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Approximate Markovian Abstractions for Linear Stochastic Systems

M. Lahijanian, S. B. Andersson, and C. Belta

Abstract— **In this paper, we present a method to generate a finite Markovian abstraction for a discrete time linear stochastic system evolving in a full dimensional polytope. Our approach involves an adaptation of an existing approximate abstraction procedure combined with a bisimulation-like refinement algorithm. It proceeds by approximating the transition probabilities from one region to another by calculating the probability from a single representative point in the first region. We derive the exact bound of the approximation error and an explicit expression for its growth over time. To achieve a desired error value, we employ an adaptive refinement algorithm that takes advantage of the dynamics of the system. We demonstrate the performance of our method through simulations.**

I. Introduction

In recent years, there has been an increasing interest in formal approaches to analysis and control for dynamical systems [1], [2]. Central to these works is the notion of finite abstraction, i.e., the process through which a system with infinitely many states is mapped to an equivalent system with finitely many states. Bisimulation and language equivalence are the prevalent notions of equivalence between the original system and its abstraction [3]. In this framework, tools from model checking [4] and automata games [5] can be used to verify or control the abstraction from temporal logic specifications. The equivalence relation guarantees the satisfaction of the specification by the original system [6], or maps the game strategy for the abstraction to a provably-correct control strategy for the original system [7].

The notions of an approximate bisimulation and of a metric to quantify the distance between the original system and its approximate abstraction were introduced in [8]–[10]. This allowed a less rigid relationship between systems, which resulted in various abstraction techniques for more complex and stochastic systems. For stochastic systems, a probabilistic Markov model is often used as the abstraction. A traditional approach to obtain such an abstraction is to use Monte Carlo simulation techniques to approximate the relationships between the states and to augment the state space to achieve the Markovian property [11]. Recent works [12], [13] developed a different approach, in which stochastic hybrid systems were abstracted to Markov chains with approximation error, also known as Markov set-chains [14]. A bound on the approximation error was determined by using a Lipschitz continuity condition on the stochastic kernels of the underlying hybrid system. A grid-based algorithm was then employed to reduce this error to any desired level. The technique, however, used a conservative bound that in general leads to a higher cardinality on the abstraction than is necessary to achieve the desired error level.

In this paper, we consider the problem of finding a Markov chain abstraction for a stochastic system with bounded noise. This abstraction model allows for simple analysis and reasoning about the complex properties of the original system. In particular, this abstraction is required for formal verification of the continuous-domain stochastic system using the existing model checking tools such as PRISM [15]. We focus on stochastic linear systems evolving in polytopic domains with polytopic partitions and with noise bounds given by polyhedral sets. Similar to [12], [13], we use a representative point in each region to approximate the transition probabilities among the regions. We develop a computational framework that allows for the calculation of exact (i.e., achievable) bounds on the approximation error. To achieve a desired error bound, we employ a bisimulation-like refinement algorithm that determines the regions with no approximation error by using the dynamics of the system.

The contribution of this work is threefold. First, we relax the Lipschitz continuity assumption on the stochastic kernels posed by [12]. Second, by directly using the probability density functions of the kernels, we compute an exact value for the error, which leads to a tighter approximation error between the original system and its abstraction. Third, we develop an adaptive refinement algorithm that exploits the dynamics of the system and the geometry of the partition. Rather than uniformly refining the entire polytope as in [12], we refine only those regions with an approximation error above a given limit. The algorithm partitions such regions by determining and preserving any subregions with probability one transitions. If these regions do not exist, the algorithm uses the system dynamics to partition them with respect to the regions to which they have transitions. As opposed to [13], which also uses adaptive gridding, we do not use the region Lipschitz constants, and therefore do not require that the kernels are given by continuous densities.

The remainder of the paper is organized as follows. In Section II, we formulate the problem and outline our approach. In Section III, we formally define polytopes and introduce the polyhedral operators that are utilized in our solution. The abstraction procedure is discussed in Section IV. Section V presents the refinement algorithm that is designed to reduce the abstraction error. We demonstrate the performance of the proposed abstraction method through illustrative case studies in Section VI. The paper concludes with final remarks in Section VII.

