function of wavelength. Although this is true for some values of $x$ and $m$, for a wide range of size parameters and indices of refraction, $m_s$ is a decreasing function of wavelength from 350 to 950 nm. For $1 < m < 1.1$ and $5 < x < 50$, $m_s \propto \lambda^{-0.37}$ as shown by Graaff \textit{et al}.\textsuperscript{22}

Figure 3 demonstrates that even for $4.6 < x < 65$, the relative amplitude of $Q_s(1 - g)$ varies from $\lambda^{-0.37}$ by less than 6\% between 400 and 900 nm.

As the size parameter $x$ decreases below 5, the wavelength dependence of $\mu_s(\lambda)$ approaches the $1/\lambda^4$ dependence expected for Rayleigh scattering. This is demonstrated in Fig. 4. Although $\mu_s$ cannot be expressed accurately by the form $\lambda^{-n}$ for all sphere sizes, there is a general trend that the wavelength dependence changes from $\lambda^{-4}$ to $\lambda^{-0.37}$ as sphere size increases. All of the calculations shown in Fig. 4 were done with $m = 1.11$ ($n_s = 1.50$ and $n_m = 1.35$). Based on the indices of refraction of the biological components of tissue discussed above it is unlikely that $m$ will be larger than this.

Smaller values of $m$ have little effect on the wavelength dependence of $\mu_s$ as shown in Fig. 5. Therefore the wavelength dependence of $\mu_s$ can be used to estimate the size of the scatterers contributing to the elastic scatter signal without concern about the relative refractive indices of the scatterers. An example of this technique for estimating the size of the dominant scattering particle in Intralipid is given in Section 3.

### 3. Multiwavelength Measurements of Tissue Phantoms

#### A. Comparison of Theory and Experiment for Polystyrene Spheres

Measurements of polystyrene sphere suspensions were made to compare predicted and measured values of $\mu_s(\lambda)$. The purpose of these measurements was to determine if the measured wavelength dependence of $\mu_s(\lambda)$ agreed with predictions of Mie theory. The infinite diffusion theory approximation was used.

The experimental system included uncooled CCD detectors and used one collection fiber that could be moved with respect to the delivery fiber [Fig. 6(b)]. A reference for the lamp was used to account for

![Fig. 3. Wavelength dependence of $Q_s(1 - g)$ for 0.5 and 3.0-μm-radius spheres. Results are shown for both $n_s = 1.46$ with $n_m = 1.33$ ($m = 1.1, x = 4.6-10.5$ for the 0.5-μm spheres and $x = 27.8-62.7$ for the 3.0-μm spheres) and for $n_s = 1.40$ with $n_m = 1.35$ ($m = 1.04, x = 4.7-10.6$ for the 0.5-μm spheres and $x = 28.3-63.6$ for the 3.0-μm spheres). The dashed line shows the $\lambda^{-4}$ wavelength dependence of scatterers with a size parameter $x \ll 1$. (On a log-log plot a $\lambda^{-n}$ dependence is a straight line.) Each curve has been multiplied by a constant to have a value of 1 at 600 nm.](image1)

![Fig. 4. Wavelength dependence of $Q_s(1 - g)$ for a wide range of sphere sizes. $n_s = 1.50$ and $n_m = 1.35$. All curves have been normalized to have a value of 1 at 600 nm. Deviations from a straight line indicate deviations from a $\lambda^{-n}$ dependence.](image2)

![Fig. 5. Effect of changes in $n_s$ on $Q_s(1 - g)$ for 0.05 and 0.1-μm-radius spheres. All curves have been normalized to have a value of 1 at 600 nm.](image3)
rately for any changes in the lamp output during the course of the measurements.

The elastic scatter signals of aqueous suspensions of three sphere sizes were measured: 1.02 ± 0.02-μm-radius spheres, 1.5 ± 0.113-μm-radius spheres, and 1.85 ± 0.113-μm-radius spheres. The concentrations by weight of the spheres were, respectively, 0.2% \( N_s = 4.18 \times 10^9 \) cm\(^{-3}\), 0.5% \( N_s = 3.45 \times 10^{10} \) cm\(^{-3}\), and 0.3% \( N_s = 1.08 \times 10^9 \) cm\(^{-3}\). On a cubic grid, this gives shortest, nearest neighbor separations of 13.37, 3.07, and 21.0 μm, respectively.

A 400-μm optical fiber (0.22 NA) was used to deliver light to the sphere suspension, and a 600-μm optical fiber (0.48 NA) was used to collect light from the sphere suspension. The diffusion approximation was valid for the geometry employed in these measurements. The sample volume was a cylinder of depth 6.5 cm and diameter 7 cm. The fibers were placed approximately 3 cm into the suspensions of spheres. Measurements of the elastic scatter signal were made as a function of fiber separation for \( d > 1 \) cm.

