Jamming and Wedging in Constrained Rigid-body Dynamics*

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Abstract

It is well known that the introduction of Coulomb friction in rigid-body dynamics can lead to problems of existence and uniqueness in the forward equations. In this paper, it is shown that these problems relate to jamming and wedging. Separate conditions for jamming and wedging are provided in the three common formulations of constrained systems. These conditions are illustrated with examples.

1 Introduction

As precision becomes more important in manipulation and assembly tasks, increasing emphasis has been placed on obtaining accurate dynamic models for motion planning and simulation. Friction is important in many robotic systems. This includes internal friction from nonbackdrivable transmission elements (e.g., screws) as well as friction arising through contact with the environment.

Many assembly and dextrous manipulation tasks have been studied as quasistatic processes under the assumption that precision assembly is inherently a slow process. In this context, jamming and wedging have been thoughtfully examined by Whitney and his colleagues [11]. However, to increase productivity and cost effectiveness, higher speeds and the inertial forces which produce them must be considered. The possibility of jamming or wedging arises when friction depends on the magnitude of the contact force. The main contribution of this paper is to provide tests for identifying the onset of jamming and wedging from the dynamic equations.

In the next section, jamming and wedging are defined and illustrated by simple examples. In section 3, three common descriptions of the constrained dynamic equations are formulated to include friction. In the following section, jamming and wedging conditions for each formulation are presented and proven. These are followed by two examples and concluding remarks.

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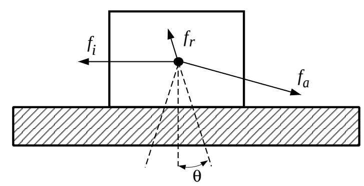

Figure 1: Block sliding to right on horizontal surface. Moment associated with friction force is neglected.

2 Background

We assume a simple Coulomb model described by the following equation in terms of normal force, f_n, friction force, f_f and velocity, v.

$$\begin{array}{l} f_n \geq 0, \text{ (unilateral contact)} \\ v \neq 0 \Rightarrow |f_f| = \mu|f_n|, \; vf_f \leq 0 \\ v = 0 \Rightarrow |f_f| = \mu'|f_n|, \; \mu' \in [0, \mu] \end{array} \quad (1)$$

The last equation indicates that during static contact, the friction force assumes the direction and magnitude necessary to prevent motion. The actual friction coefficient, μ, is fixed and depends on the materials in contact.

By definition, friction and thus acceleration are discontinuous at velocity reversals. Much more subtle is the fact that Coulomb friction can produce discontinuities in velocity as well. To understand how this can occur, consider the case of a block sliding on a horizontal surface with coefficient of friction μ as shown in Figure 1. The forces f_a, f_r and f_i are the applied, reaction and D'Alembert inertial forces, respectively. If the block starts from rest then motion will ensue only if the applied force, f_a, lies outside the "friction cone" defined by $\mu = \tan\theta$. Similarly, if the block is sliding, it will smoothly decelerate to rest if the applied force lies inside the friction cone. In both cases, the D'Alembert inertial force, f_i, is directed horizontally.

Now assume that the horizontal surface accelerates upward. Assume that motion constraints between the surface and block cause the inertial force of the block, f_i, to lie along the line labeled "Locus of inertial forces" in Figure 2. (This is analogous to the inertial coupling between a screw and nut [3]). We can express the balance of forces on the block as

$$f_a + f_i + f_r = 0. \quad (2)$$

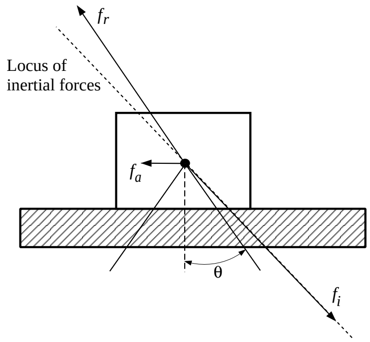

Figure 2: Block sliding to right on horizontal surface which accelerates upward.

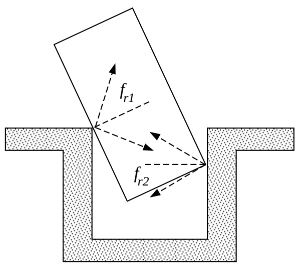

Figure 3: Wedging during peg insertion. Overlapping friction cones at contacts represent sets of possible reaction forces, f_{r1} and f_{r2}.

