The Math Toolbox

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Introduction

The physical sciences are deeply rooted in a mathematical foundation. The exact reasons why math works so well for doing science are not as obvious as we might like. In a famous essay on "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" philosopher Eugene Wigner wrote

> *The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.*

Regardless of the reasons for the correspondence between math and science, there are a handful of basic math tools that are useful in any science course. Much of the following math may be familiar from high school courses, some of it may not be as familiar. Either way, the following pages will hopefully be enough to get you started on your adventure through the physical universe.

How Scientists Use Numbers

Most people use numbers on a daily basis (even if they don't realize they are doing it). Whether it's paying for gas and groceries or reading the weather forecast before heading to work, numbers permeate our everyday life. Below, we take a look at the tricks scientists have developed for carefully handling these numbers.

Significant Figures

"Significant figures" is the term used to describe how many digits we can confidently write down for a measurement or calculated quantity. For example, the latest census reported that the population of the United States is 308,745,538 people, but does it really make sense that we can say the population is 308,745,538 and *not* 308,745,539, or even 308,745,401. Plenty of people fail to respond to the census queries, or they just flat out lie about who is living in their homes. Even if everyone responded perfectly, by the time the census bureau finally collected all the data, and added up the total population, the correct number would have changed as some people died and others were born. Thus, it is only accurate to say something like, "There are about 308 million people in the United States." The digits following the 308 million are uncertain. That is, we don't know whether or not there are 308,600,000 or 308,800,000 people living in the United States.

The fact that we can only know measured numbers with a limited amount of certainty has an impact on the way we record numbers in science. For now, you can just focus on not using too many significant figures. There are precise rules for how to use significant figures in your calculations, but here is my recommendation for a crude handling of significant figures in your homework.

Keep as many significant figures as you can during your calculations, but round your final answers to two or three significant figures. Answers with too many significant figures can be penalized.

Powers of Ten and Scientific Notation

Astronomers tend to use both *very large* numbers (sometimes much larger than a billion or trillion) and *very small* numbers (much smaller than one billionth or trillionth). These numbers are difficult to write out in full, so they are often abbreviated using powers of ten. For example, imagine you had 1,000 bags of candies, and each bag had exactly 100 candies in them. How many candies would you have? You can probably quickly figure out that you would have 100,000 candies. You also probably simply performed the multiplication $1{,}000 \times 100$ in your mind or plugged it into your calculator. However, this problem can be solved as a simple addition problem. If we express these numbers as powers of ten, then we instead have

$$1{,}000 \times 100 = 10^3 \times 10^2 = 10^{3+2} = 10^5.$$

This leads us to the first rule of dealing with powers of ten.

Rule 1: *When multiplying powers of ten, exponents can be added. When dividing powers of ten, exponents can be subtracted.*

$$10^a \times 10^b = 10^{(a+b)} \qquad \text{and} \qquad \frac{10^a}{10^b} = 10^{(a-b)}$$

Rule 1 can be used to convert a multiplication problem into addition and a division problem into a simple subtraction. For example, there are about 100 billion stars in a galaxy, and there are about 1,000 galaxies in the Virgo cluster. How many stars are in the Virgo cluster? Of course, we can simply multiply 100 billion stars per galaxy by 1,000 galaxies, but those are awfully big numbers to deal with, so let's use ***Rule 1*** to turn this into a simple addition problem. Using $100 \text{ billion} = 10^{11}$ and $1{,}000 = 10^3$, we have

$$10^{11} \text{ stars per galaxy} \times 10^3 \text{ galaxies} = 10^{11+3} \text{ stars} = 10^{14} \text{ stars}$$

Thus, there are 10^{14} stars in the Virgo cluster. Notice how this problem really only amounted to the addition of $11 + 3 = 14$. This is one of the wonderful things of multiplying and dividing powers of ten: you only need to add and subtract a few small numbers. Now that we have mastered powers of ten, we are prepared to deal with scientific notation.

Scientific notation uses multiplication to stitch together two important pieces of information: a decimal representation of the *significant figures* of a number and a power of ten indicating the *magnitude* of the number. For example, the radius of the Sun can be written as

$$696{,}000{,}000 \text{ m} = 6.96 \times 10^8 \text{ m}$$

This part indicates the first few digits of the number

This part indicates the magnitude of the number

The number on the left is written in longhand format, and the number on the right is written in scientific notation, but the two are completely equivalent. True scientific notation requires the decimal to be placed after the first digit, so you need to think carefully about which power of ten will allow you to move the decimal to that position.

Scientific notation has the added advantage of making multiplication and division of these large numbers much easier. For example, let's calculate the surface area of the Sun. The Sun is very nearly spherical, so we will need to use the formula for the surface area of a sphere, which is $S = 4\pi r^2$. We will also use two math tricks and ***Rule 1*** from above to evaluate this formula.

- Break the square term (r^2) into longhand form: $r^2 = r \times r$
- Regroup terms using the commutative property of multiplication: $a \times b = b \times a$

Using these tricks and substituting the value listed above for the radius of the Sun, we are ready to calculate a surface area.

$$S = 4\pi \times (6.96 \times 10^8 \text{m})^2$$

$$= 4\pi \times (6.96 \times 10^8 \text{m}) \times (6.96 \times 10^8 \text{m})$$ Break the square into longhand form

$$= 4\pi \times (6.96 \times 6.96) \times (10^8 \times 10^8)(\text{m} \times \text{m})$$ Regroup numbers using commutative property

$$= 4\pi \times (48.373) \times (10^{16})\text{m}^2$$ Evaluate each squared term

$$= 608. \times 10^{16}\text{m}^2 = 6.08 \times 10^{18}\text{m}^2$$ Multiply remaining numbers together

Thus, the surface area of the Sun is $6.08 \times 10^{18}\text{m}^2$.

Notice how we were able to multiply each part separately. The (6.96×6.96) part could be computed independently from the $(10^8 \times 10^8)$ part. This is a convenient feature of scientific notation, which leads us to another rule.

Rule 2: *When multiplying or dividing numbers expressed in scientific notation, the leading digits can be multiplied or divided separately from the powers of ten.*

$$(a \times 10^b) \times (c \times 10^d) = (a \times c) \times (10^b \times 10^d) = (a \times c) \times 10^{(b+d)}$$

and

$$\frac{(a \times 10^b)}{(c \times 10^d)} = \left(\frac{a}{c}\right) \times \left(\frac{10^b}{10^d}\right) = \left(\frac{a}{c}\right) \times 10^{(b-d)}$$

Units

What are Units?

Many math classes deal with numbers on a purely abstract level. For example, consider the problem: "$2x + 3 = 8$, find x." In this problem, does the number x have any physical meaning? Is it the number of fish in a pond? Is it the distance traveled by a speeding bullet? In general, without further information, x must remain a meaningless, abstract number.

In the physical sciences, we use numbers to represent quantities that carry physical meaning. For example, when I say, "The car is traveling at a speed of 30 m/s." This number has units attached to it, and those units are incredibly important. They tell us that the number represents some physical distance traveled within a certain amount of time. We could have used a different set of units to represent the exact same physical quantity. For example, if we used centimeters to measure distance, we would say "The car is traveling at a speed of $3{,}000 \text{ cm/s}$," or we could even say "The car is traveling at a speed of 67 miles per hour." Each of these sentences say the same thing just using different units of measurement.

Units Are Essential

Imagine you had gone out to an orange grove, plucked 36 oranges, and set them in your basket. If someone asked you, “How many oranges do you have?” you could correctly respond, “I have 36 oranges,” or “I have three dozen oranges.” In this case, the word dozen is functioning as a unit. It is some specified amount of a physical quantity. However, it would be incorrect to answer, “How many oranges do you have?” by simply stating, “I have three.” Without specifying what units you are using, your answer cannot be correct!