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The authors are with the Department of Mechanical Engineering, Boston University, MA, USA, E-mail: morteza@bu.edu.

M. Lahijanian is the corresponding author.

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II. Problem Formulation and Approach

We consider the following discrete-time linear stochastic system

$$x_{k+1} = Ax_k + w_k, \;\; x_k \in \mathcal{P} \subset \mathbb{R}^n, \;\; w_k \in W \subset \mathbb{R}^n \quad (1)$$

where $A \in \mathbb{R}^{n\times n}$ is invertible, $\mathcal{P}$ is a full dimensional polytope in $\mathbb{R}^n$, $k \in \mathbb{N}$, and w_k is a sample from a given probability distribution over a polyhedral subset W of $\mathbb{R}^n$. Motivated by verification problems, we are interested in the semantics of system (1) with respect to a polytopic partition $S = \{s_1, s_2, \ldots, s_{m_S}\}$ of $\mathcal{P}$, where s_i, $i = 1, \ldots, m_S$ are convex full polytopes.

The focus in this paper is to construct a finite Markov chain (MC) approximating the evolution of system (1) in partition S with an error less than a predefined level. Our long term goal, which is beyond the scope of this paper, is to develop a computational tool for verification of (1) against specifications given as Probabilistic Computation Tree Logic (PCTL) [15] and Probabilistic Linear Temporal Logic (PLTL) [16] formulas over linear predicates in its state x. In this framework, S would be a partition of $\mathcal{P}$ that does not violate any of the linear predicates occurring in the temporal logic formulas. Once an approximating Markov chain (MC) abstracting the original system is constructed, off-the-shelf tools such as PRISM [15] can be used to verify it. The satisfaction probability for the abstraction, combined with the approximation error, would provide a satisfaction probability for the original system.

The abstraction that we propose in this paper follows the approach proposed in [12]. The state space of the MC is a finite set labeling the polytopes from the partition. The transition probabilities between the states of the MC can be approximated by selecting a set of representative points, one for each region in the partition. The computation is performed by marginalizing the distribution induced by the stochastic dynamics (1). Using a representative point is clearly an approximation and induces an error in the abstraction. In fact, this error causes a distance between the distribution of system (1) and the one of the MC. This distance can be thought of as the precision of the abstraction.

In this study, we consider the following problem:

Problem 1: Given system (1) with a polytopic partition S, find a refinement of S and a corresponding MC approximating system (1) within a given precision.

Our solution to Problem 1 is based on an iterative refinement process. The initial abstraction is constructed based on the initial partition S, and it is refined if the approximation error is not below the specified threshold. One of the contributions of this work is an exact upper bound for the error as well as for its growth over time. Refinement is performed selectively only for the regions violating the error threshold. Another contribution of this paper is a refinement algorithm based on polyhedral operations that exploits the dynamics of the system together with the polyhedral bounds on the noise.

III. Preliminaries

In this section, we formally define polytopes and introduce some polyhedral operations that are used in our refinement algorithm presented in Sec. V-A.

Let $n \in \mathbb{N}$ and consider the n-dimensional Euclidean space $\mathbb{R}^n$. A full dimensional *polytope* $\mathcal{P}$ is defined as the convex hull of at least $n+1$ affinely independent points in $\mathbb{R}^n$. A set of points $v_1^{\mathcal{P}}, \ldots, v_r^{\mathcal{P}} \in \mathbb{R}^n$, $r \geq n+1$ whose convex hull gives $\mathcal{P}$ and with the property that, for any $i = 1, \ldots, r$, the point $v_i^{\mathcal{P}}$ is not in the convex hull of the remaining points $v_1^{\mathcal{P}}, \ldots, v_{i-1}^{\mathcal{P}}, v_{i+1}^{\mathcal{P}}, \ldots, v_r^{\mathcal{P}}$ is called the set of vertices of $\mathcal{P}$. A polytope is completely described by its set of vertices.