For the small applied force, f_i and f_r must take on large magnitudes in their given directions for the forces to sum to zero as shown.

If the locus of inertial forces is, in fact, collinear with the active edge of the friction cone, the magnitudes of f_i and f_r approach infinity. Considering the directions of f_r and f_i, this indicates that the block comes to rest instantaneously with respect to the horizontal surface – a discontinuity in velocity indicating the onset of *jamming* – a condition in which no motion can occur at a contact given the applied forces.

If f_i lies inside the friction cone, this also corresponds to jamming. To understand this important point, note that during static contact, the Coulomb model produces the force necessary to inhibit sliding. Consequently, the reaction force produced is collinear with the inertial force and thus produces jamming.

Wedging is a static phenomenon which occurs when the resultant forces at the constraints are linearly dependent. An example of a planar peg-in-hole insertion presented by Whitney [11] is shown in Figure 3. If the friction cones at the two contact points overlap, static contact allows four independent reaction components to constrain only three degrees of freedom. This is the familiar problem of a statically indeterminate system.

Starting with Painlevé, examples of rigid-body systems with Coulomb friction were published which produced either no solution or multiple solutions to the forward dynamics problem [2,4,5,6,8]. While not stated, the examples described all correspond to jamming. This is made especially clear in [2] and [5] which indicate the impact-like transition which occurs.

Recently, it has been shown that the inclusion of compliance at the contacts resolves the existence and uniqueness problems [3,10]. Of course, this is the method used to resolve statically indeterminate problems and so compliant models eliminate the rigid-body ambiguities of both jamming and wedging. Issues related to jamming and wedging are discussed in [1,7,9,11].

3 Constrained Rigid-body Dynamics

Consider an n degree of freedom system with generalized coordinates given by the vector q which is subject to the m holonomic (rheonomic) constraints

$$\Phi(q,t) = 0. \tag{3}$$

Using Lagrange multipliers, the constrained dynamic equation is

$$M(q)\ddot{q} + h(q,\dot{q}) = \tau + \Phi_q^T(q,t)\lambda + f_f \tag{4}$$

where $M(q)$ is the inertia matrix and h consists of centrifugal, Coriolis and gravity terms. The generalized input forces and torques are given by τ, λ is the vector of constraint forces and f_f is the friction forces and torques.

We can write the normal force at contact i, f_{ni}, in terms of the generalized coordinates as

$$f_{ni} = \Phi_{qi}^T(q,t)\ \lambda \tag{5}$$

Following the Coulomb model, we assume friction at the i^{th} contact, f_{fi}, to depend linearly on the contact forces,

$$f_{fi} = \hat{\Phi}_{qi}^T(q,\dot{q},\mu_i,t)\ \lambda \tag{6}$$

where $\hat{\Phi}_{qi}^T$ depends linearly on the coefficient of friction μ_i. In contact coordinates p_i, the friction force is orthogonal to the constraint direction, i.e.,

$$\hat{\Phi}_{qi} J_i^{-1} J_i^{-T} \Phi_{qi} = 0 \tag{7}$$

assuming the inverse of the Jacobian matrix $J_i = \partial p_i / \partial q$ exists.

Collecting the row vectors $\hat{\Phi}_{qi}$ into the matrix $\hat{\Phi}_q$ and the coefficients of friction into the vector $\mu = [\mu_1, \mu_2, \ldots, \mu_m]^T$, the dynamic equation can be written as

$$M(q)\ddot{q} + h(q,\dot{q}) = \tau + \left[\Phi_q^T(q,t) + \hat{\Phi}_q^T(q,\dot{q},\mu,t)\right]\lambda \tag{8}$$

Three common matrix representations of (3) and (8) are developed below.