In the example above with the traveling car, it would be incorrect to state, “The car is traveling at a speed of 67.” Units are a vital piece of information. Without them, others must *assume* a set of units to attach to the number, and it is likely that they will assume the wrong units because there are many possible choices, but only one choice is correct. For example, if they incorrectly assumed you were using standard metric units, they might think you meant “The car is traveling at a speed of $67\ \mathrm{m/s}$, which is 150 miles per hour!” Thus, you must always specify any units associated with your answers. Without units, it is impossible to know whether or not you have actually gotten the correct answer (or whether or not you deserve a speeding ticket).

Units Are Helpful

In order for an equation to be physically true, the units attached to the numbers in the equation must make logical sense. Let’s consider a simple equation of motion.

$$\text{velocity} = \frac{\text{distance}}{\text{time}} \qquad \text{or} \qquad v = \frac{d}{t}$$

From this equation, we can directly see that the units of velocity *must* be some distance traveled per unit time. For example, if a ball rolls a distance of 10 meters in a timespan of 5 sec, then substituting these values into the equation above gives an average velocity of

$$\text{velocity} = \frac{10\ \text{meters}}{5\ \text{sec}} = 2\ \frac{\mathrm{m}}{\mathrm{s}}$$

This equation is how we *define* velocity, so if you are working through a problem, and you get a velocity with units of m^2/s, you know you have done something wrong. Go back and look at your units to see where things went awry. In this sense, *units are like a “spellcheck” on your math.*

Let’s consider a more complicated equation of motion.

$$\Delta x = v_i \cdot \Delta t + \tfrac{1}{2} a \cdot (\Delta t)^2$$

Put into plain English, this equation says, “For an object moving with an initial velocity (v_i) and experiencing a constant acceleration (a), for a duration of time (Δt), the change in position (Δx) is equal to the initial velocity multiplied by the amount of time elapsed plus one-half the acceleration multiplied by the square of the amount of time elapsed.” Let’s use this equation to solve a problem and see how units can help us get to the correct answer. First we will do it the wrong way just to see how things turn out when we don’t pay close attention to units.

Free Fall (pt. 1): A ball is thrown into a bottomless pit with an initial downward velocity of $50\ \mathrm{cm/s}$. Gravity continues to pull on the ball and accelerates it downward at a rate of about $10\ \mathrm{m/s^2}$. How far will the ball have fallen after half a minute?

A Common Error: This is a fairly basic problem where we simply need to substitute the provided numbers into the equation of motion. Let’s just go ahead and plug in the numbers we have been given and see what comes out the other side of the equation.

$$v_i = 50\ \text{cm/s} \qquad \Delta t = 0.5\,\text{min} \qquad a = 10\ \text{m/s}^2$$

$$\Delta x = 50\ \frac{\text{cm}}{\text{s}} \cdot 0.5\ \text{min} + \frac{1}{2} \cdot \left(10 \frac{\text{m}}{\text{s}^2}\right)(0.5\ \text{min})^2 = 26.25\ \text{m? or } 26.25\text{cm?}$$

This answer is incorrect because we have not properly handled units. If you get to the end of a problem and are not sure what the units are, then there is a good chance you have made a mistake. *Incorrect use of units is one of the most common mistakes in science courses*. In this example, we have attempted to simply plug in the numbers we were given without any thought about if those numbers were expressed using the correct units. An equation will only give you a correct answer if you have first performed all the unit conversions you need. We will discuss that process next.

How to Convert to Different Units

It is possible to convert from one set of units to another by multiplying by special numbers called "conversion factors." A conversion factor is simply a fraction where the numerator (top number) is a quantity measured in one set of units and the denominator (bottom number) is the equivalent quantity measured in a different set of units. Here are some example conversion factors.

$$\frac{100\ \text{cm}}{1\ \text{m}} \ \text{or}\ \frac{1\ \text{m}}{100\ \text{cm}} \qquad \frac{1.61\ \text{km}}{1\ \text{mi.}} \ \text{or}\ \frac{1\ \text{mi.}}{1.61\ \text{km}}$$

$$\frac{60\ \text{s}}{1\ \text{min}} \ \text{or}\ \frac{1\ \text{min}}{60\ \text{s}} \qquad \frac{3.16 \times 10^7\ \text{s}}{1\ \text{yr}} \ \text{or}\ \frac{1\ \text{yr}}{3.16 \times 10^7\ \text{s}}$$

In all of these examples, the quantity in the numerator is equal to the quantity in the denominator: $100\ \text{cm} = 1\ \text{m}$, $1.61\ \text{km} = 1\ \text{mi.}$, and $3.16 \times 10^7\text{s} = 1\ \text{yr}$. By carefully multiplying the necessary set of conversion factors, we can convert a given quantity from one set of units into whatever other units would be most convenient. The key to choosing the correct set of conversion factors is finding the ones which form an interlocking chain from the units you currently have to the units you want to have.

Rule 3.a: *Units in a numerator cancel the same units in the denominator of another term, and units in a denominator cancel the same units in the numerator of another term.*

$$\frac{Unit\ A}{\cancel{Unit\ B}} \times \frac{\cancel{Unit\ B}}{Unit\ C} = \frac{Unit\ A}{Unit\ B} \qquad \text{and} \qquad \frac{\cancel{Unit\ A}}{Unit\ B} \times \frac{Unit\ C}{\cancel{Unit\ A}} = \frac{Unit\ A}{Unit\ B}$$

Let's take a look at some examples to illustrate the correct use of these conversion factors.

Length: Convert 3.11 mi into km.

$$3.11\ \cancel{\text{mi}} \times \frac{1.61\ \text{km}}{1\ \cancel{\text{mi}}} = 5.00\ \text{km}$$

Miles in the top…

…cancel miles in the bottom.

Time: Convert 12,960 seconds into days.

$$12{,}960\ \cancel{\text{s}} \times \frac{1\ \cancel{\text{min}}}{60\ \cancel{\text{s}}} \times \frac{1\ \cancel{\text{hr}}}{60\ \cancel{\text{min}}} \times \frac{1\ \text{day}}{24\ \cancel{\text{hr}}} = 0.1500\ \text{day}$$

Speed: What is $0.01\ \text{cm/s}$ expressed in km/yr?

$$\frac{0.0100\ \cancel{\text{cm}}}{1\ \cancel{\text{s}}} \times \frac{1\ \cancel{\text{m}}}{100\ \cancel{\text{cm}}} \times \frac{1\ \text{km}}{1{,}000\ \cancel{\text{m}}} \times \frac{3.16 \times 10^7\ \cancel{\text{s}}}{1\ \text{yr}} = \frac{3.16\ \text{km}}{1\ \text{yr}}$$

In this last example, note how the units in the numerators cancel with units in the denominator. The only units that survive are the units we wanted: km in the numerator and yr in the denominator.

Acceleration: What is $360\ \text{m/min}^2$ as measured in cm/s^2?

Acceleration is a more complicated quantity because of the time^2 in the denominator. First, let's write out each factor of "minutes" separately, so $\frac{360\ \text{m}}{(1\ \text{min})^2} = \frac{360\ \text{m}}{1\ \text{min} \times 1\ \text{min}}$. Now we can apply whatever conversion factors we need.

$$\frac{360\ \cancel{\text{m}}}{1\ \cancel{\text{min}} \times 1\ \cancel{\text{min}}} \times \frac{100\ \text{cm}}{1\ \cancel{\text{m}}} \times \frac{1\ \cancel{\text{min}}}{60\ \text{s}} \times \frac{1\ \cancel{\text{min}}}{60\ \text{s}} = \frac{10\ \text{cm}}{\text{s}^2}$$

Notice how we needed two multiples of the $1\ \text{min}/60\ \text{s}$ conversion factor. This was because we needed to cancel *both* of the minute units in the denominator of the original quantity. We could have just as easily done this by simply squaring the entire conversion factor.

$$\frac{360\ \cancel{\text{m}}}{(1\ \cancel{\text{min}})^2} \times \frac{100\ \text{cm}}{1\ \cancel{\text{m}}} \times \left(\frac{1\ \cancel{\text{min}}}{60\ \text{s}}\right)^2 = \frac{10\ \text{cm}}{\text{s}^2}$$

Rule 3.b: *If a set of units are raised to some power (e.g., m^3 or s^2), then the entire conversion factor applied to that unit must be raised to the same power.*

$$\frac{Unit\ A}{(\cancel{Unit\ B})^d} \times \left(\frac{\cancel{Unit\ B}}{Unit\ C}\right)^d = \frac{Unit\ A}{Unit\ C} \quad \text{and} \quad \frac{(\cancel{Unit\ A})^d}{Unit\ B} \times \left(\frac{Unit\ C}{\cancel{Unit\ A}}\right)^d = \frac{Unit\ C}{Unit\ B}$$

Let's do one final example using ***Rule 3.b***.