$$\mathcal{P} = conv(v_1^{\mathcal{P}}, \ldots, v_r^{\mathcal{P}}), \quad (2)$$

where $conv$ denotes the convex hull. Alternatively, $\mathcal{P}$ can be described as the intersection of at least $n+1$ closed half spaces. In other words, there exists a $t \geq n+1$, $h_i \in \mathbb{R}^n$, and $l_i \in \mathbb{R}$, $i = 1, \ldots, t$ such that

$$\mathcal{P} \;\; = \;\; \{x \in \mathbb{R}^n | h_i^T x \leq l_i, \, i = 1, \ldots, t\}. \quad (3)$$

The above definition can be written as the matrix inequality $Hx \leq L$, where $H \in \mathbb{R}^{t\times n}$, $L \in \mathbb{R}^t$, and the ith row of H and the ith entry of L are h_i^T and l_i, respectively. Forms (2) and (3) are referred to as V- and H-representations of the polytope, respectively.

Given a linear stochastic system (1) and an arbitrary polytope $\mathcal{P}$, we define $Pre(\mathcal{P})$ as the set of all points that make a transition to $\mathcal{P}$ for *some* values of w in one time step:

$$Pre(\mathcal{P}) = \{x \in \mathbb{R}^n | \exists w \in W, \; (Ax + w) \in \mathcal{P}\}. \quad (4)$$

Similarly, we define $PreR(\mathcal{P})$ (robust $Pre(R)$) to be the set of all points that make a transition to $\mathcal{P}$ for *all* possible values of w in one time step:

$$PreR(\mathcal{P}) = \{x \in \mathbb{R}^n | \forall w \in W, \; (Ax + w) \in \mathcal{P}\}. \quad (5)$$

It is important to note that $Pre(\mathcal{P})$ is the set of points that have a non-zero probability of making a transition to region $\mathcal{P}$, while $PreR(\mathcal{P})$ is the set of all the points that make a transition to $\mathcal{P}$ with probability one.

We use $Post(\mathcal{P})$ to denote the set of points that can be reached from $\mathcal{P}$ in one step for some values of w, *i.e.*,

$$Post(\mathcal{P}) = \{x \in \mathbb{R}^n | \text{ for each } x' \in \mathcal{P}, \exists w \in W, \; x = (Ax' + w)\}. \quad (6)$$

If polytope $\mathcal{P}$ has V-form $\mathcal{P} = conv(v_1^{\mathcal{P}}, \ldots, v_r^{\mathcal{P}})$ and matrix form $Hx \leq L$, and polytope W has V-form $W = conv(v_1^W, \ldots, v_{m_W}^W)$, then the above sets are all convex polytopes that can be computed as (see [17], [18]):

$$\begin{aligned} Pre(\mathcal{P}) &= conv(\{Av_i^{\mathcal{P}} + v_j^W, \text{ for all } i \leq r, \, j \leq m_W\}), \\ Post(\mathcal{P}) &= conv(\{A^{-1}(v_i^{\mathcal{P}} - v_j^W), \text{ for all } i \leq r, \; j \leq m_W\}), \\ PreR(\mathcal{P}) &= \{x \in \mathbb{R}^n | H_R x \leq L_{R_i}, \, i = 1, \ldots, m_W\}, \end{aligned} \quad (7)$$

where $H_R = HA$ and $L_{R_i} = L - Hv_i^W$.

IV. ABSTRACTION

In this section, we describe our method of generating a Markov chain abstraction for stochastic system (1) evolving in a polytopic partitioned domain $\mathcal{P}$. This approach is an adaptation of the general abstraction framework for stochastic hybrid systems presented in [12]. More specifically, we employ the method of approximating the transition probabilities using a representative point from that framework. In our approach, however, we quantify the exact bound on the one-step transition error by directly using the distribution of the stochastic system instead of the conservative bounds computed in [12] by assuming Lipschitz continuity distribution for the underlying system. It is important to note that we do not exploit the linear dynamics and polytopic noise bound of system (1) in the abstraction approach (we do, however, exploit these features in the refinement process described in Sec. V). Thus, this method can be applied to any discrete-time stochastic system with a well-defined distribution.