Derivations of equations to determine \( \mu_s \) and \( \mu_s' \) from measurements of amplitude-modulated light transport in the diffusion approximation have been given by several authors and include the derivations for cw light.\(^{23,24} \) These derivations rely on the fact that measurements of light intensity are made at multiple fiber separations, \( \rho \). Expressions for \( \mu_s(\lambda) \) and \( \mu_s'(\lambda) \) are given in Eqs. (4) and (5) where \( b_1(\lambda) \) is the slope of \( \rho I(\rho) \), \( b_0(\lambda) \) is the intercept of \( \rho I(\rho) \), and \( c \) is a function of the amplitude of the incident light and the system response function. Because the measured intensities were referenced to a measurement of a diffuse reflector with a flat wavelength response, \( c \) does not depend on wavelength.

\[
\mu_s(\lambda) = \frac{cb_1^2(\lambda)}{\exp[(b_0(\lambda))]}, \quad (4)
\]

\[
\mu_s'(\lambda) = \frac{b_1^3(\lambda)}{3\mu_s(\lambda)} - \mu_s(\lambda). \quad (5)
\]

We determined the constant \( c \) by measuring the elastic scatter signal as a function of fiber separation in 10% whole milk or suspensions of polystyrene spheres with and without the addition of a solution of red food coloring with a known absorption coefficient. The change in absorbance can then be used to evaluate \( c \).

The experimentally determined values for \( \mu_s' \) for suspensions of polystyrene spheres are compared with calculations of \( \mu_s' \) in Fig. 7. The efficiency factors \( Q_s, Q_a \) and the distribution of scattering angles \( P(\theta) \) were calculated for each size of sphere with a value of 1.59 used for the index of the polystyrene spheres and a value of 1.33 for the index of the suspending medium. The reduced scattering coefficient \( \mu_s' \) was then determined according to Eq. (2). Calculations for both spheres of uniform dimension and for a mixture of discrete sphere sizes are shown in Fig. 7. The curves have been scaled such that the area under the theoretical curves is equal to the area under the corresponding experimental curves. The wavelength dependencies of the theoretical and experimental curves are quite close. The theoretical curve of \( \mu_s' \), calculated for a single sphere size, oscillates as a function of wavelength. The result of our
summing the contributions of several sizes of spheres is that this oscillation is damped out.

In the derivation of Eqs. (4) and (5) the wavelength dependence of the speed of light in the medium that is due to the dispersion of the medium has been ignored. This is a reasonable approximation. Based on measurements of the dispersion of water, \( v \) is expected to change by \(<0.8\%\) over the wavelength range 400-700 nm. Therefore, the wavelength dependence of \( c \) that is due to this effect is very weak. Wavelength-dependent errors in \( c \) may also be due to differences in dispersion of air and the medium being measured. The system response function is determined with the fibers in air above a piece of diffuse reflecting material; however, the measurements were performed with the fibers in an aqueous medium. The reflectance of photons at the fiber interface is a function of the indices of refraction of the fiber and the medium it is in contact with. Because air and water have different dispersions, measurement of the system response with an air-to-fiber instead of a water-to-fiber interface causes small wavelength-dependent errors in \( c \). These errors are \(<1\%\) for aqueous media.

B. Measurements at Large Source Detector Separations

The experimental system was comprised of a xenon arc lamp (Optical Radiation Corporation compact lamp, Azusa, Calif.), optical fibers for light delivery to and collection from the tissue phantom, a spectrograph and a thermoelectrically cooled CCD detector [Fig. 6(a)]. The fiber-optic probe consisted of several collection fibers arranged linearly from the source fiber. The bottom surface of the probe had dimensions of 4.5 cm \( \times \) 3 cm and was painted flat black. Ten fibers could be imaged simultaneously onto the CCD array. The relative efficiency of light collection by each fiber was determined with the use of an integrating sphere and a separate light-delivery fiber. We determined the entire optical system response for a subsystem containing only one collection fiber by replacing the tissue phantom with a diffuse reflecting material, such as Spectralon, that has a flat response as a function of wavelength. By combining the optical system response function measured using one collection fiber, and the measurement of the relative light collection efficiency of the collection fibers, we measured the light intensity as a function of distance from the collection fibers to within a constant.