Density of Granite: You found a slab of granite on your weekend hike, and you want to know the density of the rock. After hauling it back to your house, you measure the mass to be $344\ \mathrm{kg}$ and its volume to be $0.125\ \mathrm{m}^3$. What is the density of the slab of granite measured in $\mathrm{g/cm}^3$?

Solution: To solve this problem, we will need to convert kilograms to grams and m^3 to cm^3. We can do this in one quick step. I will put the conversion factors relating to mass on the *left* of the original value and the conversion factors relating to volume on the *right* of the original value.

This part converts kg into g

This is the original density

This part converts m^3 into cm^3

$$\frac{1000\ \mathrm{g}}{1\ \mathrm{kg}} \times \frac{344\ \mathrm{kg}}{0.125\ \mathrm{m}^3} \times \left(\frac{1\ \mathrm{m}}{100\ \mathrm{cm}}\right)^3 = 2.75\frac{\mathrm{g}}{\mathrm{cm}^3}$$

Final answer

Now we have enough information and enough practice to correctly solve the "Free Fall" problem from above.

Free Fall (pt. 2): A ball is thrown into a bottomless pit with an initial downward velocity of $50\ \mathrm{cm/s}$. Gravity continues to pull on the ball and accelerates it downward at a rate of about $10\ \mathrm{m/s}^2$. How far will the ball have fallen after half a minute?

Correct Solution: First, we need to decide which units we want to use. Let's use seconds to keep track of time and meters to keep track of distances.

$$v_i = 50\ \frac{\mathrm{cm}}{s} \times \frac{1\ \mathrm{m}}{100\ \mathrm{cm}} = 0.5\frac{\mathrm{m}}{s}$$

Convert velocity into $\frac{\mathrm{m}}{\mathrm{s}}$

$$\Delta t = 0.5\ \mathrm{min} \times \frac{60\ \mathrm{s}}{1\ \mathrm{min}} = 30\ \mathrm{s}$$

Convert time into s

$$a = 10\ \frac{\mathrm{m}}{\mathrm{s}^2}$$

Leave acceleration in $\frac{\mathrm{m}}{\mathrm{s}^2}$

Now, substituting these units into the equation $\Delta x = v_i \cdot \Delta t + \frac{1}{2}a \cdot (\Delta t)^2$, gives us

$$\Delta x = 0.5\ \frac{\mathrm{m}}{\cancel{s}} \cdot 30\cancel{s} + \frac{1}{2} \cdot \left(10\frac{\mathrm{m}}{\cancel{s^2}}\right)(30\cancel{s})^2 = 465\ \mathrm{m}$$

Notice how everything except the meters units canceled. This should make sense. If you were asked how far something has traveled, then you should only have units of length at the end of your problem. For example, it would never be correct to say "the ball fell $65\ \mathrm{m/s}$" or "the ball fell $20\ \mathrm{m}^2$" because those are not a measures of distance: the former is a speed and the latter is an area.

Adding and Subtracting with Units

Units provide another useful check on your math. When adding or subtracting two numbers, the units on those numbers must be the same. To illustrate this, imagine you built a replica of the Empire State Building using LEGO® bricks, and you measured its height to be $1.2\ \mathrm{m}$. Now, your pet monkey, K. K., walks over and takes off the top $10\ \mathrm{cm}$ of the tower. What is the height of the building now? You cannot simply subtract the numbers: $1.2\ \mathrm{m} - 10\ \mathrm{cm} = -8.8\ \mathrm{m}?\ \text{or}\ \mathrm{cm}?$ This is a negative number, which doesn't make sense for the height of the building. First, you must convert $10\ \mathrm{cm}$ into $0.1\ \mathrm{m}$. Now, subtraction gives a new height of $1.2\ \mathrm{m} - 0.1\ \mathrm{m} = 1.1\ \mathrm{m}$.

Rule 4: *Only quantities measured in the* ***same units*** *can be added to each other or subtracted from each other.*

Physical Constants with Units

Many scientific numbers come with special units already attached. When those numbers appear in an equation, you must convert all other quantities to match the units of that special number. One of my favorite examples of this is Newton's law of gravitation. Using this law, we can calculate the downward acceleration of falling objects on Earth:

$$a_g = \frac{G \cdot M_E}{R_E^2}.$$

The numbers in this equation are as follows.

$$G = 6.67 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg\ s}^2} \qquad \text{the universal gravitational constant}$$

$$M_E = 5.97 \times 10^{27}\ \mathrm{g} \qquad \text{the mass of Earth}$$

$$R_E = 6{,}378\ \mathrm{km} \qquad \text{the radius of Earth}$$

It might be tempting to simply put these numbers into the equation, but that would be incorrect! First, we need to convert the mass and radius of Earth into the units contained in G (i.e. kg and m, respectively). Using two common conversion factors, we have

$$M_E = 5.97 \times 10^{27}\ \mathrm{g} \times \frac{1\ \mathrm{kg}}{1{,}000\ \mathrm{g}} = 5.97 \times 10^{24}\ \mathrm{kg}, \quad \text{and}$$

$$R_E = 6{,}378\ \mathrm{km} \times \frac{1000\ \mathrm{m}}{1\ \mathrm{km}} = 6.378 \times 10^{6}\ \mathrm{m}.$$

Now that we have the numbers in the correct units, we can go ahead and plug everything in.

$$a_g = \frac{G \cdot M_E}{R_E^2} = \frac{6.67 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg}\ s^2} \cdot 5.97 \times 10^{24}\ \mathrm{kg}}{(6.378 \times 10^6\ \mathrm{m})^2} = 9.79\ \frac{\mathrm{m}}{\mathrm{s}^2}$$

Unit Notation

The units of a quantity can be written using negative exponents for those units which appear in the denominator. Here are some examples of how this works.

$$\text{gravitational acceleration at Earth} = 9.79\ \frac{\mathrm{m}}{\mathrm{s}^2} = 9.79\ \mathrm{m\ s}^{-2}$$

$$\text{density of granite} = 2.75\ \frac{\mathrm{g}}{\mathrm{cm}^3} = 2.75\ \mathrm{g\ cm}^{-3}$$

$$\text{universal gravitational constant} = 6.67 \times 10^{-11}\ \frac{\mathrm{m}}{\mathrm{kg\ s}^2} = 6.67 \times 10^{-11}\ \mathrm{m\ kg}^{-1}\ \mathrm{s}^{-2}$$

Solving Scientific Problems

Now that we have discussed how scientists use numbers, let's take a look at how scientists solve the problems they face on a daily basis. In particular, we are going to review how scientists use symbols (e.g., Roman letters and Greek letters) to manage complex problems before ever touching a calculator. We will also review some other common tools such as trigonometry and angles, and proportionality and ratios.

Basic Algebra

The Oxford English Dictionary defines algebra as "the branch of mathematics in which letters are used to represent numbers in formulae and equations." All the physical sciences use a fair amount of algebra, so here are some reminders of the basic rules.