3.1 Differential-algebraic Equations

In order to obtain a set of $(n+m)$ differential-algebraic equations, we differentiate (3) twice to obtain the constraint acceleration equation.

$$\Phi_q \dot{q} = -\Phi_t \tag{9}$$

$$\Phi_q \ddot{q} = -(\Phi_q \dot{q})_q \dot{q} - 2\Phi_{qt}\dot{q} - \Phi_{tt} = \Gamma(q, \dot{q}, t) \qquad (10)$$

(10) and the dynamic equation (8) can be expressed together in matrix form as

$$\begin{bmatrix} M & -(\Phi_q^T + \hat{\Phi}_q^T) \\ \Phi_q & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \lambda \end{bmatrix} = \begin{bmatrix} \tau - h \\ \Gamma \end{bmatrix} \qquad (11)$$

It is easily seen that the choice of generalized coordinates does not affect the rank of the leading matrix. Provided that this matrix is invertible, the equation can be solved for the n accelerations, $\ddot{q}$, and the m contact forces, λ.

3.2 Embedding Method

The preceding equation (11) is redundant since, with m constraints, there are only $(n-m)$ independent generalized coordinates. It is common to reduce these to a set of n equations. To do so, we partition q into $(n-m)$ independent coordinates, q_i, and m dependent coordinates, q_d. The acceleration constraint equation can be written as

$$\begin{bmatrix} \Phi_{q_i} & | & \Phi_{q_d} \end{bmatrix} \begin{bmatrix} \ddot{q}_i \\ - \\ \ddot{q}_d \end{bmatrix} = \Gamma \qquad (12)$$

Assuming the existence of $\Phi_{q_d}^{-1}$, the accelerations can be written as

$$\ddot{q} = \begin{bmatrix} \ddot{q}_i \\ \ddot{q}_d \end{bmatrix} = \begin{bmatrix} I \\ -\Phi_{q_d}^{-1}\Phi_{q_i} \end{bmatrix} \ddot{q}_i + \begin{bmatrix} 0 \\ \Phi_{q_d}^{-1}\Gamma \end{bmatrix} \qquad (13)$$

Defining

$$A = \begin{bmatrix} I \\ -\Phi_{q_d}^{-1}\Phi_{q_i} \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \Phi_{q_d}^{-1}\Gamma \end{bmatrix} \qquad (14)$$

the dynamic equation (8) combined with constraint equation (13) can be written as

$$\begin{bmatrix} MA, & -(\Phi_q^T + \hat{\Phi}_q^T) \end{bmatrix} \begin{bmatrix} \ddot{q}_i \\ \lambda \end{bmatrix} = \tau - h - Mb \qquad (15)$$

involving the $(n-m)$ independent generalized accelerations and the m reaction forces. It can be shown that the rank of the matrix in (15) does not depend on the choice of independent coordinates. [1] is an example of the embedding method.

3.3 Coordinate Reduction

Without friction, it is customary to premultiply (15) by A^T since $A^T \Phi_q^T = 0$. This eliminates the contact forces λ and produces $(n-m)$ equations involving the independent accelerations while preserving the positive definite property of the inertia matrix.

With friction, it is still possible to eliminate the contact forces, λ. This can be accomplished by premultiplying (15) by an orthogonal complement of $(\Phi_q + \hat{\Phi}_q)^T$. Define

$$T^\perp = \left\{ T \in \Re^{(n-m)\times n} \mid T(\Phi_q + \hat{\Phi}_q)^T = 0, \ \mathrm{rank}(T) = n - m \right\} \qquad (16)$$

In other words, $T^\perp$ is the set of matrices whose null space is spanned by the columns of $(\Phi_q + \hat{\Phi}_q)^T$.

For $T \in T^\perp$, (15) is reduced to

$$[TMA] \ \ddot{q}_i = T(\tau - h - Mb) \qquad (17)$$

[9] is an example of this approach. Note that it is not possible to define a change of coordinates $q_i = Bp$ such that $B^T A^T \in T^\perp$ so as to ensure a positive definite inertia matrix $B^T A^T MAB$.

If the contact forces are of interest, the accelerations, $\ddot{q}_i$, can be eliminated from (15) by premultiplying it by $S \in S^\perp$ where

$$S^\perp = \left\{ S \in \Re^{m \times n} \mid S(MA) = 0, \ \mathrm{rank}(S) = m \right\} \qquad (18)$$

This yields the equation

$$\left[S(\Phi_q^T + \hat{\Phi}_q^T) \right] \ \lambda = -S(\tau - h - Mb) \qquad (19)$$

4 Jamming and Wedging Conditions

The conditions for jamming and wedging are presented in two theorems below.