The phantom consisted of 15\% Intralipid-10\% in a roughly cylindrical container with a diameter and height of 30 cm and 9 cm, respectively. Small amounts of red food coloring were added. The Intralipid still appeared white after the addition of the food coloring. The absorbance of the red food coloring solution was also measured on a Cary spectrophotometer. We made wavelength-dependent measurements of scattered light at delivery-collection fiber center-to-center separations of 1.62, 1.8, 2.0, 2.2, and 2.4 cm by placing the optical probe on the surface of the medium. The light-delivery fiber was 1000 \( \mu \)m in diameter and the collection fibers were 400 \( \mu \)m, although they were coupled into a harness of fibers that were 200 \( \mu \)m in diameter. The numerical aperture of the fibers was 0.22. The integration time for measurement of the Intralipid was 4 s.

The light intensity as a function of distance \( \rho \) from the delivery fiber was modeled with an expression given below that is similar to that of Farrell et al.\textsuperscript{25} The mismatch in boundary conditions was assumed to be small, and \( A \) was 1.06 for all of the fits. As pointed out by Haskell et al., this expression neglects the fluence rate term.\textsuperscript{26} For the parameters used here, this term is negligible. Nonlinear fits of light intensity as a function of fiber separation were performed. The three parameters of the fits were \( \mu_s'(\lambda) \), \( \mu_e'(\lambda) \), and the overall amplitude factor, \( I_0 \). At the distances measured, the shape of these curves can be used to determine \( \mu_e'(\lambda) \). However, different pairs of \( \mu_s(\lambda) \) and \( \mu_s'(\lambda) \) with the same values of \( \mu_e \) do not result in differently shaped curves. The amplitude, in contrast, does change. Therefore, to determine \( \mu_s'(\lambda) \) and \( \mu_s(\lambda) \), it is necessary to determine \( I_0 \). Because both the response of a single delivery-collection pair of fibers to Spectralon was measured and because the relative response of all of the collection fibers was measured, it was possible to treat the data so that \( I_0 \) was not a function of wavelength. To determine \( I_0 \), the intensity as a function of fiber separation and wavelength was fit iteratively with different values of \( I_0 \). When the correct value of \( I_0 \) was reached, the difference in absorbance between measurements made with different amounts of red dye present agreed with the predictions based on the spectrophotometer measurements.

\[
I(\rho) = I_0 \left[ \frac{1}{\mu'_t} \left( \mu_{\text{eff}} + \frac{1}{r_1} \right) \exp(-\mu_{\text{eff}}r_1) \right. \\
\left. + \left( \frac{1}{\mu'_t} + 2z_b \right) \left( \mu_{\text{eff}} + \frac{1}{r_2} \right) \exp(-\mu_{\text{eff}}r_2) \right],
\]

where

\[
r_1 = \left[ \frac{1}{\mu'_t} + \rho^2 \right]^{1/2}, \quad r_2 = \left[ \frac{1}{\mu'_t} + 2z_b \right]^{1/2} + \rho^2,
\]

\[
\mu_{\text{eff}} = \frac{\mu_e}{D}, \quad \mu'_t = \mu_s + \mu_s' \quad \text{and} \quad z_b = 2AD,
\]

where \( D = (1/3\mu_e') \) and \( A \) is related to the internal reflection.

The absorber was added to Intralipid in small increments. Figure 8 shows the change in absorbance that is due to the incremental addition of the red dye and the expected change in absorbance based on measurements of the solution of the red food coloring made in a standard spectrophotometer. As pointed out above, \( I_0 \) was chosen to make these curves agree in amplitude. The shift in the peak of the absorbance has also been seen in measurements with milk used as the scattering medium and is consistent with
of literature values reported by Flock et al.\textsuperscript{27} The wavelength dependence of the reduced scattering coefficient can be used to estimate the average size of the scattering particles. From 450 to 600 nm, the wavelength dependence of the $\mu_s' (\lambda)$ is expressed accurately by $\lambda^{-2}$. A comparison with Fig. 4 finds that the scattering particles are of the order of 75 nm in radius. The average radius of particles in Intralipid was found to be 46.5 nm by van Staveren et al. using transmission electron microscopy.\textsuperscript{28}

C. Measurements with Small Fiber Separations: Sensitivity to Particle Size

In this subsection we demonstrate that sensitivity to structural features of the scatterers can be varied by changing the separation $d$ between light delivery and collection fibers. For small values of $d (d < 1/\mu_s')$, measurements are more sensitive to sphere size than for large values of $d (d > 1/\mu_s')$.

Aqueous solutions of 0.2485-, 0.483-, and 1.5-µm-radius polystyrene spheres were measured with fiber separations of 0.05 and 1.649 cm. The concentrations of the spheres by weight were, respectively, 0.119%, 0.154%, and 0.34%. The reduced scattering coefficients were 4.64, 4.85, and 1.53 cm\(^{-1}\), respectively, at 600 nm. A 600-µm delivery fiber (0.48 NA) and a 400-µm collection fiber (0.22 NA) were used. The fibers penetrated the surface of the suspension less than 1 mm. Sample volumes greater than 30 mL were measured in black-walled cylinders with diameters of ~3.5 cm.