Rule 5: *We may add or subtract the* ***same*** *number or symbol to* ***both*** *sides of an equation.*

Let's just look at one example of how to use this rule to get the answers we want.

Lemonade Litigation: You own a lemonade stand and have sold 30 cups of lemonade for $\$1.50/\text{cup}$ and paid your two workers \$10 each. At the end of the day you find \$15 in your cash box. Has someone stolen from you?

Solution:

$$\text{Revenue} - \text{Expenses} - \text{Stolen} = \text{Profit}$$

Begin with profit equation

$$30\text{ cups} \times \frac{\$1.50}{\text{cup}} - 2\text{ workers} \times \frac{\$10}{\text{worker}} - \text{Stolen} = \$15$$

Calculate the revenue and expenses

$$\text{Stolen} + (\$45 - \$20 - \text{Stolen}) = \$15 + \text{Stolen}$$

Add "stolen" quantity to both sides

$$\$25 - \$15 = (\$15 + \text{Stolen}) - \$15$$

Subtract \$15 from both sides

$$\$10 = \text{Stolen}$$

It would seem that someone has taken \$10 from you!

Rule 6.a: *We may multiply or divide* ***both*** *sides of an equation by the* ***same*** *number or symbol.*

Here is an example illustrating ***Rule 6.a***.

Sailing in Space: You are the captian of a rocket ship with mass $m_{\text{ship}} = 2{,}000$ kg. Your ship is floating in free space, far away from any gravitating planets or stars. To start your trip home, you fire the ship's engines, which put out $15{,}000$ N of thrust. What is the ship's rate of acceleration?

Solution: We will use Newton's second law, $F_{\text{thrust}} = m_{ship} \cdot a$, to solve for the acceleration. We also need to recognize that the thrust, $15{,}000\text{ N} = 15{,}000\text{ kg m s}^{-2}$, is equal to the force (F_{thrust}) in this equation.

$F_{\text{thrust}} = m_{\text{ship}} \cdot a$	Begin with Newton's second law
$\dfrac{F_{\text{thrust}}}{m_{\text{ship}}} = \dfrac{m_{\text{ship}} \cdot a}{m_{\text{ship}}}$	Divide both sides by m_{ship}
$\dfrac{F_{\text{thrust}}}{m_{\text{ship}}} = a$	You have found the acceleration
$\dfrac{15{,}000 \text{ kg m s}^{-2}}{2{,}000 \text{ kg}} = 7.5 \dfrac{\text{m}}{\text{s}^2} = a$	Plug in the given numbers

Rule 6.b: *Division by a number is the same as multiplying by the reciprocal of that number.*

The reciprocal of a number (e.g., x) is simply one divided by that number (e.g., $1/x$). For example, the reciprocal of 2 is $1/2$, the reciprocal of 5 is $1/5$, and the reciprocal of $2/73$ is $\frac{1}{(2/73)} = \frac{73}{2}$. Reciprocals can also be used to flip entire fractions upside down. This turns out to be a very useful trick, so let's take a closer look how this works.

Reciprocal Recipe: You have been teleported to a foreign planet. Using some geometry you learned in an astronomy class, you determine the radius of the planet to be $R_{\text{pl}} = 8{,}250$ km. You also measure the gravitational acceleration on the surface of the planet to be $a_g = 6.5 \text{ m s}^{-2}$. What is the mass of the planet (M_{pl})to which you have been teleported?

Solution: To solve this problem, we will reuse the equation for gravitational acceleration from page 9. First, let's solve that equation to find the mass of the planet.

$a_g = \dfrac{G}{R_{\text{pl}}^2} \times M_{\text{pl}}$	To isolate what we want (M_{pl}), we need to divide by $\frac{G}{R_{\text{pl}}^2}$
$\dfrac{R_{\text{pl}}^2}{G} \times a_g = \left(\dfrac{G}{R_{\text{pl}}^2} \times M_{\text{pl}}\right) \times \dfrac{R_{\text{pl}}^2}{G}$	So we multiply by the reciprocal: $\frac{R_{\text{pl}}^2}{G}$
$\dfrac{R_{\text{pl}}^2 \cdot a_g}{G} = M_{\text{pl}}$	This cancels all the unwanted constants and isolates M_{pl}

Notice how we were able to simultaneously move both the G and the R_{pl}^2 across the equal sign. By recognizing which pieces of the equation need to be moved, we can use reciprocals to move them all together. Finally, converting the radius of the planet into meters, $R_{\text{pl}} = 8.25 \times 10^6$ m, we have

$$M_{\text{pl}} = \frac{R_{\text{pl}}^2 \cdot a_g}{G} = \frac{(8.25 \times 10^6 \text{ m})^2 \cdot 6.5 \text{ m s}^{-2}}{6.67 \times 10^{-11} \dfrac{\text{m}^3}{\text{kg } s^2}}$$

$$M_{\text{pl}} = 6.6 \times 10^{24} \text{ kg.}$$

Now that we have a full set of algebraic tools, let's look at one more example illustrating how to isolate the desired quantity even when it is surrounded by a bunch of other numbers and symbols. Newton's version of Kepler's 3$^{\text{rd}}$ law relates the period of an orbit (P), the average distance between the two orbiting bodies (a), and the masses of the orbiting bodies (M_1 and M_2).

$$P^2 = \frac{4\pi^2 a^3}{G(M_1 + M_2)}$$

Now, let's solve this equation for the mass of the secondary body (M_2).

$$\frac{1}{P^2} \times P^2 = \frac{4\pi^2 a^3}{G(M_1 + M_2)} \times \frac{1}{P^2} \qquad \text{Multiply by } \tfrac{1}{P^2}.$$

$$(M_1 + M_2) \times 1 = \frac{4\pi^2 a^3}{G(M_1 + M_2)P^2} \times (M_1 + M_2) \qquad \text{Multiply by } (M_1 + M_2)$$

$$(M_1 + M_2) - M_1 = \frac{4\pi^2 a^3}{G \cdot P^2} - M_1 \qquad \text{Subtract } M_1$$

$$M_2 = \frac{4\pi^2 a^3}{G \cdot P^2} - M_1$$

Exponentiation

We often need to raise an entire number (including units and powers of ten) to some power. Some common mistakes to avoid when doing this are illustrated below.

Numbers Combined by Multiplication or Division

When a product of two (or more) numbers is raised to some power, each member of the product is individually raised to the same power.

$$(a \times b)^c = a^c \times b^c$$

The multiplication rule is also useful when handling numbers scientific notation.

$$(2 \times 10^5)^3 = 2^3 \times (10^5)^3 = 2^3 \times 10^{5 \cdot 3} = 8 \times 10^{15}$$

Similarly, when the quotient of two numbers is raised to some power, each member of the quotient is individually raised to the same power.

$$\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$$

Numbers with Units

The units attached to a number can be treated as though they are attached by multiplication.

$$\left(50\ \frac{\mathrm{m}}{s}\right)^2 = \left(50 \times \frac{\mathrm{m}}{s}\right)^2 = 50^2 \times \left(\frac{\mathrm{m}}{s}\right)^2 = 2{,}500\,\frac{\mathrm{m}^2}{\mathrm{s}^2}$$

Numbers Combined by Addition or Subtraction

These numbers are often incorrectly treated as though they were multiplied.

$$(a + b)^c \neq a^c + b^c$$

By simply writing the exponent out as a multiplication expression, it becomes clearer that we actually need to use a process such as "F.O.I.L" (**F**irst + **O**uter + **I**nner + **L**ast) to handle these.

$$(a + b)^2 = (a + b) \times (a + b) = a^2 + a \cdot b + b \cdot a + b^2 = a^2 + 2 \cdot a \cdot b + b^2$$

How to Solve Your Problem

Before describing another set of tools for problem solving, let's outline how to use basic algebra to solve any mathematical science problem. The basic procedure involves three steps:*

1. **Translate**
   a. Translate the question into variables, equations, and pictures, and ask
      i. What quantities do you know, and what quantity(ies) are unknown?
      ii. What are you trying to solve for?
2. **Equate**
   a. Determine the relevant equations to use to solve the problem
3. **Solve**
   a. Use the equations to solve the problem
   b. Work with variables to get the answer, then plug in values

Although you may not have noticed, this is the exact procedure we have been using in all of the examples. Let's see how each of these steps work as we solve yet another problem.

