GEOMETRIC PHASES IN DISSIPATIVE SYSTEMS WITH SYMMETRY

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Abstract: Classical dissipative systems with symmetry can exhibit a geometric phase effect wherein an adiabatic variation of a parameter drives a shift in the symmetry direction. Viewing the parameter as a control variable, the effect may be useful in the parametric control of dissipative systems, many of which exhibit pattern-forming solutions. Earlier work by A.S. Landsberg developed a theory for this effect in systems admitting an *abelian* symmetry. In this paper we present a generalization allowing for arbitrary continuous symmetries. This generalization is achieved by defining a new principal connection, here called the Landsberg connection, on an appropriate principal fiber bundle. A simple example is presented to illustrate the theory.

Keywords: nonlinear systems, phase shift, perturbation analysis, dissipation, geometric approaches

1. INTRODUCTION

Dissipation is a common feature in physical systems and can give rise to interesting dynamics such as exponentially stable equilibria and attracting limit cycles. If these systems also admit a symmetry then they can exhibit a geometric phase effect due to an adiabatic variation of a parameter. The essential idea is as follows. Under appropriate assumptions, a system with a symmetry group action on state space can be factored into dynamics on a reduced space, independent of the group variables, and dynamics on the group. Assume that the reduced system has an exponentially stable equilibrium point which depends on a parameter. Since this point is exponentially stable, as the parameter is slowly varied the system will remain close to equilibrium at all times. If the parameter is brought back to its original value, the reduced system will return to the original equilibrium point. However, there may be a net shift in the group variables which depends only on the path followed by the parameter. This shift is called the geometric phase.

It is not uncommon to find the existence of pattern-forming solutions in systems with dissipation (Cross and Hohenberg, 1993). If these systems exhibit a translational symmetry then any shift of the pattern by the action of the symmetry group will also be a solution. An adiabatic variation of the parameter will result in a deformation of the pattern. After returning the parameter to its original value the initial pattern will be recovered but it may have shifted in the symmetry direction. The geometric phase is this displacement of the pattern.

Interest in the effect of geometric phases in physical systems was spurred by work of Michael Berry (Berry, 1984; Shapere and Wilczek, 1989). Early work on geometric phases in classical dissipative systems includes (Kepler and Kagan, 1991) in which the authors considered systems with stable

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limit cycles undergoing an adiabatic variation of a parameter around a closed path. They showed the existence of a geometric shift in the variable parametrizing the limit cycle and expressed this shift as the integral of a two-form over a surface bounded by the loop in parameter space. Together with Epstein they applied these ideas to explore geometric phase shifts in chemical oscillators (Kagan *et al.*, 1991).

Landsberg developed a general theory of geometric phases in classical dissipative systems with *abelian* symmetries (Landsberg, 1992; Landsberg, 1993). However, many symmetry groups are non-abelian and it is the purpose of this work to develop a theory allowing for arbitrary symmetries. It is hoped that this will lead to novel control techniques for systems of this type and particularly of pattern-forming systems.

The remainder of this paper is organized as follows. The natural framework in which to discuss geometric phases is that of connections on fiber bundles and in Section 2 we present a brief review of the essential concepts. In Section 3 we give the basic model for a dissipative system with symmetry and develop the Landsberg connection under the assumption that the group dynamics are at an equilibrium whenever the reduced dynamics are. We relax this assumption and introduce the notion of the dynamic phase in Section 4. This permits us to describe the behavior of systems exhibiting propagating patterns. We then present a simulation example in Section 5 before concluding in Section 6.

2. PRINCIPAL CONNECTIONS AND THE GEOMETRIC PHASE

To establish notation, we provide a brief review of principal bundles and principal connections. A standard reference is (Nomizu, 1956).

Let P be a smooth manifold and let G be a Lie group that acts freely and properly on P on the left. A principal fiber bundle with structure group G is a fiber bundle $\pi : P \to P/G$ whose fibers are diffeomorphic to the group G. We often denote the base space P/G by B. Let $\mathfrak{g}$ denote the Lie algebra associated to G.