Let $Q = \{q_1, \ldots, q_{m_Q}\}$, for some $m_Q \in \mathbb{N}$, denote a refinement of S. We choose Q as the set of states for the MC. Next, we describe how to approximate the transition probabilities between the regions in Q.

A. Calculating Transition Probabilities

To compute the transition probabilities, we assume that the stochastic system evolves in $\mathcal{P}$ with a Borel-measurable stochastic kernel on $\mathcal{P}$ given by $T : \mathcal{B}(\mathcal{P}) \times \mathcal{P} \to [0, 1]$. The stochastic kernel assigns to each $x \in \mathcal{P}$ a probability measure $T(\cdot|x)$ on the Borel space $(\mathcal{P}, \mathcal{B}(\mathcal{P}))$.

We denote the one-step transition probability from a point in q_i to q_j by $p(q_j|q_i)$, *i.e.*

$$p(q_j|q_i) = Prob(x_{k+1} \in q_j | x_k \in q_i), \tag{8}$$

where $q_i, q_j \in Q$. Let us select any point $\bar{x}_i \in q_i$ to be the representative point of region q_i (*e.g.* geometric center of q_i). We denote the transition probability from $\bar{x}_i$, to q_j by

$$\bar{p}(q_j|q_i) = Prob(x_{k+1} \in q_j | x_k = \bar{x}_i). \tag{9}$$

These transition probabilities depend on the kernel T and can be obtained by marginalizing the probability distribution of (1) over each polytope in the set Q (see Sec. IV-B). We use the value in (9) to approximate the transition probability of $p(q_j|q_i)$. In other words, we construct the transition probability matrix of the MC using (9).

The difference between the values in (8) and (9) introduces an error between the distribution of the MC and the one of the underlying system. The value of the upper-bound for this error defines the abstraction precision and depends on the density of the kernel T and the representative point. Thus, it is important to choose a representative point that minimizes the error given T. The method of selecting this optimal representative point is beyond the scope of this paper. However, we quantify the precision of the abstraction by finding the exact upper-bound for the error for any choice of the representative point.

B. 1-Step Error

We define and bound the one-step error e_1 as follows.

$$\begin{aligned} e_1 &= |p(q_j|q_i) - \bar{p}(q_j|q_i)| \\ &= \left| \int_{q_j} T(dx_{k+1}|x_k \in q_i) - \int_{q_j} T(dx_{k+1}|\bar{x}_i) \right| \\ &\leq \max_{x_k \in q_i} \left| \int_{q_j} T(dx_{k+1}|x_k \in q_i) - \int_{q_j} T(dx_{k+1}|\bar{x}_i) \right| \end{aligned}$$

This bound on e_1 is exact in the sense that there exists some $x_k \in q_i$ such that the error is equal to that bound. For simplicity, let $\bar{p}_{ij} = \bar{p}(q_j|q_i)$. Since the stochastic transition kernel can be derived from the dynamics of the underlying system, and subpolytopes q_j are well-defined convex regions, the above integral is computable. Define ε from $e_1 \leq \varepsilon$. Using the inequality above, the bound ε can be written as

$$\begin{aligned} \varepsilon = \max \Bigg\{ & \bar{p}_{ij} - \min_{x_k \in q_i} \int_{q_j} T(dx_{k+1}|x_k \in q_i) \ , \\ & \max_{x_k \in q_i} \int_{q_j} T(dx_{k+1}|x_k \in q_i) - \bar{p}_{ij} \Bigg\} . \end{aligned} \tag{10}$$

Next, we examine the evolution of error over time.