Theorem 1: Jamming - Assuming no wedging, the rigid-body system described by (11), or (15), or (17) and (19) is *jammed* if there exists a vector μ_J with components μ_{Ji} in the intervals $[0, \mu_i]$ such that a linear combination of columns of $\left[\Phi_q^T + \hat{\Phi}_q^T\right]$ lie in the subspace of admissible inertial forces and torques spanned by the columns of MA in (15). If in motion, the generalized accelerations and contact forces associated with the linearly dependent columns become infinite and the sliding velocities at these contacts jump discontinuously to zero. Evaluated at μ_J, the matrices in (11), (15), (17) and (19) are singular.

Proof: The theorem is proven for jamming at the i^{th} contact. It can be generalized for simultaneous jamming at multiple contacts.

Based on our assumption of no wedging and $\mathrm{rank}(A) = n - m$, we know

$$\mathrm{rank}(\bar{\Phi} = \left[\Phi_q^T + \hat{\Phi}_q^T\right]) = m \qquad (20)$$

$$\mathrm{rank}\,[MA] = n - m \qquad (21)$$

Clearly, the columns of MA span the subspace of inertial forces and torques expressed in the generalized coordinate directions. Assume that the i^{th} column of $\bar{\Phi}$ is almost a linear combination of the columns of MA. That is,

$$\bar{\Phi}_i = [MA]\,\alpha + \epsilon v \qquad (22)$$

where $\alpha \in \Re^{n-m}$, $\epsilon \in \Re > 0$ and $v \in \Re^n$. The latter is chosen so that $\left[MA, \ \bar{\Phi}\right]$ has full rank when $\epsilon \neq 0$.

Using the properties of determinants,

$$\begin{aligned} &\det\left[MA, \ \bar{\Phi}\right] \\ &= \det\left[MA, \ \bar{\Phi}_1, \ldots, \bar{\Phi}_{i-1}, \epsilon v, \bar{\Phi}_{i+1}, \ldots, \bar{\Phi}_m\right] \\ &= \epsilon \det\left[MA, \ \bar{\Phi}_1, \ldots, \bar{\Phi}_{i-1}, v, \bar{\Phi}_{i+1}, \ldots, \bar{\Phi}_m\right] \\ &= \epsilon\Delta \end{aligned} \qquad (23)$$

with $\Delta \neq 0$.

Using Cramer's rule to solve for λ_i,

$$\lambda_i = \frac{\det\left[MA,\ \bar{\Phi}_1, \ldots, \bar{\Phi}_{i-1}, \tau - h - Mb, \bar{\Phi}_{i+1}, \ldots, \bar{\Phi}_m\right]}{\epsilon\Delta} \tag{24}$$

Note that $\tau - h - Mb$ is a function of the input forces and torques and so can be chosen arbitrarily. Assuming it is linearly independent of the other columns in the numerator,

$$\lim_{\epsilon \to 0} \lambda_i = \infty \tag{25}$$

If $\tau - h - Mb$ was not independent of the other columns,

$$\lim_{\epsilon \to 0} \lambda_i = \frac{0}{\lim_{\epsilon \to 0} \epsilon\Delta} = 0 \tag{26}$$

However, any input perturbation with a linearly independent component will cause $\lambda \to \infty$.

Now solving for the k^{th} acceleration,

$$\ddot{q}_k = \frac{\det\left[(MA)', \bar{\Phi}'\right]}{\epsilon\Delta} \tag{27}$$

where $(MA)'$ is MA with its k^{th} column replaced by $\tau - h - Mb$ and $\bar{\Phi}'$ is $\bar{\Phi}$ with its i^{th} column replaced by $\epsilon v + \alpha_k (MA)_k$, $(MA)_k$ being the k^{th} column of MA. When $\alpha_k = 0$, ϵ can be factored out of the numerator of (27) and $\lim_{\epsilon \to 0} \ddot{q}_k$ is finite. When $\alpha_k \neq 0$, $\lim_{\epsilon \to 0} \ddot{q}_k = \infty$.

In particular, we can determine that a degree of freedom is lost since the infinite accelerations are caused by friction forces which, according to Coulomb's model, are directed opposite the relative velocity. Analogous to the original holonomic velocity constraints described by (9), the jammed system possesses a new constraint at the i^{th} contact,

$$\hat{\Phi}_{qi}\dot{q} = -\hat{\Phi}_{ti}. \tag{28}$$

The rigid-body assumption indicates that this constraint is imposed instantaneously.