Lunar Loops: You have used RADAR technology to determine that the Moon is 3.8×10^8 m away from the Earth. Assuming the Moon orbits the Earth in a perfect circle, and measuring the period of the Moon's orbit to be 27.3 days, what is the orbital speed of the Moon?

Solution: Let's explicitly practice each of the three steps.

1. **Translate**
   a. We know the period of the Moon ($P = 27.3$ days) and the radius of its circular orbit ($r = 3.8 \times 10^8$ m).
   b. We are trying to find the speed at which the Moon travels through space as it orbits the Earth.
2. **Equate**
   a. We know that a speed can be determined by using $v = \frac{\text{distance}}{\Delta t}$. We also know that the distance traveled is the circumference of a circle ($C = 2\pi r$). Finally, we can use conversion factors to convert the period into a more common unit of time:

$$P = 27.3 \text{ days} \times \frac{24 \text{ hr}}{1 \text{ day}} \times \frac{60 \text{ min}}{1 \text{ hr}} \times \frac{60 \text{ sec}}{1 \text{ min}} = 2.36 \times 10^6 \text{ sec}$$

3. **Solve**
   a. We will use the circumference as the distance traveled by the Moon, and we will use the period as the time it takes the Moon to travel that distance.
      i. $v = \frac{\text{distance}}{\Delta t} = \frac{C}{\Delta t} = \frac{2\pi r}{P}$
      ii. We have already solved the problem; this is the hard part! Now, we only need to punch numbers into a calculator to find a numerical answer.
      iii. $v = \frac{2\pi(3.8\times10^8 \text{ m})}{2.36\times10^6 \text{ sec}} = 1.0 \times 10^3 \ \frac{\text{m}}{\text{s}} = 1.0 \ \frac{\text{km}}{\text{s}}$

* This outline was graciously provided by Mason Keck (keckm@bu.edu).

Proportionality and Ratios

Many basic algebra problems can be simplified by concentrating on the proportionality of the important variable. By taking a ratio (i.e., think division) of two equations, we can cancel out all the unchanging parts and only worry about the changing part. For example, the amount of light leaving the surface of a star is called "flux" and is usually denoted by the letter 'F.' The flux from a star with temperature T, is given by the Stefan-Boltzmann law,

$$F = \sigma T^4,$$

where σ is just a constant number with some units. Because σ is just a constant, we can express the flux from a hot star as a proportionality.

$$F \propto T^4$$

Any time you see the proportionality symbol ($\propto$), you can just think "the left side is equal a constant number times the right side." Let's practice this with an example.

Flux from Two Stars: Imagine two stars, one with a temperature $T_A = 6{,}000$ K, and the other with $T_B = 3{,}000$ K. Compare the flux from these two stars.

Solution: Let's begin by simply writing down the flux for each star.

$$F_A = \sigma T_A^4$$

$$F_B = \sigma T_B^4$$

We can compute the ratio F_A/F_B by dividing both sides of the top equation by F_B.

$$\frac{F_A}{F_B} = \frac{\sigma T_A^4}{F_B}$$

Now, we can replace F_B on the right-hand side with σT_B^4 because these two expressions are perfectly equal. This gives us

$$\frac{F_A}{F_B} = \frac{\sigma T_A^4}{\sigma T_B^4} = \left(\frac{T_A}{T_B}\right)^4.$$

Notice how the σ disappeared from the right-hand side; it could have been any number with any units (e.g., 25 m s^{-2}, 42 kg m^{-2}, or 81 joule · sec), but by taking the ratio of the fluxes, we were able to get rid of the constant number and all of its units. This highlights the power of proportionality. In this case, only the proportional dependence on temperature mattered.

Substituting the temperatures provided gives us the final answer.

$$\frac{F_A}{F_B} = \left(\frac{T_A}{T_B}\right)^4 = \left(\frac{6{,}000 \text{ K}}{3{,}000 \text{ K}}\right)^4 = (2)^4 = 16$$

Thus, the flux from star A is 16 times greater than the flux from star B. Next, let's use proportionality to compute the gravitational acceleration on the surface of Jupiter.

Note: The following sections use the astronomical symbols for the Sun ($\odot$) *and the Earth* ($\oplus$).

Gravity on Jupiter: Recall from a previous example that the gravitational acceleration at the surface of Earth is $a_{g,\oplus} = 9.79 \text{ m s}^{-2}$. Given that the mass of Jupiter is $M_J = 317\ M_\oplus$ and the radius of Jupiter is $R_J = 11.2\ R_\oplus$, find the gravitational acceleration on the surface of Jupiter.

Solution: We begin by writing down the complete equation for the gravitational acceleration at the surface of Jupiter and at the surface of Earth.

$$a_{g,J} = \frac{G \cdot M_J}{R_J^2}$$

$$a_{g,\oplus} = \frac{G \cdot M_\oplus}{R_\oplus^2}$$

Next we divide the top equation by $a_{g,\oplus}$ to form the ratio of Jupiter's surface gravity to Earth's surface gravity.

$$\frac{a_{g,J}}{a_{g,\oplus}} = \frac{\left(\frac{G \cdot M_J}{R_J^2}\right)}{a_{g,\oplus}}$$

Again, we can replace the $a_{g,\oplus}$ on the right-hand side with $\frac{G \cdot M_\oplus}{R_\oplus^2}$.

$$\frac{a_{g,J}}{a_{g,\oplus}} = \frac{\left(\frac{G \cdot M_J}{R_J^2}\right)}{\left(\frac{G \cdot M_\oplus}{R_\oplus^2}\right)}$$

Next, we are going to make use of those reciprocals we talked about on page 11. If we concentrate on the right-hand-side of the above equation, we see that there is a group of numbers on the top divided by a group a numbers on the bottom. Now, according to ***Rule 6.b***, dividing be a number is the same as multiplying by the reciprocal. Thus, instead of dividing by $\left(\frac{G \cdot M_\oplus}{R_\oplus^2}\right)$, let's multiply by its reciprocal: $\left(\frac{R_\oplus^2}{G \cdot M_\oplus}\right)$. Thus,

$$\frac{\left(\frac{G \cdot M_J}{R_J^2}\right)}{\left(\frac{G \cdot M_\oplus}{R_\oplus^2}\right)} \Longrightarrow \frac{G \cdot M_J}{R_J^2} \times \frac{R_\oplus^2}{G \cdot M_\oplus}.$$

Let's use this to simplify the ratio of surface gravity we found before.

$$\frac{a_{g,J}}{a_{g,\oplus}} = \frac{G \cdot M_J}{R_J^2} \times \frac{R_\oplus^2}{G \cdot M_\oplus}$$

This clearly allows us to cancel the physical constant G from the numerator and denominator, but it also allows us to cleverly use the reciprocal rule one last time. Let's pause to regroup the masses and radii together.

$$\begin{aligned} \frac{a_{g,J}}{a_{g,\oplus}} &= \frac{M_J}{R_J^2} \times \frac{R_\oplus^2}{M_\oplus} && \text{We have canceled } G \\ &= \frac{M_J}{M_\oplus} \times \frac{R_\oplus^2}{R_J^2} && \text{Group the masses and radii together} \\ &= (M_J/M_\oplus) \times \left(\frac{R_\oplus}{R_J}\right)^2 && \text{Focus on the radii} \end{aligned}$$

We are going to use the reciprocal rule one last time, but now we will use it to go back to division, so that multiplying by $\left(\frac{R_\oplus}{R_J}\right)^2$ becomes a division by its reciprocal, $\left(\frac{R_J}{R_\oplus}\right)^2$.