C. k-Step Error

Let us denote the probability that the state after k steps will be in q_j given an initial condition in q_i by $p^k(q_j|q_i) = Prob(x_{k_0+k} \in q_j | x_{k_0} \in q_i)$ for any integers $k_0 \geq 0$ and $k \geq 2$. Similarly, $\bar{p}^k(q_j|q_i) = Prob(x_{k_0+k} \in q_j | x_{k_0} = \bar{x}_i)$. The distribution $p^k(q_j|q_i)$ can be associated with the actual trajectory of the stochastic system evolving on Q. The quantity $\bar{p}^k(q_j|q_i)$ is the distribution over the state space of Q generated by the MC. We are interested in quantifying the error between these two distributions as time progresses.

Let us define this error at the k-th time step as

$$e_k = \left| p^k(q_j|q_i) - \bar{p}^k(q_j|q_i) \right| .$$

The following theorem determines an upper-bound for the growth rate of the error over time.

Theorem 1: Consider a discrete stochastic system evolving in a domain partitioned into m_Q regions $Q = \{q_1, q_2, \ldots, q_{m_Q}\}$. The k-th step error between the probability distribution of the system over Q and that of the approximating Markov chain generated by using the representative points for each region in Q is upper-bounded by

$$e_k \leq N^{k-2}(N + k - 1)\varepsilon, \tag{11}$$

where $k \geq 2$ is the time index, ε is the upper-bound of the 1-step error, and N is the upper-bound of the maximum number of outgoing transitions from q_i, $i = 1, 2, \ldots, m_Q$.

Proof: We prove this theorem by induction. For $k = 2$ and any $q_i, q_j \in Q$, we have:

$$\begin{aligned} e_2 &= \left|p^2(x_2 \in q_j | x_0 \in q_i) - \bar{p}^2(x_2 \in q_j | x_0 = \bar{x}_i)\right| \\ &\leq \sum_{b=1}^{m_Q} |p(x_2 \in q_j | x_1 \in q_b) p(x_1 \in q_b | x_0 \in q_i) \\ &\quad -\bar{p}(x_2 \in q_j | x_1 = \bar{x}_b) \bar{p}(x_1 \in q_b | x_0 = \bar{x}_i)| \\ &= \sum_{b=1}^{m_Q} [p(x_2 \in q_j | x_1 \in q_b)\, |p(x_1 \in q_b | x_0 \in q_i) \\ &\quad -\bar{p}(x_1 \in q_b | x_0 = \bar{x}_i)| + \bar{p}(x_1 \in q_b | x_0 = \bar{x}_i) \\ &\quad |p(x_2 \in q_j | x_1 \in q_b) - \bar{p}(x_2 \in q_j | x_1 = \bar{x}_b)|] \\ &= \varepsilon \sum_{b=1}^{m_Q} \left[p(x_2 \in q_j | x_1 \in q_b) + \bar{p}(x_1 \in q_b | x_0 = \bar{x}_i)\right]. \end{aligned}$$

It is evident that $\sum_{b=1}^{m_Q} p(x_2 \in q_j | x_1 \in q_b) \leq N$ and $\sum_{b=1}^{m_Q} \bar{p}(x_1 \in q_b | x_0 = \bar{x}_i) = 1$. Thus, $e_2 \leq (N+1)\varepsilon$. The induction step, that is proving that (11) holds for $k+1$ given that it holds for k, is similar to the $k = 2$ case and is omitted for due to space limitations. ■

V. Refinement

In this section, we describe the refinement algorithm that exploits the dynamics of the system and the geometry of the partition. Recall that as the first step of the abstraction procedure, the approximation error bounds over the transition probabilities corresponding to each region are computed. For any region whose error is larger than a given desired value, the following refinement sequence is performed. The region with the largest error bound is selected as the target polytope. This polytope is then partitioned into smaller regions according to the refinement algorithm described in Sec. V-A. This refinement affects not only the target polytope but also those to which it has a transition. After refinement, the error bounds are recomputed. If the error bounds are still too large, a new target polytope is selected and the process is repeated until all of the error bounds are less than the desired value.

