Minimum Time Control of A Second-Order System

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Abstract—A new algorithm for determining the switching time and final time for the minimum-time control of a second-order system is described. We show that if there is only one switch in the bang-bang control, then the switching time and final time are related through an affine mapping. This mapping is determined by the system dynamics, the initial and target states, and the control bounds on the control input. The resulting time-optimal controller is easy to design and implement, making it suitable for online implementation. Since the algorithm also produces the final time, it becomes feasible to optimize the switching sequence over a collection of states to be visited so as to minimize the total visiting time.

I. INTRODUCTION

It is common to take measurements using a point or short-range sensor in many applications. For example, in scanning probe microscopy (SPM), the interaction between the probe and the sample surface is local. To image the sample, the generic approach is to raster scan the probe over the region of interest and rebuild the sample image pixel by pixel using the data collected on the regular grid. In recent years, non-raster scanning approaches have been proposed by researchers to reduce the imaging time by reducing the area to be sampled [1], [2]. Under such schemes, the next sampling point is not necessarily near the current position of the probe. To collect the data as quickly as possible, one should move the probe from rest to rest in minimum-time.

It is well known that by applying Pontryagin’s maximum principle (PMP) [3] or Bellman’s principle of optimality [4], it can be proved that the solution to the time-optimal control problem of a linear system with bounded control is always a bang-bang control law [3]. Though bang-bang control can transfer the system from an initial state to a target state in minimum time by simply switching the control between the minimum and maximum value, it is not practical in most applications because of its sensitivity to disturbance, parameters variation and unmodeled dynamics [5].

Motivated by the above limitations, researchers have been striving to increase the robustness of the controllers while maintaining approximate time optimal performance for over forty years. The proximate time optimal servomechanism (PTOS) was developed for second-order and third-order system [5], [6]. Extended PTOS (XPTOS) and adaptive PTOS (APTOS) were proposed to address flexible dynamics in the system [7]–[9]. Recently, the shaped time-optimal servomechanism (STOS) was developed to modify the bang-bang control signal before it was applied to the system to eliminate the residual vibration due to the flexible modes [10], [11]. A comparison of input shaping and time-optimal flexible-body control was presented in [12]. The robust and state-constrained time optimal control problem were also considered in [13]–[15]. Other approaches have also been proposed by researchers to achieve proximate time-optimal output transition. The feedforward minimum-time control of non-minimum-phase linear scalar systems for set-point regulation was presented in [16] by using the stable input-output dynamic inversion technique. The linear quadratic minimum time (LQMT) output-transition problem was also solved in [17] with energy constraints and pre- and post-actuation. The near time-optimal controller for nonlinear second order systems was also presented in [18], [19].

The above controllers have been applied successfully in many applications, such as to disk drive systems [10], [14], [20]. However, they do not provide an estimate of the transition time from one state to another. In applications that include a collection of states to be visited by the controller, such information is useful. A primary motivating example is that of 3-D particle tracking in a confocal microscope [21]–[23]. In this application, a collection of measurements is needed to estimate the position of the particle and the overall tracking speed is limited in part by how fast the system can move through the sequence of positions. To improve scanning speed and measurement accuracy, one not only needs to achieve the minimum-time transition from one state to another, but also to optimize the visiting sequence over all states that need to be visited. This requires us to estimate the transition time online between any two states before they are actually visited. In this paper, we developed a new method to numerically calculate online the switching and final time of a bang-bang control law for the case where only one switch is needed to transfer a stable second-order system optimally from one state to another.

This paper is organized as follows. In Section II, the solution of the time-optimal control problem is given. In Section III, it is shown that the solution of the switching and final time is equivalent to finding a crossing point of two spiral curves under an affine mapping. From this mapping, the switching and final time can be calculated quickly based on the geometry of the mapping. In Section IV, simulation and experimental results are shown to demonstrate this approach. A brief discussion and concluding remarks are made in Section V.

II. PROBLEM FORMATION

A. System Model

Consider a stable second-order system transfer function

\[
\frac{X(s)}{U(s)} = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2}, \tag{1}
\]

where \(b_1\), \(b_2\), \(a_1\), and \(a_2\) are constants. This transfer function can be written in the form of the system equation:

\[
x''(t) + a_1 x'(t) + a_2 x(t) = b_1 u'(t) + b_2 u(t),
\]

where \(x(t)\) is the state of the system, \(u(t)\) is the control input, and \(x'(t)\) and \(x''(t)\) are the first and second derivatives of \(x(t)\) with respect to time, respectively.