Definition 2.1. A **principal connection** on the principal bundle $\pi : P \to P/G$ is a $\mathfrak{g}$-valued one form $\mathcal{A} : TP \to \mathfrak{g}$ satisfying:

(1) $\mathcal{A}(\xi_P(p)) = \xi \quad \forall \xi \in \mathfrak{g}$ and $p \in P$ where $\xi_P(p)$ is the infinitesimal generator corresponding to ξ.
(2) $\mathcal{A}$ is Ad$-$equivariant. That is,

$$\mathcal{A}(T_p\Phi_g(v_p)) = \mathrm{Ad}_g\mathcal{A}(v_p) \qquad (1)$$

for all $v_p \in T_pP$ and $g \in G$ where Φ_g is the action of G on P and Ad_g is the adjoint action of G on $\mathfrak{g}$.

■

Given $p \in P$, there is a natural subspace of T_pP called the vertical space at p, denoted by V_p and defined by $V_p = \ker T_p\pi$. The connection defines a horizontal space at each point p given by

$$H_p = \{v_p \in T_pP | \mathcal{A}(v_p) = 0\} \qquad (2)$$

and thus gives a splitting of the tangent space at each p, $T_pP = V_p \oplus H_p$.

Given a principal connection $\mathcal{A}$, a point $p \in P$, and a tangent vector $w \in T_{\pi(p)}B$, the horizontal lift of w to T_pP is defined as the unique tangent vector in H_p which projects to w under $T_p\pi$.

Let $b(t)$, $t \in [0,1]$, be a piecewise differentiable curve in P/G. A horizontal lift of $b(\cdot)$ with respect to $\mathcal{A}$ is a curve $p(\cdot)$ in P such that $\pi(p(t)) = b(t)$ and such that the tangent vector $\frac{dp(t)}{dt}$ is horizontal for each $t \in [0,1]$. Consider now a closed curve at $b_0 \in B$, i.e. $b(0) = b(1) = b_0$. The diffeomorphism of the fiber $\pi^{-1}(b_0)$ onto itself given by parallel transport along $b(t)$ is called the **holonomy** or **geometric phase** of the path $b(\cdot)$.

The holonomy can be identified as an element of G as follows (Yang, 1992). Assume $b(\cdot)$ is contained in an open set U of B and let $p_0 \in \pi^{-1}(b(0))$. Let $\sigma : U \to P$ be an arbitrary local section of the bundle and let $p(\cdot)$ be the horizontal lift of $b(\cdot)$ with $p(0) = p_0$. Let $g(\cdot)$ be the curve in G such that $p(t) = \Phi(g(t), \sigma(b(t)))$. Then

$$\begin{aligned}\frac{dp(t)}{dt} &= T_{\sigma(b(t))}\Phi_{g(t)}[T_{b(t)}\sigma(\dot{b}(t))] \\ &\quad + T_{\sigma(b(t))}\Phi_{g(t)}\xi_P(t)(\sigma(b(t)))\end{aligned} \qquad (3)$$

where $\xi(t) \stackrel{\triangle}{=} g(t)^{-1}\dot{g}(t)$ and $\xi_P(t)(\sigma(b(t)))$ is the corresponding infinitesimal generator. Since $p(t)$ is horizontal, applying $\mathcal{A}$ to both sides of (3) yields

$$\begin{aligned}0 &= \mathcal{A}\Big[T_{\sigma(b(t))}\Phi_{g(t)}[T_{b(t)}\sigma(\dot{b}(t))] \\ &\quad + T_{\sigma(b(t))}\Phi_{g(t)}\xi_P(t)(\sigma(b(t)))\Big] \\ &= \mathrm{Ad}_g\Big[\mathcal{A}(T_{b(t)}\sigma(\dot{b}(t))) + \mathcal{A}(\xi_P(t)(\sigma(b(t))))\Big] \\ &= \mathrm{Ad}_g\Big[(\sigma^*\mathcal{A})(\dot{b}(t)) + \xi(t)\Big]\end{aligned}$$

where we have used the Ad$-$equivariance of $\mathcal{A}$ and the fact that $\mathcal{A}$ maps infinitesimal generators to the corresponding Lie algebra elements. The $\mathfrak{g}$-valued form $\sigma^*\mathcal{A}$ is called the local connection form and is denoted $\mathcal{A}_{loc}$. Thus

$$\xi(t) = -\mathcal{A}_{loc}(\dot{b}(t)). \qquad (4)$$

By the definition of $\xi(t)$ we have

$$\dot{g}(t) = g(t)\xi(t) = -g(t)\mathcal{A}_{loc}(\dot{b}(t)) \tag{5}$$

and the solution of this differential equation at $t = 1$ is the geometric phase. For concreteness, in the remainder of this paper we restrict ourselves to matrix Lie groups.