To complete the proof, we note from (1) that friction employs the minimum force necessary to resist sliding. Since we have shown that sliding velocity jumps to zero at the jamming contact for $\mu_{Ji} \leq \mu_i$, it is clear that jamming also occurs at μ_i.

Clearly, the matrix in (15) evaluated at μ_J is singular. By the definition of $T^{\perp}$ in (16), TMA in (17) also loses rank at μ_J. Similarly, the definition of $S^{\perp}$ indicates that $\left[S(\Phi_q^T + \hat{\Phi}_q^T)\right]$ in (19) loses rank at μ_J. Lastly, for (11), the existence of μ_J implies the existence of $\beta \in \Re^n$ such that

$$\begin{bmatrix} M \\ \Phi_q \end{bmatrix} \beta = \begin{bmatrix} \bar{\Phi}_i \\ 0 \end{bmatrix} \tag{29}$$

Selecting $\beta = A\alpha$ gives the desired result since $\Phi_q A = 0$. □

At the onset of jamming, despite finite input forces and torques, the relevant contact and friction forces, as well as accelerations, approach infinity producing loss of a degree of freedom under impact-like conditions.

Theorem 2: Wedging - Assuming no jamming, the rigid-body system described by (11), or (15), or (17) and (19) is *wedged* if there exists a vector μ_W with components μ_{Wi} in the intervals $[0, \mu_i]$ such that

$$\text{rank}\left[\Phi_q^T(q,t) + \hat{\Phi}_q^T(q, \dot{q}, \mu_W, t)\right] < m, \tag{30}$$

i.e., its columns are linearly dependent. In this case, sticking occurs for at least one of the contacts corresponding to the linearly dependent columns and the rigid-body system becomes underconstrained. Evaluated at μ_W, the matrices in (11), (15) and (19) are singular.

Proof: According to the rank condition given above, it is clear that the matrices in (11), (15) and (19) are singular. This condition appears to indicate that it is only possible to solve for a linear combination of those contact forces associated with the linearly dependent columns. Furthermore, the equations appear overconstrained. In particular, the forces generated during sliding contact cannot balance all possible right hand sides in (11), (15) and (19). However, since Φ_q^T has full column rank m and given the linear dependence of the columns of $\hat{\Phi}^T$ on their respective coefficients of friction, the rank of $(\Phi_q + \hat{\Phi}_q)^T$ will be restored to m if one or more of the components of μ_W is reduced. This corresponds to zero sliding velocity at the affected contacts.

For each static contact, we have a new variable given by its component of μ. Thus, by considering at least one of the linearly dependent contacts to be sticking, we have not only restored full rank to the matrices of (11), (15) and (19), we have at least one more unknown than we have equations.

Clearly, any contact k with $\mu_{Wk} < \mu_k$ will stick. In particular, consider the case when columns j and k of $(\Phi_q + \hat{\Phi}_q)^T$ corresponding to scleronomic (time independent) constraints are linearly dependent. For some $\alpha \in \Re$,

$$\alpha(\Phi_{qj} + \hat{\Phi}_{qj})^T = (\Phi_{qk} + \hat{\Phi}_{qk})^T \tag{31}$$

Transposing this equation, postmultiplying by $\dot{q}$ and applying (9) yields

$$\alpha\hat{\Phi}_{qj}\dot{q} = \hat{\Phi}_{qk}\dot{q} = 0 \tag{32}$$

This equation indicates that when two columns of $(\Phi_q + \hat{\Phi}_q)^T$ are linearly dependent, static contact at one constraint implies static contact at the other. Two additional equations are needed to solve uniquely for the $n+2$ variables $\ddot{q}_i$, λ, μ_j and μ_k. □

In a similar fashion to the previous proof, we can consider the singularity as the limit as the friction coefficients approach μ_W from below and show that the linearly dependent contact forces go to infinity while the accelerations remain finite. The latter is possible because the infinite contact forces effectively cancel each other. However, the rigid body assumption becomes implausible under these conditions. As in statically indeterminate problems, the additional equations needed to solve for the unknowns can be obtained by relaxing the rigid body assumption.