This finally leads to the very simple expression:

$$\frac{a_{g,J}}{a_{g,\oplus}} = \frac{(M_J/M_\oplus)}{(R_J/R_\oplus)^2}.$$

Notice how we only need to know the ratios $(M_J/M_\oplus)$ and $(R_J/R_\oplus)$. You could measure the masses in any units you like: kilogram, grams, or even the Imperial unit "sulgs" ($1\ \text{sulg} = 14.6\ \text{kg}$). Regardless of what unit you use, the truth remains that Jupiter is 317 times more massive than Earth, thus $(M_J/M_\oplus) = 317$. Similarly, we could also measure the planet radii in any units we like: centimeters, inches, meters, or even light years. Regardless of the units, it remains true that the radius of Jupiter is 11.2 times greater than the radius of Earth, thus $(R_J/R_\oplus) = 11.2$. Substituting these ratio values into the equation, we find that

$$\frac{a_{g,J}}{a_{g,\oplus}} = \frac{(317)}{(11.2)^2} = 2.53.$$

Thus, the surface gravity on Jupiter is 2.53 times stronger than it is on Earth. Since $a_{g,\oplus} = 9.79\ \text{m s}^{-2}$, we find that the surface gravity on Juiter is $24.7\ \text{m s}^{-2}$.

Kepler's 3rd Law Revisited: You have just found a brand new planet in orbit about the a distant star with an average orbital distance of $a_{\text{pl}} = 2\ \text{AU} = 2\ a_\oplus$. By carefully studying the light from that star, you determine it has a mass $M_* = 4\ M_\odot$. What is the orbital period of this planet?

Solution: Let's compare this new planet to one we are already very familiar with: the Earth.

First, let's write down Kepler's 3rd law for both the new planet and for Earth.

$$P_{\text{pl}}^2 = \frac{4\pi^2 a_{\text{pl}}^3}{G(M_* + M_{\text{pl}})}$$

$$P_\oplus^2 = \frac{4\pi^2 a_\oplus^3}{G(M_\odot + M_\oplus)}$$

Before we go any further, let's make a small simplification. The mass of nearly any star is much, much greater than the mass of any orbiting planet. Thus, we can ignore the mass of the planet in the above equations, so we can replace $M_* + M_{\text{pl}}$ with M_*, and $M_\odot + M_\oplus$ with $M_\odot$.

$$P_{\text{pl}}^2 = \frac{4\pi^2 a_{\text{pl}}^3}{G \cdot M_*}$$

$$P_\oplus^2 = \frac{4\pi^2 a_\oplus^3}{G \cdot M_\odot}.$$

As we did above, let's begin by setting the ratio of the left-hand-sides equal to the ratio of the right hand-sides.

$$\frac{P_{\mathrm{pl}}^2}{P_\oplus^2} = \frac{\left(\dfrac{4\pi^2 a_{\mathrm{pl}}^3}{G \cdot M_*}\right)}{\left(\dfrac{4\pi^2 a_\oplus^3}{G \cdot M_\odot}\right)}$$

$$= \frac{4\pi^2 a_{\mathrm{pl}}^3}{G \cdot M_*} \times \frac{G \cdot M_\odot}{4\pi^2 a_\oplus^3}$$

Division by a number becomes multiplication by its reciprocal.

We can again cancel the constants, such as $4\pi^2$ and G. Now, rearranging using the same steps we used above, this becomes

$$\frac{P_{\mathrm{pl}}^2}{P_\oplus^2} = \frac{a_{\mathrm{pl}}^3}{M_*} \times \frac{M_\odot}{a_\oplus^3} = \frac{a_{\mathrm{pl}}^3}{a_\oplus^3} \times \frac{M_\odot}{M_*}$$

Group the masses and orbital distances together

$$= \left(a_{\mathrm{pl}}/a_\oplus\right)^3 \times \left(\frac{M_\odot}{M_*}\right)$$

Transforming the multiplication by $\left(\frac{M_\odot}{M_*}\right)$ into division by its reciprocal, $\left(\frac{M_*}{M_\odot}\right)$, we finally get

$$\left(\frac{P_{\mathrm{pl}}}{P_\oplus}\right)^2 = \frac{\left(a_{\mathrm{pl}}/a_\oplus\right)^3}{(M_*/M_\odot)}.$$

We can solve for P_{pl} by substituting $\left(a_{\mathrm{pl}}/a_\oplus\right) = (2\, a_\oplus/a_\oplus) = 2$ and $(M_*/M_\odot) = (4\, M_\odot/M_\odot) = 4$ and taking the square root of both sides of the equation.

$$\left(\frac{P_{\mathrm{pl}}}{P_\oplus}\right)^2 = \frac{(2)^3}{(4)} = \frac{8}{4} = 2$$

$$\sqrt{\left(\frac{P_{\mathrm{pl}}}{P_\oplus}\right)^2} = \sqrt{2}$$

$$\frac{P_{\mathrm{pl}}}{P_\oplus} = \sqrt{2}$$

Thus, the orbital period of the newfound planet is $\sqrt{2}$ times greater than the orbital period of Earth. Since $\sqrt{2} \cong 1.41$ and the orbital period of Earth is 1 yr, the orbital period of the planet is 1.41 yr.

IMPORTANT EQUATION: We have just derived an incredibly useful version of Kepler's 3rd law. I will rewrite it here using a more explicit notation.

$$\left(\frac{P_{\mathrm{pl}}}{1\ \mathrm{yr}}\right)^2 = \frac{\left(a_{\mathrm{pl}}/1\ \mathrm{AU}\right)^3}{[(M_1 + M_2)/M_\odot]}$$

This equation is valid for any objects in orbit about each other: a planet around the Sun, an asteroid around a distant star, or even two stars in orbit about each other. If you measure the period in years, the average orbital distance in AU, and the mass of the two bodies (one of which is sometimes small enough to be ignored) in "solar masses," then this equation will always be true. Astronomers use this equation to quickly find the masses and/or orbits of planets, stars, black holes, etc.

Trigonometry

Although trigonometry is often considered everyone everyone's least favorite high school math class, it also turns out to be one of the most practical parts of mathematics. Trigonometry can be used to relate nearly any set of angles and distances in the real world, and you can solve some really incredible problems using only the three basic trigonometric functions: sin, cos, and tan.

It is possible that the three most useful words you ever learned in math class, are

$$\text{SOH} \quad \left(\text{Sine of an angle} = \frac{\text{Opposite}}{\text{Hypotenuse}}\right)$$

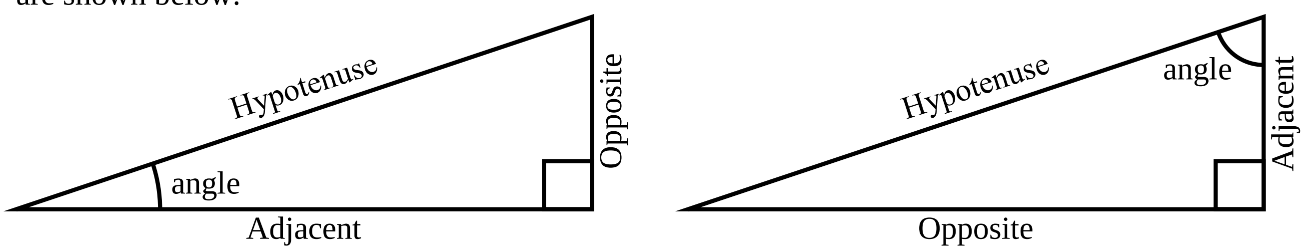
$$\text{CAH} \quad \left(\text{Cosine of an angle} = \frac{\text{Adjacent}}{\text{Hypotenuse}}\right)$$

$$\text{TOA} \quad \left(\text{Tangent of an angle} = \frac{\text{Opposite}}{\text{Adjacent}}\right).$$

These three words describe three equations (shown in the parentheses above) relating the length of two sides of a triangle to one of the angles on the inside of that triangle.