3. THE GEOMETRIC PHASE

We now give a brief derivation of the Landsberg connection. For details see (Andersson, 2003).

3.1 Dissipative systems with symmetry

Let P be a smooth manifold. A vector field X on P is called dissipative with respect to a smooth function h on P if

i) $(X(h))(p) \leq 0$, for all p,
ii) $(X(h))(p) = 0$ if and only if $X = 0$ for all p.

Let Φ be a free and proper left action of a matrix Lie group G on P and construct the principal bundle $\pi : P \to P/G$. Assume that X is equivariant with respect to the group action, that is for every $g \in G$ and for every $p \in P$ we have

$$X(\Phi_g(p)) = (T_p\Phi_g)\, X(p) \tag{6}$$

The system defined by X is then said to admit G as a symmetry.

Using standard reduction arguments (see, e.g. (Marsden *et al.*, 1990)), in a local trivialization of the fiber bundle the curve $p(\cdot)$ starting at $p_0 = (g_0, y_0)$ can be locally defined by the system

$$\begin{aligned} \dot{g} &= g\xi(y,\lambda) \\ \dot{y} &= f(y,\lambda) \end{aligned} \tag{7}$$

where we have introduced the parameter $\lambda \in U \subset \mathbb{R}^m$. Here f is the projection of the dissipative vector field X onto the base space and $\xi(\cdot) \in \mathfrak{g}$ is a curve in the Lie algebra. (The left invariance of the group system follows from the equivariance of X.)

3.2 The Landsberg connection

Now assume there exists a family of exponentially asymptotically stable equilibria $y^*(\lambda)$, i.e. $f(y^*(\lambda), \lambda) = 0$ for all $\lambda \in U$, and that $\xi(y^*(\lambda), \lambda) = 0$ for all λ. (This second condition will be relaxed in Section 4.)

We wish to understand the behavior of system (7) as the parameter λ is varied adiabatically. To do so, take $\lambda = \lambda(\tau)$ where $\tau = \epsilon t$, $\epsilon > 0$, and carry out an asymptotic analysis of the system. Begin by assuming y can be expressed as

$$y(t) = y_0(t,\tau) + \epsilon y_1(t,\tau) + \cdots \tag{8}$$

with initial condition $y(0) = y^*$. From (8) we have

$$\dot{y} = \frac{\partial y_0}{\partial t} + \epsilon\left[\frac{\partial y_0}{\partial \tau} + \frac{\partial y_1}{\partial t}\right] + O(\epsilon^2). \tag{9}$$

Setting $\epsilon = 0$ and using $\dot{y} = f(y,\lambda)$ yields

$$f(y_0,\lambda) = \frac{\partial y_0}{\partial t}, \quad y_0(0) = y^* \tag{10}$$

and therefore $y_0 \equiv y^*$. Now expand f in a Taylor series about the solution $y = y^*$. This gives

$$f(y,\lambda) = \epsilon\,(T_{y^*}f)\, y_1 + O(\epsilon^2) \tag{11}$$

where $(T_{y^*}f)$ denotes the linearization of f at y^*. Combining (9) and (11) we find

$$\epsilon\,(T_{y^*}f)\, y_1 + O(\epsilon^2) = \epsilon\left[\frac{\partial y^*}{\partial \tau} + \frac{\partial y_1}{\partial t}\right] + O(\epsilon^2). \tag{12}$$

At first order in ϵ we have

$$\frac{\partial y_1}{\partial t} = (T_{y^*}f)\, y_1 - \frac{\partial y^*}{\partial \tau}. \tag{13}$$