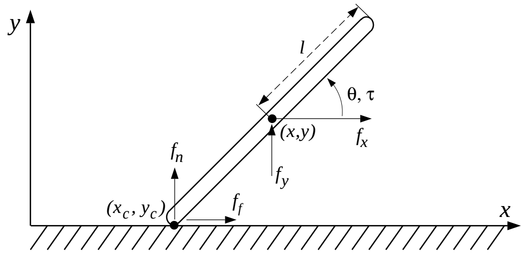

Figure 4: Planar rod of mass m and moment of inertia J in contact with immobile wall.

5 Examples

Example 1: Sliding Rod – Jamming

This problem has been used as an example of frictional ambiguities in many papers including [4,5,8]. As shown in Figure 4, the rigid rod possesses three degrees of freedom in planar motion which can be described by the x and y coordinates of its center of mass and by its angular orientation, θ. Thus, we take $q = [x, y, \theta]^T$. The rod has a mass, m, and a moment of inertia, J, about its mass center.

The constraint for the rod to maintain contact with the wall can be written as

$$\Phi(x, y, \theta) = y - l\sin\theta = 0 \tag{33}$$

This can be differentiated to obtain the velocity and acceleration constraint equations of (9) and (10) giving

$$\Phi_q = \begin{bmatrix} 0 & 1 & -l\cos\theta \end{bmatrix} \tag{34}$$

$$\Gamma = -l\dot{\theta}^2 \sin\theta \tag{35}$$

Noting that the Coulomb friction force on the rod is

$$f_f = [-\mu\ \mathrm{sgn}(\dot{x}_c)\lambda, 0, 0] \tag{36}$$

where $\dot{x}_c$ denotes the x component of rod velocity at the point of contact, we have

$$\hat{\Phi}_q = \begin{bmatrix} -\mu\ \mathrm{sgn}(\dot{x}_c) & 0 & -\mu\ \mathrm{sgn}(\dot{x}_c) l\sin\theta \end{bmatrix} \tag{37}$$

we can write (11) for the rod as

$$\begin{bmatrix} m & 0 & 0 & \mu\ \mathrm{sgn}(\dot{x}_c) \\ 0 & m & 0 & -1 \\ 0 & 0 & J & l(\cos\theta + \mu\ \mathrm{sgn}(\dot{x}_c)\sin\theta) \\ 0 & 1 & -l\cos\theta & 0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \\ \lambda \end{bmatrix} = \begin{bmatrix} f_x & f_y & \tau & -l\dot{\theta}^2\sin\theta \end{bmatrix}^T \tag{38}$$

To obtain the form of (15), we select $q_i = [x, y]^T$ and $q_d = \theta$. For $\theta \neq \pi/2$, this choice yields A and b of (14) to be

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1/l\cos\theta \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}^2 \tan\theta \end{bmatrix} \tag{39}$$

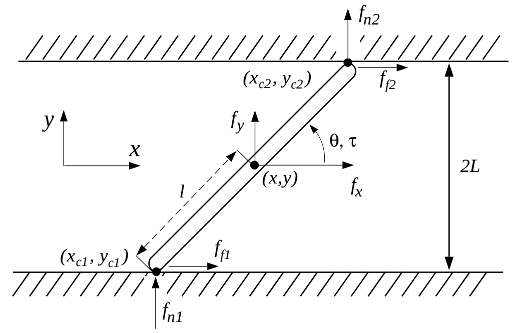

Figure 5: Planar rod of mass m and moment of inertia J in contact with two immobile walls.

and $\begin{bmatrix} MA, & -(\Phi_q^T + \hat{\Phi}_q^T) \end{bmatrix}$

$$= \begin{bmatrix} m & 0 & \mu\ \mathrm{sgn}(\dot{x}_c) \\ 0 & m & -1 \\ 0 & J/l\cos\theta & l(\cos\theta + \mu\ \mathrm{sgn}(\dot{x}_c)\sin\theta) \end{bmatrix} \tag{40}$$