We note that this procedure is *adaptive* in that it targets only those regions whose error bounds are too large rather than refining all regions equally.

A. Refinement Algorithm

Our refinement algorithm consists of two main functions, both of which utilize the polyhedral operators $PreR$, Pre, and $Post$ defined in Sec. III. The first function identifies the regions within the target polytope that have outgoing transitions with probability one for all their points. Since these subregions have an error of zero, the refinement of the target polytope does not modify them. The second function refines the target polytope with respect to the Pre of the adjacent regions. The refinement algorithm calls the second function only if no region with probability one transition is identified; $PreR$ of all adjacent regions returns an empty set.

To find the regions with probability one transitions within a target polytope, we perform the $PreR$ operation on the adjacent regions. Recall from (7) that $PreR$ returns a convex polytope. By definition of $PreR$, these are the regions with zero error; they are included in the refinement. The $Post$ of these regions is then found and these subregions are also included in the refinement to ensure the probability one transitions are maintained. These two operations carve a new set of subpolytopes out of the existing ones. The remaining regions of the affected subpolytopes (the target and any regions to which probability one areas move) are then convexified.

In the case that regions with zero error do not exist (*i.e.* $PreR$ of the adjacent regions returns an empty set), the algorithm refines the target polytope using the Pre operator of its adjacent subpolytopes. Finally, if the target region cannot be partitioned into smaller regions using Pre (*i.e.* intersection of the Pre of the adjacent subpolytopes with the target region is the target region itself), it is refined by triangulation.

VI. Case Studies

In this section, we demonstrate the abstraction method and the refinement algorithms proposed in this paper through illustrative case studies. We analyzed the performance of the methods in two scenarios. In the first case, we assumed unbounded noise given by a normal distribution. In this framework, we compared the performance of the error calculation method presented in this paper with the one in [12]. However, since our refinement algorithm requires the noise to be bounded, we employed a grid-base refinement method instead for this example as proposed by [12]. In the second case, we demonstrated our abstraction algorithm including the refinement algorithm presented in Sec. V-A by assuming a truncated normal distribution for the noise term.

A. Unbounded Noise

For the case of unbounded noise, we considered a square domain $\mathcal{P}$ centered at the origin with the length of 2 per side. The dynamics were given by (1) with

$$A = \begin{pmatrix} 0.4 & 0.1 \\ 0 & 0.5 \end{pmatrix}, \quad w \sim \mathcal{N}(0, 0.04\mathbb{I}),$$

where $\mathbb{I}$ is the identity matrix, .

As the first step in the abstraction, we chose the geometric center of each subpolytope as the representative point. Then, we computed the approximation error using two methods: (1) the exact method described in this paper, and (2) the method from [12]. The latter utilizes a Lipschitz condition on the stochastic dynamics to establish an upper bound. In order to be able to compare the results, we employed a grid-based refinement method rather than the adaptive scheme described in Sec. V-A. Grid-based refinement simply subdivides each region into four equally-sized squares (right) until the error is below a given bound.

From the system dynamics, the probability distribution of x_{k+1} is $\mathcal{N}(Ax_k, 0.04\mathbb{I})$. Letting f denote the normal density formula, the continuous transition kernel is $T(\cdot|x) = f(\cdot; Ax, 0.04\mathbb{I})$ and the transition probability $\bar{p}_{ij} = \int_{q_j} f(x; A\bar{x}_i, 0.04\mathbb{I})dx$.

1) Exact Method: To calculate the exact error bound ε, we found the minimum and maximum transition probabilities in (10) as described below. It can be shown that the point that maximizes $p(q_j|q_i)$ is $A^{-1}\bar{x}_j$ if this point is in q_i. If it is not, then the point that maximizes $p(q_j|q_i)$ is the one on the boundary of q_i with the shortest Euclidean distance to $\bar{x}_j$ after one step evolution. Similarly, the point that minimizes $p(q_j|q_i)$ always lives on the facet of q_i that is the farthest from $\bar{x}_j$ after one step evolution. The proof of this result is omitted for space reasons.