Which Side Is What?

Which of the three triangle sides gets called opposite, adjacent, or hypotenuse depends on which of the three inner angles you use to set up your equation. Two examples illustrating the relationship are shown below.



The adjacent, opposite, and hypotenuse triangle sides are defined by the following criteria.

- Hypotenuse: the side opposite the right angle
- Opposite: the side opposite to the angle we are interested in
- Adjacent: the side that is in contact with the angle we are interested in and the right angle.

Since only right triangles contain a right angle and have a hypotenuse, the sine, cosine, and tangent functions only relate to triangle side-lengths for right triangles. Now, let us take a look at two examples of how we can use trigonometry to do some simple science problems.

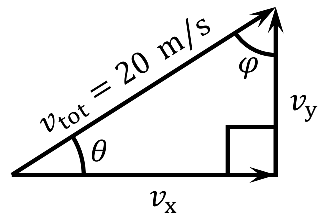
Ballistic Balloons: Your friends have just launched a water balloon at your wide open dormitory window. The balloon was launched 50° above horizontal at a total speed of $20\ \mathrm{m/s}$.

(a) What is the vertical part of the balloon's velocity as it leaves the launcher?

(b) What is the horizontal part of the balloon's velocity as it leaves the launcher?

Solution: We can imagine the total velocity of the balloon is like the hypotenuse of a right triangle and the horizontal and vertical parts of the velocity are the other two triangle sides. To start, we will just draw any old right triangle. The side lengths and angles do not need to be accurate. We just need something to help us visualize the problem.

The initial velocity of the balloon is 50° above the horizontal direction. Thus, the angle labeled θ, connecting the horizontal triangle side and the total velocity (the hypotenuse) is 50°.



Now, let's think about how to solve for the triangle sides labeled v_{x} and v_{y}. Since we know that $\theta = 50°$, we can just write down all three SOH, CAH, TOA equations for this angle.

$$\sin\theta = \frac{v_{\text{y}}}{v_{\text{tot}}} \qquad \cos\theta = \frac{v_{\text{x}}}{v_{\text{tot}}} \qquad \tan\theta = \frac{v_{\text{y}}}{v_{\text{x}}}$$

(a) We were asked to find the vertical part of the velocity (represented by v_{y}). Since we know the total velocity (v_{tot}), and the angle θ, we can use the sine equation to solve for v_{y}.

$$\sin\theta = \frac{v_{\text{y}}}{v_{\text{tot}}}$$ Write down starting equation

$$v_{\text{tot}} \times \sin\theta = \frac{v_{\text{y}}}{v_{\text{tot}}} \times v_{\text{tot}}$$ Multiply both sides by v_{tot}

$$v_{\text{tot}} \times \sin\theta = v_y$$ You've found v_{y}

$$20\,\frac{\text{m}}{\text{s}} \times \sin(50°) = 15.3\,\frac{\text{m}}{\text{s}} = v_y$$ Substitute in the given values

(b) We follow similar steps to find the horizontal part of the velocity (represented by v_{x}). Since we know the value of v_{tot} and the angle θ, we can use the cosine equation to solve for v_{x}.

$$\cos\theta = \frac{v_{\text{x}}}{v_{\text{tot}}}$$ Write down starting equation

$$v_{\text{tot}} \times \cos\theta = \frac{v_{\text{x}}}{v_{\text{tot}}} \times v_{\text{tot}}$$ Multiply both sides by v_{tot}

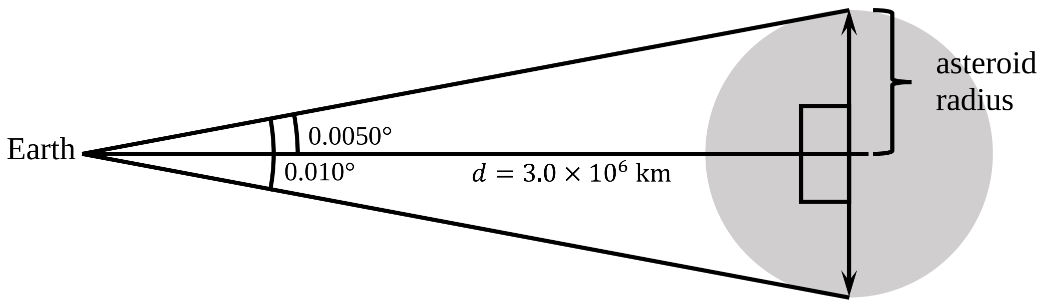
$$v_{\text{tot}} \times \cos\theta = v_x$$ You've found v_{x}

$$20\,\frac{\text{m}}{\text{s}} \times \cos(50°) = 12.9\,\frac{\text{m}}{\text{s}} = v_y$$ Substitute in the given values

Let's try another example using basic trigonometry.

Catastrophic Collision: You have just discovered an asteroid, and it is headed straight for Earth! You bounce radio signals off the asteroid and discover that it is 3.0×10^6 km away and appears to be 0.010° across. What is the radius of this asteroid?

Solution: To solve this, let's again start by drawing a triangle with the Earth at one vertex and the asteroid extremities at the other two vertices.



The large triangle encompassing the entirety of the asteroid is not a right triangle. Since the SOH, CAH, TOA equations only work for right triangles, let's cut right through the center to make two smaller right triangles. The angle at the Earth vertex is now half its original value: 0.0050°.

We were asked to find the radius of the asteroid. This length is the side *opposite* to the 0.0050° angle. We also know the distance to the asteroid, which is the side *adjacent* to the 0.0050° angle. Given the quantities we know, we can use the TOA equation to solve for the asteroid radius.

$$\tan(0.0050°) = \frac{\text{radius}}{\text{distance}}$$

$$\text{distance} \times \tan(0.0050°) = \frac{\text{radius}}{\text{distance}} \times \text{distance}$$

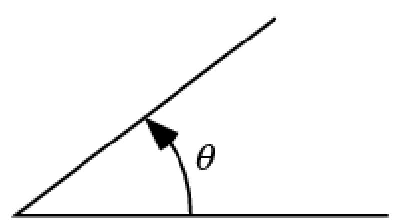
$$\text{distance} \times \tan(0.0050°) = \text{radius}$$

$$(3.0 \times 10^6\text{ km}) \times (8.7 \times 10^{-5}) = \text{radius}$$

$$260\text{ km} = \text{radius}$$

Radians, Degrees, and Arcseconds

Angles may seem like an intuitive concept, but they have a very precise definition. Here is a definition provided by the Wolfram MathWorld website:



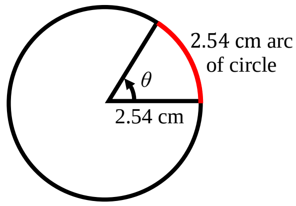
"Given two intersecting lines or line segments, the amount of rotation about the point of intersection (the vertex) required to bring one into correspondence with the other is called the angle (θ) between them."

There are several common ways of measuring the amount of rotation specified by an angle. The most common units for measuring angles are radians, degrees, and arcseconds. Each of these units measure the same thing (angles) just like days, hours, minutes, and seconds all measure the same thing (time).

Radians: This measure of angles is defined using a fraction—the portion of a circle swept out by an angle divided by the radius of the circle. This relationship is expressed in equation

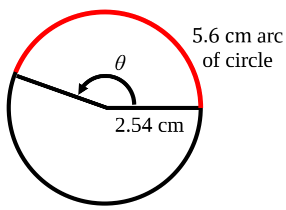
$$\theta = \frac{S}{r},$$

where S is some arc length (i.e., a portion of the circle's circumference), r is the radius of the circle, and θ is the angle, measured in radians. Consider a circle with a radius of 1 in., or 2.54 cm. Let's grab a stretch of that circle also 2.54 cm long and paint it red.