To obtain (17), we compute an orthogonal complement of $\Phi_q^T + \hat{\Phi}_q^T$. One such matrix is

$$T = \begin{bmatrix} 1 & \mu\ \mathrm{sgn}(\dot{x}_c) & 0 \\ -l\sin\theta & l\cos\theta & 1 \end{bmatrix} \tag{41}$$

from which $[TMA]$ is found to be

$$[TMA] = \begin{bmatrix} m & \mu\ \mathrm{sgn}(\dot{x}_c)m \\ -ml\sin\theta & ml\cos\theta + J/l\cos\theta \end{bmatrix} \tag{42}$$

Since there is one friction contact, only jamming is possible. From Theorem 1, the minimum coefficient of friction necessary for jamming, μ_J corresponds to the value at which the matrices of (38), (40) and (42) lose rank. This value is

$$\mu_J = -\mathrm{sgn}(\dot{x}_c)\left(\frac{J + ml^2\cos^2\theta}{ml^2\sin\theta\cos\theta}\right) \tag{43}$$

which is the critical value obtained in [5,8]. From this equation, jamming is only possible for $\theta \in (0, \pi/2)$ if $\dot{x}_c < 0$. This is analogous to the situation in pole vaulting.

Example 2: Sliding Rod – Wedging

As an example of wedging, we consider the case of the sliding rod in contact with two walls depicted in Figure 5. This is analogous to geometries studied in [8,11].

There are now two constraints which are given by

$$\Phi(x, y, \theta) = \begin{bmatrix} y - l\sin\theta + L \\ y + l\sin\theta - L \end{bmatrix} = 0 \tag{44}$$

The velocity and acceleration constraint equations of (9) and (10) become

$$\Phi_q = \begin{bmatrix} 0 & 1 & -l\cos\theta \\ 0 & 1 & l\cos\theta \end{bmatrix} \tag{45}$$

$$\Gamma = \begin{bmatrix} -l\dot{\theta}^2\sin\theta \\ l\dot{\theta}^2\sin\theta \end{bmatrix} \tag{46}$$

The Coulomb friction force at contact i is

$$f_{fi} = [-\mu \ \mathrm{sgn}(\dot{x}_{ci})\lambda_i, 0, 0] \tag{47}$$

where $\dot{x}_{ci}$ denotes the x component of rod velocity at contact i and the signs of λ to maintain contact are taken to be $\lambda_1 > 0$ and $\lambda_2 < 0$. Thus,

$$\hat{\Phi}_q = \begin{bmatrix} -\mu \ \mathrm{sgn}(\dot{x}_{c1}) & 0 & -\mu \ \mathrm{sgn}(\dot{x}_{c1}) l \sin\theta \\ \mu \ \mathrm{sgn}(\dot{x}_{c2}) & 0 & -\mu \ \mathrm{sgn}(\dot{x}_{c2}) l \sin\theta \end{bmatrix} \tag{48}$$

The matrix of (11) can now be assembled from

$$\begin{bmatrix} M \\ \Phi_q \end{bmatrix} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \\ 0 & 1 & -l\cos\theta \\ 0 & 1 & l\cos\theta \end{bmatrix} \tag{49}$$

and $\begin{bmatrix} (\Phi_q + \hat{\Phi}_q), & 0 \end{bmatrix}^T =$

$$\begin{bmatrix} \mu \ \mathrm{sgn}(\dot{x}_{c1}) & -\mu \ \mathrm{sgn}(\dot{x}_{c2}) \\ -1 & -1 \\ l(\cos\theta + \mu \ \mathrm{sgn}(\dot{x}_{c1})\sin\theta) & -l(\cos\theta - \mu \ \mathrm{sgn}(\dot{x}_{c2})\sin\theta) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{50}$$

Selecting $q_i = x$ and $q_d = [y, \theta]^T$ yields the following for A and b of (14).

$$A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \dot{\theta}^2 \tan\theta \end{bmatrix} \tag{51}$$

The matrix of (15) is $\begin{bmatrix} MA, & -(\Phi_q^T + \hat{\Phi}_q^T) \end{bmatrix} =$

$$\begin{bmatrix} m & \mu \ \mathrm{sgn}(\dot{x}_{c1}) & -\mu \ \mathrm{sgn}(\dot{x}_{c2}) \\ 0 & -1 & -1 \\ 0 & l(\cos\theta + \mu \ \mathrm{sgn}(\dot{x}_{c1})\sin\theta) & -l(\cos\theta - \mu \ \mathrm{sgn}(\dot{x}_{c2})\sin\theta) \end{bmatrix} \tag{52}$$