For this example, the above result was used to calculate the maximum and minimum value for the probability $p(q_j|q_i)$. The value of ε was then found through (10). The error as a function of the cardinality of Q is shown in Fig. 1.

2) Lipschitz Method: To implement the abstraction method in [12], we use the following assumption.

Assumption 1 (Continuity of Stochastic Kernels): Suppose that the continuous stochastic kernel T of the system (1) admit a density F_T. Assume that the following Lipschitz property holds:

$$|F_T(y|x) - F_T(y|x')| \leq L\|x - x'\|, \quad \forall x, x', y \in \mathcal{P},$$

where L is a finite positive constant, and $\|\cdot\|$ is the Euclidean norm. ■

Given a grid partition of the state space and Assumption 1, the authors in [12] show that the one-step error is upper-bounded by $e_1 \leq \delta^{n+1}L$, where δ is the diagonal of each square cell, n is the dimension of vector x, and L is the Lipschitz constant.

In this example, the continuous stochastic kernel is given by a normal density function. Hence, Assumption 1 holds, and the Lipschitz constant is bounded by

$$L \leq \frac{1}{(2\pi)^{\frac{n}{2}} 0.2^{n+1}} \|A\|_2 e^{-\frac{1}{2}},$$

where $\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$ is the matrix 2-norm. For the proof of the above expression for the Lipschitz constant see [19].

The error as a function of cardinality is shown in Fig. 1, together with the exact bound found by our method. While the two error calculations converge as the cardinality increases, there is a significant different at lower cardinalities (or, equivalently, larger subpolytopes). The result illustrates that for a given error level, the exact bound will uniformly yield an abstraction with lower cardinality.

The precision in the error calculation comes at a cost. Fig. 2 shows the ratio of the computation times for the two methods. The approximate method involves only the calculation of the diagonal length of the cells in the refinement and is thus constant as the cardinality increases. The exact method, however, requires the evaluation of the probability density function over the polyedral regions.

B. Bounded Noise

For the case of bounded noise, we considered the same A matrix for the stochastic system as used above. The

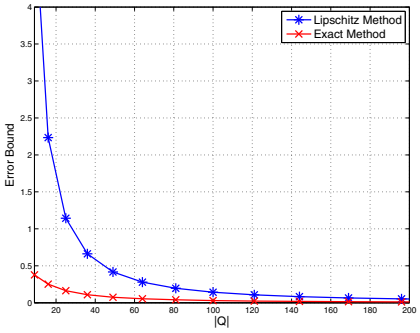

Fig. 1. Comparison of the error bounds computed through the exact method described in Sec. IV of this paper with the Lipschitz method [12]. The exact method uniformly finds tighter bounds than the Lipschitz method.

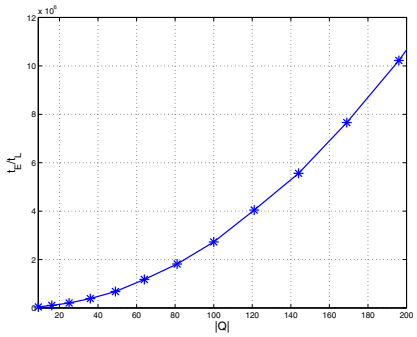

Fig. 2. Ratio of the computation times for the error bounds using the method presented in this paper (exact) to the Lipschitz method [12].

polyhedral set W was defined by

$$W = conv\left(\begin{bmatrix} 0.4 \\ 0.4 \end{bmatrix}, \begin{bmatrix} 0.4 \\ -0.4 \end{bmatrix}, \begin{bmatrix} -0.4 \\ -0.4 \end{bmatrix}, \begin{bmatrix} -0.4 \\ 0.4 \end{bmatrix}\right).$$