The collected elastic scatter signal $I$ is shown in Fig. 10. The curves have been normalized to a value of 1 at 500 nm. The three curves at a fiber center-to-center separation of 500 µm are quite distinctive. The oscillations seen in the elastic scatter signal of the 1.5-µm spheres are not an artifact. Similar oscillations for 1.85-µm-radius spheres have been seen in Monte Carlo simulations that employ Mie theory.\textsuperscript{29}

At a source-to-detector separation of 1.649 cm, the wavelength dependence of the elastic scatter signal of the three sizes of spheres are all similar.

4. Discussion and Conclusions

The main conclusions from Section 2 are: (1) Small changes in size and index of refraction cause greater changes in the wavelength dependence of $P(\theta, \lambda)$ and $\mu_s' (\lambda)$ than in $\mu_s'' (\lambda)$ for scatterers relevant to tissue. (2) Measurements in geometries where high-angle scattering is enhanced are expected to be more useful for the differentiation of morphological features. (3) As particle size increases from the Rayleigh regime to a regime in which $r \geq \lambda$ but $x < 70$, the wavelength dependence of $\mu_s'' (\lambda)$, which is approximately a monotonically decreasing function of wavelength, changes from $\lambda^{-4}$ to $\lambda^{-0.37}$. For relative refractive indices typical of tissue, this wavelength dependence does not change with refractive index, but rather depends only on particle size.

In Subsection 3.C, we demonstrated that measurements made with small source detector separations are more sensitive to the size of the
measurements of \( \mu_s' \) for three suspensions of polystyrene microspheres were also made with a less accurate experimental system. There was good agreement between the experiment and calculations with Mie theory used for the prediction of the wavelength dependence of \( \mu_s' \). However, agreement between measured and predicted values of the magnitude of \( \mu_s' \) was less accurate and differed by approximately a factor of 2. This illustrates an important point. The measurement of the absolute values of \( \mu_s' \) is more difficult than the measurement of the wavelength dependence. Therefore techniques for understanding optical properties based on the wavelength dependence rather than the absolute value of \( \mu_s' \) are important.

Other groups have also begun investigating the use of cw and or white light systems for measuring \( \mu_s' \) and \( \mu_a' \). Chance et al. have also demonstrated the use of a cw system for measuring \( \mu_s' \) and \( \mu_a' \).30 Their method differed from ours in that the calibration used was a turbid medium for which both \( \mu_s' \) and \( \mu_a' \) were known, whereas our calibrations required only knowledge of \( \mu_a' \). Liu et al. have also made measurements of optical properties and introduced an approximation to the Farrell expression.31 Recently, measurements of light intensity as a function of distance from the source and of \( \mu_s' \) and \( \mu_a' \) with the use of an experimental system similar to the one used to make the measurements of Intralipid have been reported.32,33

Throughout this paper we have approximated the scatterers in tissue as homogeneous spheres, and we used Mie theory to predict scattering characteristics. Many of the expected scatterers in tissue are not homogenous and are not spherical. Therefore the question arises as to the range of validity of this approximation. Comparison of experiment and theory in measurements of red blood cells and yeast cells partially answers this question. Measurements of \( \mu_a' \) and \( \mu_s' \) have been compared to Mie theory calculations for red blood cells.34 Deviations in \( \mu_a' \) typically were less than 10%, but differences in measured and predicted values of \( g \) caused larger (factor of 2) discrepancies in \( \mu_s' \). Nonetheless, trends that were seen in the experimental data (as a function of osmolality or RBC species) were also seen in the Mie calculations. Measurements of \( \mu_s' \) of yeast cells have also been compared with Mie calculations of \( \mu_s' \).35 By varying the cell refractive index, good agreement was found for a variety of yeast cells. Therefore, we believe that the trends we report in this paper are relevant to scattering from real cells in tissue.

We have demonstrated that measurements of \( \mu_s' \) and \( \mu_a' \) can be made rapidly with a cw, white light system and that these measurements are sensitive to changes in absorbance of \( \sim 0.001 \, \text{cm}^{-1} \). We have also shown that the wavelength dependence of \( \mu_s(\lambda) \) can be used to estimate the size of the scatterers in Intralipid. Possible applications for this technique are monitoring real-time changes of multiple, physiologically important chromophores in vivo. The sys-
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References and Notes

12. The spheres are actually a mixture of polystyrene, polystyrene-diving-benzene, polyvinyltoluene, and/or butadiene.
18. This is the definition of the size parameter used by Bohren and Huffman (Ref. 19, p. 100). The definition $x = 2\pi r / \lambda$ is also found frequently in the literature.