To investigate wedging in the third formulation, we must obtain the form of (19). An orthogonal complement of MA is

$$S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{53}$$

from which $-S(\Phi_q^T + \hat{\Phi}_q^T)$ is found to be

$$\begin{bmatrix} -1 & -1 \\ l(\cos\theta + \mu \ \mathrm{sgn}(\dot{x}_{c1})\sin\theta) & -l(\cos\theta - \mu \ \mathrm{sgn}(\dot{x}_{c2})\sin\theta) \end{bmatrix} \tag{54}$$

From inspection of (49) and (50) or (52), it is easy to verify that jamming is not possible. Examination of (50) or (52) or (54) provides two conditions for wedging.

$$\dot{x}_{c1} < 0, \ \dot{x}_{c2} > 0 \tag{55}$$

$$\mu_W = [1/\tan\theta, \ 1/\tan\theta] \tag{56}$$

The condition on the contact velocities is equivalent to $\dot{\theta} < 0$ which violates the contact constraints of (45). However, the friction forces were written in terms of velocity direction only for convenience and it is recognized that the same forces can be produced by static contact which does not violate the constraints. At μ_W, the contact forces are collinear.

6 Conclusion

Given the Coulomb model, singularities in the dynamic equations associated with jamming and wedging are due to a breakdown of the rigid-body assumption. At the onset of jamming, the assumption of adequate separation between the time scales governing the contact force dynamics and those governing the "rigid-body" dynamics is violated. In wedging, the rigid-body reaction forces are indeterminate and the system is capable of storing elastic energy. By adding compliance to the model, the reaction forces can be made functions of system state and the dynamics of jamming and wedging become well-posed problems.

7 References

1. L. Cai and A. Goldenberg, "A New Approach to Position and Force Control of Robot Manipulators." *Proc. IEEE Int. Conf. Control and Applications*, Israel, April 1989, RA-3-5, pp. 1-7.
2. P. Dupont, "The Effect of Coulomb Friction on the Existence and Uniqueness of the Forward Dynamics Problem." *Proc. 1992 IEEE Int. Conf. Robotics and Automation*, Nice, France, May 1992, pp. 1442-1447.
3. P. Dupont, "Existence and Uniqueness of Rigid-Body Dynamics with Coulomb Friction." *Trans. Canadian Society for Mech. Eng.*, Vol. 17, No. 4A, 1993, pp. 513-525.
4. P. Lötstedt, "Coulomb Friction in Two-dimensional Rigid Body Systems." *ZAMM*, 61, pp. 605-615.
5. M.T. Mason and Y. Wang, "On the Inconsistency of Rigid-body Frictional Planar Mechanics." *Proc. 1988 IEEE Int. Conf. Robotics and Automation*, Philadelphia, PA, April, 1988.
6. P. Painlevé, "Sur les Lois du Frottement de Glissement." *Comptes Rendus de l'Académie des Sciences*, 121, pp. 112-115, 1895.
7. F. Pfeiffer and C. Glocker, "Dynamical Systems with Unsteady Processes." *Friction-induced Vibration, Chatter, Squeal and Chaos*, ASME WAM, Anaheim, CA, November 1992, DE-Vol. 49, pp. 65-74.
8. V.T. Rajan, R. Burridge and J.T. Schwartz, "Dynamics of a Rigid Body in Frictional Contact with Rigid Walls: Motion in Two Dimensions." *Proc. 1987 IEEE Int. Conf. Robotics and Automation*, Raleigh, North Carolina, 1987.
9. K. Tahboub and P. Müller, "A General Reduced Dynamic Model for Control and Simulation of Constrained Robots." *Proc. 1992 IEEE Int. Conf. Intelligent Robots and Systems*, Raleigh, N.C., July 1992.
10. Y.-T. Wang, V. Kumar and J. Abel, "Dynamics of Rigid Bodies Undergoing Multiple Frictional Contacts." *Proc. 1992 IEEE Int. Conf. Robotics and Automation*, Nice, France, May, 1992, pp. 2764-69.
11. D. E. Whitney, "Quasi-Static Assembly of Compliantly Supported Rigid Parts." *ASME J. Dyn. Sys., Meas. and Control*, Vol. 104, pp. 65-77, March 1982.