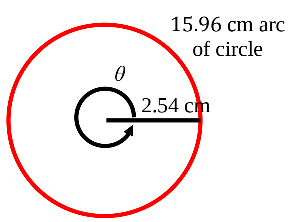


The length of the red curve divided by the length of the radius gives the angle marked between the two lines, measured in radians. Thus, $\theta = 2.54\text{ cm}/2.54\text{ cm} = 1\text{ radian}$. Notice how the numerator and denominator have the same units, and they cancel each other. Radians are what is called a "dimensionless unit," which means that multiplying or dividing by some number of radians will not affect on the units.

How many radians would be swept out if we painted 5.6 cm of the circle?



We can again divide the length of the red curve by the length of the radius to find the angle. This gives an angle $\theta = 5.6\text{ cm}/2.54\text{ cm} = 2.2\text{ rad}$. Let's consider one last example by painting the entire circumference of the circle.



Dividing the circumference $(C = 2\pi \cdot r)$ by the radius gives an angle

$$\theta = \frac{2\pi \cdot 2.54\text{ cm}}{2.54\text{ cm}} = 2\pi\text{ rad}.$$

Thus, a full circle encompasses 2π rad.

Degrees: This is usually the most familiar unit of angle. A full circle in this unit system encompasses $360°$. Combining this with the measure of a full circle in radians, we find that $2\pi\text{ rad} = 360°$. This can be expressed as a conversion factor:

$$\frac{360°}{2\pi\ \text{rad}} \quad \text{or} \quad \frac{57.3°}{1\ \text{rad}}$$

Arcseconds: Astronomers often measure very small angles—much smaller than 1°. Therefore, a different unit for measuring angles is needed. This unit is the arcsecond.

To visualize how small arcsecond is, imagine that you were served a *tiny* slice of pie 1° wide. Now, cut that slice of pie into into 60 equally sized *very tiny* pieces. Each one of those *very tiny* pieces of pie is 1 arcminute (written as $1'$) wide. Now, once again cut one of those *very tiny* pieces into 60 equally sized *very*, *very tiny* pieces of pie. Each one of those *very*, *very tiny* pieces of pie is 1 arcsecond (written as $1''$) wide. Let's summarize these relationships using conversion factors. For good measure, we will even include the conversion factor relating radians and arcseconds.

$$\frac{60'}{1°} \quad \text{or} \quad \frac{1°}{60'} \qquad\qquad \frac{60'}{1°}\times\frac{60''}{1'}=\frac{3600''}{1°} \quad \text{or} \quad \frac{1°}{3600''}$$

$$\frac{60''}{1'} \quad \text{or} \quad \frac{1'}{60''} \qquad\qquad \frac{360°}{2\pi\ \text{rad}}\times\frac{3600''}{1°}=\frac{206265''}{1\ \text{rad}} \quad \text{or} \quad \frac{1\ \text{rad}}{206265''}$$

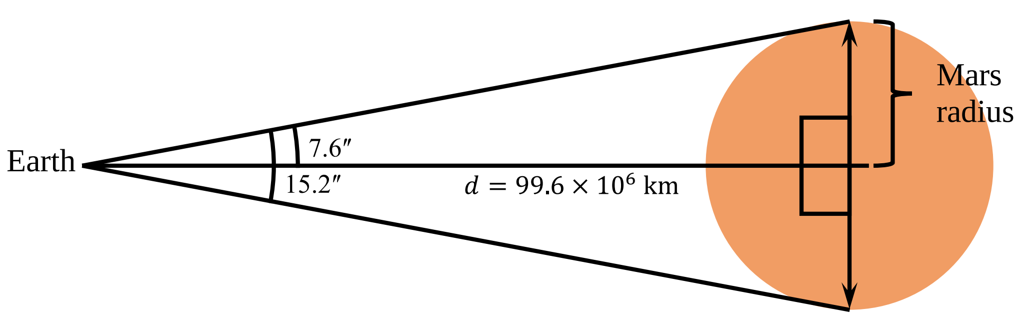
Small Angle Formulae

Many trigonometry problems can be quickly solved by using approximations known as "the small angle formulae." These formulae simplify the computation of sine, cosine, and tangent for very small angles, less than about $5°$ or 0.1 rad. It is important to note that *the small angle formulae only work if you measure the angle in radians!* If you are given a small angle in degrees or arcseconds, you will need to convert it into radians before plugging it into these formulae.

$$\text{If } \theta \lesssim 0.1\ \text{rad, then} \begin{cases} \sin(\theta) \cong \theta \\ \cos(\theta) \cong 1 \\ \tan(\theta) \cong \theta \end{cases}$$

Martian Madness: On April 14th, 2014, Earth reached its point of closest approach to Mars. At that time, Mars was only 92×10^6 km away from Earth. On the night of April 14th, you measured the angular diameter of Mars to be $15.2''$. What is the physical diameter of Mars?

Solution: We can solve this problem using the same method as the "Catastrophic Collisions" problem on page 20. We will also use the small angle formula to simplify the procedure.



Using the smaller, right triangle as we did before, we can set up a sine relationship, but first we will need to convert the 7.6″ angle into radians so that we can use a small angle formula.

$$\theta = 7.6'' \times \frac{1\ \text{rad}}{206265''} = 3.7 \times 10^{-5}\ \text{rad}$$

$$\sin(\theta) = \frac{r_{\text{Mars}}}{d}$$

$$\theta \cong \frac{r_{\text{Mars}}}{d}$$ Use a small angle formula

$$d \times \theta \cong \frac{r_{\text{Mars}}}{d} \times d$$ Multiply by d

$$d \times \theta \cong r_{\text{Mars}}$$ You have found r_{Mars}

$$(92 \times 10^{6}\ \text{km}) \cdot (3.7 \times 10^{-5}\ \text{rad}) = r_{\text{Mars}}$$ Substitute values

$$3.4 \times 10^{3}\ \text{km} = r_{\text{Mars}}$$ (Note how multiplying by radians did not affect the units of the final answer.)

Now finding the diameter is simple:

$$D_{\text{Mars}} = 2 \times r_{\text{Mars}}$$

$$D_{\text{Mars}} = 2 \times 3.4 \times 10^{3}\ \text{km} = 6.8 \times 10^{3}\ \text{km}$$

The End

You made it to the end of this workbook! Congratulations, you now have all of the tools you will need to succeed in any math-science course. Refer to these examples whenever you get lost on a homework problem. There are an infinity of possible homework problems in science courses, but the above examples cover the basic types of problems you'll encounter in this course, and they should help you get started on the right track.

Now, take a break, go get some ice-cream, watch a movie, or play Frisbee with some friends. You have earned it.

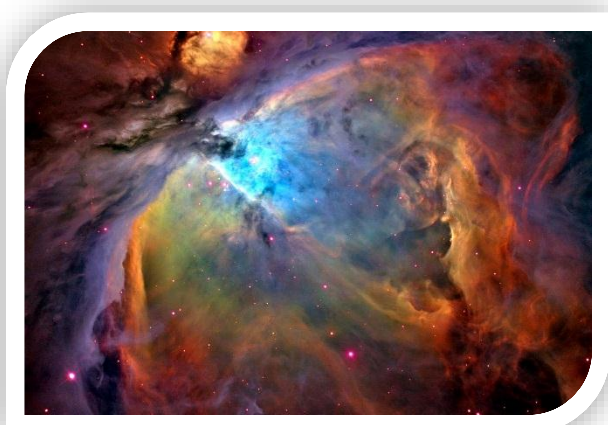


Image of the Orion Nebula: a nearby stellar nursery.
From: Astronomy Picture of the Day
Credit & Copyright: Russell Croman