For fixed τ this is a linear ordinary differential with constant coefficients. Using the variation of constants formula, the solution is

$$y_1(t) = (T_{y^*}f)^{-1}\left[\frac{\partial y^*}{\partial \tau} - e^{(T_{y^*}f)t}\frac{\partial y^*}{\partial \tau}\right] \tag{14}$$

where the fact that $y_i(0) = 0$ for all $i \neq 0$ has been used. Since the equilibrium y^* is assumed exponentially stable we know $(T_{y^*}f)$ is Hurwitz and thus $(T_{y^*}f)^{-1}$ exists. From the Hurwitz property the second term in (14) decays to zero exponentially. The rate of this decay determines the dissipative time scale of the system. For times long with respect to this time scale the second term in (14) can be neglected and thus

$$y(t) \approx y^* + (T_{y^*}f)^{-1}\frac{\partial y^*}{\partial t} \tag{15}$$

Recalling that y^* depends on time only through its dependence on λ we write

$$y(t) \approx y^* + (T_{y^*}f)^{-1}\,\nabla_\lambda y^* \frac{d\lambda}{dt}. \tag{16}$$

To determine the effect of the parameter variation on the group variables, expand the map $\xi(\cdot)$ in a Taylor series around y^* and truncate to first order. This yields

$$\xi(y) \approx (T_{y^*}\xi)(T_{y^*}f)^{-1}\nabla_\lambda y^* \frac{d\lambda}{dt} \tag{17}$$

where we have used the assumption that $\xi(y^*, \lambda) = 0$. Define the map $\mathcal{A}_{loc} : T\mathbb{R}^m \to \mathfrak{g}$ by

$$\mathcal{A}_{loc}(\lambda)(v) = \left((T_{y^*}\xi)(T_{y^*}f)^{-1}\nabla_\lambda y^*\right)v. \tag{18}$$

Consider the principal bundle $G \times U \to U$ and the action of G on $G \times U$ defined by

$$\widetilde{\Phi} : G \times (G \times U) \to G \times U,\ \widetilde{\Phi}_h((g, \lambda)) = (hg, \lambda)$$

The infinitesimal generator corresponding to an element $\eta \in \mathfrak{g}$ is given by

$$\eta_{G\times U} = \left.\frac{d}{ds}\right|_{s=0} (\exp(s\eta)g, \lambda) = (\eta g, 0). \tag{19}$$

Proposition 3.1. The $\mathfrak{g}$-valued one form given by

$$\mathcal{A}_L(g, \lambda)(\dot{g}, \dot{\lambda}) = \mathrm{Ad}_g\left(g^{-1}\dot{g} - \mathcal{A}_{loc}(\lambda)\dot{\lambda}\right) \tag{20}$$

is a principal connection on $\pi : G \times U \to U$. The corresponding geometric phase equation is

$$\dot{g} = g\mathcal{A}_{loc}(\lambda)\dot{\lambda}. \tag{21}$$

Proof A simple calculation shows that $\mathcal{A}_L$ meets the conditions in Definition 2.1. (21) follows from (20) and (18). ■

The geometric phase corresponding to an adiabatic variation of λ around a loop parametrized by $s \in [0, 1]$ is the solution to (21) at $s = 1$.

Remark 3.2. Since Landsberg first carried out the perturbation analysis (for systems with abelian symmetries), we refer to this connection as the *Landsberg* connection.

4. THE DYNAMIC PHASE

Consider once again system (7). We would like to remove the restriction that $\xi(y^*, \lambda) = 0$ by expressing g as the product of a component capturing the geometric evolution, denoted g_{gp}, and a component capturing the dynamic evolution, denoted g_{dp}. Since the symmetry group may be nonabelian, there are multiple ways both to define the systems for g_{gp} and g_{dp} and to combine these terms to get g. The most natural choice is to define

$$\dot{g}_{gp} = g_{gp}(\xi(y, \lambda) - \xi(y^*, \lambda)) \stackrel{\triangle}{=} g_{gp}\xi_{gp}(y, \lambda) \tag{22}$$

and to set

$$g(t) = g_{gp}(t)g_{dp}(t). \tag{23}$$

Taking the derivative of (23) and using (22) yields

$$\dot{g}_{dp} = g_{dp}\mathrm{Ad}_{g_{gp}}(\xi(y^*, \lambda)),\quad g_{dp}(0) = \mathbb{I}. \tag{24}$$