The random variable w in this case was given by the truncated normal density function

$$g(x; W, 0, 0.04\mathbb{I}) = \begin{cases} \frac{f(x;0,0.09\mathbb{I})}{\int_W f(y;0,0.09\mathbb{I})dy} & \text{if } x \in W, \\ 0 & \text{Otherwise.} \end{cases}$$

Here f is a Gaussian distribution with a variance of 0.09 chosen to ensure that the truncated distribution g has a variance of 0.04. The domain polytope $\mathcal{P}$ is given by the vertices $[2, 2]^T$, $[2, -2]^T$, $[-2, -2]^T$, and $[-2, 2]^T$ partitioned by the 9 linear predicates $x_1 \geq 1$, $x_1 \geq 0$, $x_1 \geq -1$, $x_2 \geq 1$, $x_2 \geq 0$, and $x_2 \geq -1$ inducing 16 subpolytopes.

Here, we constructed a MC using the geometric center of each subpolytope as the representative point. We computed the error of the approximation using the exact method and reduced it by the refinement algorithm described in Sec. V-A. For this system, the transition probability $\bar{p}_{ij} = \int_{q_j} g(x; W, A\bar{x}_i, 0.04\mathbb{I})dx$. We found the error of the abstraction ε by discretizing q_i and evaluating (10) for each point in q_i. Then, we determined the region with the largest error and refined it using our adaptive algorithm to decrease the error.

The value of the error after each iteration of the algorithm is shown in Fig. 3. For the initial partition of $\mathcal{P}$, the method

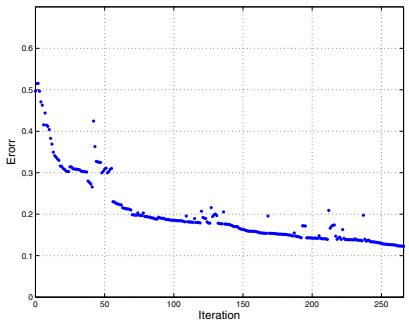

Fig. 3. Error bounds as a function of the number of refinement iteration.

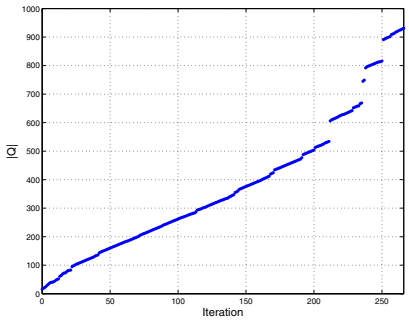

Fig. 4. Cardinality of the state space as a function of the number of refinement iteration.

yields a Markov chain with the approximation error of 0.50. After the first iteration of the refinement, the error actually increased. That is because by refining the target region, there is a chance of increasing the error corresponding to the regions that have transitions to the target region. However, this jump in the error value was quickly smoothened by more refinement iterations since the target region is always partitioned with respect to the regions that it has transitions to. Thus, the overall value of the error generally decreases with more refinement iterations as Fig. 3 illustrates. These sudden jumps are also observed in the number of regions (cardinality of Q) as shown in Fig. 4. This is due to partitioning of a (large) region which has transitions to many (smaller) regions. Since the refinement is done by the Pre of the regions to which the target region has transitions to, one iteration of the algorithm could result in a sudden increase in the number of regions.

VII. Conclusion

In this paper, we introduced a method for finite Markovian abstraction for a discrete-time linear stochastic system in a full-dimensional convex polytope with a polytopic partition. This method is an adaption of the approach presented in [12] to the framework of linear stochastic systems with polyhedral noise bounds. The main contributions of this work are an exact bound for the approximation error, an expression for the error growth over time, and an efficient refinement algorithm that exploits the linearity of the dynamics and of the partition.

Our case studies suggest that, in general, our method leads to abstractions with smaller cardinality for the same desired error level when compared to the method from [12]. However, this comes at the cost of computation time since the error must be determined by computing a set of integrals numerically.

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