In the adiabatic approximation, we replace the group equation in system (7) by(24) together with

$$\dot{g}_{gp} = g_{gp}\mathcal{A}_{loc}(\lambda)\dot{\lambda},\quad g_{gp}(0) = \mathbb{I} \tag{25}$$

where ξ_{gp} in (22) is used in the definition for $\mathcal{A}_{loc}$. The group evolution is reconstructed from (23). Letting T denote the time at which the parameter returns to its original value, the geometric and dynamic phases are the solutions to (25) and (24) respectively at time T. With these choices, the errors introduced by the adiabatic approximation go to zero in the adiabatic limit. For details see (Andersson, 2003).

5. EXAMPLE

We now illustrate the techniques developed in this paper with an example containing both a geometric and a dynamic phase. Let $G = SE(2)$ and as a basis for the Lie algebra $se(2)$ choose

$$\mathcal{A}_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \mathcal{A}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consider a damped harmonic oscillator with forcing (in state space form) where the natural frequency and the driving force are parameter-dependent.

$$\dot{y} = \begin{pmatrix} 0 & \omega(\lambda_1) \\ -\omega(\lambda_1) & -k \end{pmatrix} y + \begin{pmatrix} 0 \\ f(\lambda_2) \end{pmatrix} = Ay + b \tag{26}$$

for $k > 0$. The example system we consider is

$$\begin{aligned} \dot{g} &= g(\mathcal{A}_1 y_1 + \mathcal{A}_2 y_2), \\ \dot{y} &= A(\lambda)y + b(\lambda) \end{aligned} \tag{27}$$

The parameter-dependent equilibrium point is

$$y^*(\lambda) = -A^{-1}(\lambda)b(\lambda) = \begin{pmatrix} \dfrac{f(\lambda_2)}{\omega(\lambda_1)} \\ 0 \end{pmatrix}. \tag{28}$$

At this equilibrium point we have

$$\xi(y^*, \lambda) = \mathcal{A}_1 \frac{f(\lambda_2)}{\omega(\lambda_1)} \tag{29}$$

and so the group dynamics are not stationary when the dynamics on the reduced space are at equilibrium. Following the technique outlined in Section 4, we replace (27) by the system

$$\begin{aligned} \dot{g}_{gp} &= g_{gp}(\mathcal{A}_1(y_1 - y_1^*) + \mathcal{A}_2 y_2), \\ \dot{y} &= A(\lambda)y + b(\lambda), \\ \dot{g}_{dp} &= g_{dp}\mathrm{Ad}_{g_{gp}}\left(\mathcal{A}_1 \frac{f(\lambda_2)}{\omega(\lambda_1)}\right) \stackrel{\triangle}{=} g_{dp}\xi_{dp}. \end{aligned} \tag{30}$$

The natural frequency and the driving force are taken to have the following forms.

$$\omega(\lambda_1) = \bar{\omega} + \lambda_1, \quad f(\lambda_2) = \frac{\lambda_2^2}{2}. \tag{31}$$

The rate of dissipation is given by the largest real part of the eigenvalues of the matrix A. These eigenvalues are

$$\mu_{l,s} = \frac{-k \pm \sqrt{k^2 - 4\omega^2}}{2}. \tag{32}$$

5.1 The geometric and dynamic phase equations

The Lie algebra $se(2)$ is solvable and thus the solution to a left invariant system $\dot{g} = g\xi$ can be expressed globally as a product of exponentials

$$g(t) = e^{\gamma_1(t)\mathcal{A}_1} e^{\gamma_2(t)\mathcal{A}_2} e^{\gamma_3(t)\mathcal{A}_3} \tag{33}$$

where the Wei-Norman parameters are defined by

$$\begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \gamma_3 & 1 & 0 \\ -\gamma_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}, \gamma(0) = 0, \tag{34}$$

and are solvable by quadrature (Wei and Norman, 1964). Here $\xi_i(t)$ are the components of $\xi(t) \in \mathfrak{g}$ in the given basis.

Beginning with system (30) and a curve $\lambda(\cdot)$ in parameter space, we use the Landsberg connection to define the geometric phase, as in (21). Expressing the solution in terms of the Wei-Norman coordinates we find

$$\gamma_1(t) = \int_0^t \left[\frac{k\lambda_2^2(\tau)\dot{\lambda}_1(\tau)}{2(\bar{\omega} + \lambda_1(\tau))^4} - \frac{k\lambda_2(\tau)\dot{\lambda}_2(\tau)}{(\bar{\omega} + \lambda_1(\tau))^3} \right] d\tau,$$

$$\gamma_2(t) = \int_0^t \left[\frac{-\lambda_2^2(\tau)\dot{\lambda}_1(\tau)}{2(\bar{\omega} + \lambda_1(\tau))^3} + \frac{\lambda_2(\tau)\dot{\lambda}_2(\tau)}{(\bar{\omega} + \lambda_1(\tau))^2} \right]$$
$$\cdot \cos\left(\int_\tau^t \left[\frac{k\lambda_2^2(\sigma)\dot{\lambda}_1(\sigma)}{2(\bar{\omega} + \lambda_1(\sigma))^4} - \frac{k\lambda_2(\sigma)\dot{\lambda}_2(\sigma)}{(\bar{\omega} + \lambda_1(\sigma))^3} \right] d\sigma \right) d\tau,$$

$$\gamma_3(t) = -\int_0^t \left[\frac{-\lambda_2^2(\tau)\dot{\lambda}_1(\tau)}{2(\bar{\omega} + \lambda_1(\tau))^3} + \frac{\lambda_2(\tau)\dot{\lambda}_2(\tau)}{(\bar{\omega} + \lambda_1(\tau))^2} \right]$$
$$\cdot \sin\left(\int_\tau^t \left[\frac{k\lambda_2^2(\sigma)\dot{\lambda}_1(\sigma)}{2(\bar{\omega} + \lambda_1(\sigma))^4} - \frac{k\lambda_2(\sigma)\dot{\lambda}_2(\sigma)}{(\bar{\omega} + \lambda_1(\sigma))^3} \right] d\sigma \right) d\tau.$$

If the loop is completed at time T, the geometric phase (in the Wei-Norman coordinates) is $\gamma(T)$. Consider a smooth loop $\mathcal{C}$ in parameter space. Using Stokes' theorem, the integral of γ_1 over $\mathcal{C}$ can be written

$$\gamma_1(T) = \int_{\mathcal{D}} \frac{2k\lambda_2}{(\bar{\omega} + \lambda_1)^4} d\lambda_1 \wedge d\lambda_2 \tag{35}$$

where $\mathcal{D}$ is the region in parameter space bounded by $\mathcal{C}$. The one-form in the equation for γ_1 is thus not exact and the geometric phase is not necessarily trivial.

Using the solution to the geometric phase in the equation for the dynamic phase (30) we have

$$\begin{aligned} \xi_{dp} = \mathcal{A}_1 &\frac{f(\lambda_2)}{\omega(\lambda_1)} \\ &+ \mathcal{A}_2 \frac{f(\lambda_2)}{\omega(\lambda_1)}(\gamma_2(t)\sin(\gamma_1(t)) + \gamma_3(t)\cos(\gamma_1(t))) \\ &- \mathcal{A}_3 \frac{f(\lambda_2)}{\omega(\lambda_1)}(\gamma_2(t)\sin(\gamma_1(t)) + \gamma_3(t)\cos(\gamma_1(t))) \end{aligned}$$

where the γ_i are the Wei-Norman parameters for the geometric phase.

5.2 An elliptical loop

We now choose the closed loop given by

$$\lambda_1 = a\cos\theta, \quad \lambda_2 = b\sin\theta, \quad \theta \in [0, 2\pi] \tag{36}$$

The solution to the geometric phase is then

$$\gamma_1(2\pi) = \gamma_2(2\pi) = \gamma_3(2\pi) = 0 \tag{37}$$

and thus for an elliptical loop the geometric phase is zero. (For loops in which the geometric phase is not zero see (Andersson, 2003).) To solve for the dynamic phase we need γ_2 and γ_3 as functions of θ, not just at $\theta = 2\pi$. Due to the complexity of the equations we turn to numerical simulation.

To vary the parameter, set $\theta = \frac{2\pi}{T}t$ where T is taken so as to satisfy the adiabatic condition. In the simulations that follow we chose $\bar{\omega} = 100$, $k = 200$, $a = 50$, $b = 25$, and $T = 10$.

In Figure 1 we show the evolution of the linear system. As y_2 is close to zero at all times it is clear the system remains close to equilibrium. In Figures 2 and 3 we show the evolution in the group of the full system (true evolution), the evolution of the geometric and dynamic phases, and the evolution of the reconstructed system. The results are expressed using the standard $SE(2)$ coordinates (ϕ, x, y). (The evolution of y is similar to that of x and thus for space reasons its plot is omitted.) The figures show, as expected, that the geometric phase is zero and that the reconstructed trajectory is a good approximation of the true trajectory.

6. CONCLUSIONS

In this paper we presented a theory describing geometric phases in dissipative systems with symmetry. By appealing to the framework of connections on fiber bundles, we were able to allow for non-abelian symmetry groups. It is hoped that

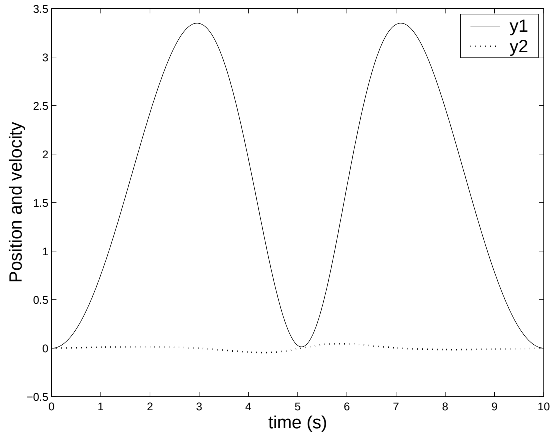

Fig. 1. Evolution of linear system

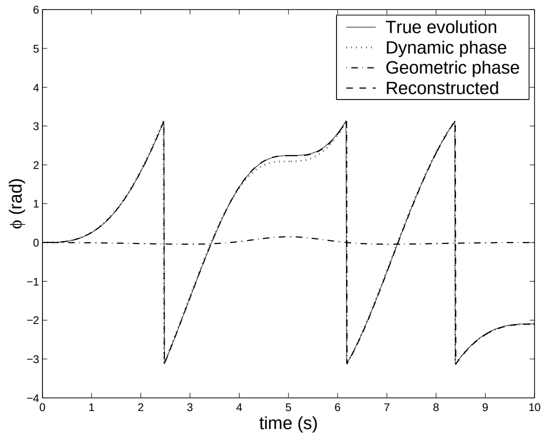

Fig. 2. Evolution of ϕ

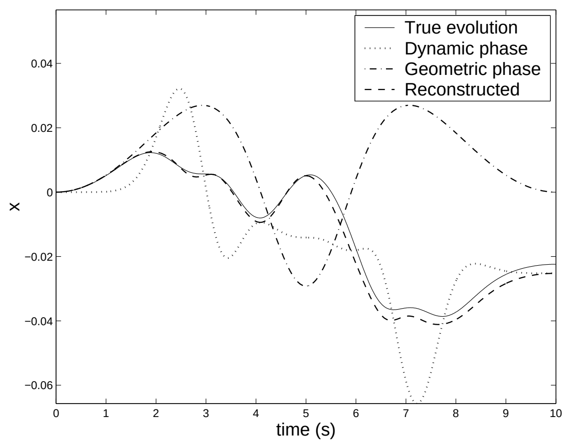

Fig. 3. Evolution of x

this work will prove useful in the control of dissipative systems with symmetry. One intriguing class of systems are Euler-Poincaré systems with double bracket dissipation where energy is dissipated but a Casimir functional of the angular momentum is conserved (Bloch *et al.*, 1996). Many interesting physical systems may be modelled in this way, including processes in ferromagnetics, geophysics, plasma physics, and stellar dynamics.

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