The minimum time control problem is to find the control input \(u(t)\) such that the state \(x(t)\) transitions from an initial state \(x_i\) to a target state \(x_f\) in the minimum time possible, subject to the system equation and control constraints. The control constraints are typically bounded, such as \(|u(t)| \leq u_{max}\).

The goal is to find a control input that satisfies the system equation and control constraints and minimizes the total time of transition from \(x_i\) to \(x_f\). This problem can be formulated as a time-optimal control problem, which is a special case of the optimal control problem. The solution to the time-optimal control problem will provide the switching times and final time for the minimum-time control of a second-order system.

The minimum-time control problem is a non-linear optimization problem due to the non-linear system dynamics. The solution to this problem can be obtained using numerical optimization techniques, such as gradient-based methods or heuristic methods. Alternatively, analytical solutions can be obtained for special cases, such as linear systems or systems with specific constraints.

The minimum-time control problem is an important problem in control theory and has applications in various fields, such as robotics, aerospace engineering, and manufacturing systems. The solution to the minimum-time control problem provides a fundamental understanding of the behavior of the system and helps in designing efficient control strategies.
where \( x_i \in \mathbb{R} \) is the system position and \( u \in [u_{\text{min}}, u_{\text{max}}] \subset \mathbb{R} \) is the (bounded) input control signal. Define the system states as

\[
\begin{align*}
  x_1 &= x_2, \\
  x_2 &= x_1 + \beta_1 u. 
\end{align*}
\]

Then the canonical controllable state-space model is given by

\[
\dot{X} = AX + Bu,
\]

where 

\[
X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.
\]

\( \beta_i \) is given implicitly by

\[
\begin{bmatrix} 0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_2 & a_1 & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}.
\]

We assume that the eigenvalues of \( A \) are given by

\[
\lambda_{1,2} = -\frac{a_1}{2} \pm i\omega
\]

where \( \omega \) can be calculated as

\[
\omega = \sqrt{4a_2 - a_1^2}
\]

and assume that \( \omega \neq 0 \). (We note that the case of two pure real eigenvalues is a simpler version of the scheme presented here.) With these assumptions, the eigenvectors of \( A \) are

\[
P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}.
\]

Thus,

\[
P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.
\]

\[ \text{B. Minimum Time Control} \]

We consider the following problem

\textit{Problem 1:} Given the state-space model (3) of system (1) and the initial system state \( X_0 \), design a control law to drive system (3) to the target state \( X_f \) in minimum time.

To solve this problem, we define a new system state \( \dot{X} \) and rewrite system (3) and the initial state \( X_0 \) as

\[
\begin{align*}
  \dot{X} &= X - X_f, \\
  \dot{\dot{X}} &= AX + Bu + AX_f, \\
  \dot{X}_0 &= X_0 - X_f. 
\end{align*}
\]

Then Problem 1 can be stated equivalently as:

\textit{Problem 2:} Given system (10), design a control law to drive the system from the initial state \( \dot{X}_0 \) to the origin in minimum time.

This problem can be solved by using the PMP [3]. To minimize the switching time, the cost function is taken as

\[
J = \int_0^{t_f} 1 \, dt.
\]

Now, we apply the PMP as follows. Define the Hamiltonian of (10) as

\[
H = 1 + \lambda^T (AX + Bu + AX_r),
\]

where \( \lambda \) is the state of the adjoint system of (10). The combined system is then given by

\[
\begin{align*}
  \dot{X} &= \frac{\partial H}{\partial \lambda} = AX + Bu + AX_r, \\
  \dot{\lambda} &= -\frac{\partial H}{\partial X} = -A^T \lambda,
\end{align*}
\]

with the additional condition that

\[
H(t_f) = 1 + \lambda(t_f)^T (AX(t_f) + Bu(t_f) + AX_f) = 0.
\]

(Note that while 14 is true for all time, we will explicitly use it at the final time.) Since the target state is the origin, we have

\[
\dot{X}(t_f) = 0,
\]

yielding, from (14),

\[
1 + \lambda(t_f)^T (Bu(t_f) + AX_f) = 0.
\]

A control signal giving the minimum time is found by minimizing the Hamiltonian in (12). This yields

\[
u(t) = \begin{cases} u_{\text{min}}, & \text{if } \lambda^T B > 0, \\ u_{\text{max}}, & \text{if } \lambda^T B < 0, \end{cases}
\]

with the value of \( u \) arbitrary when \( \lambda^T B = 0 \). This is the bang-bang control law for the linear system. In general, there is no analytical solution for the combined system in (13) due to the lack of sufficient boundary conditions on the adjoint system. As a result, there is also no analytical expression for the switching function \( \lambda^T B \) in the bang-bang control law (17). Though one can numerically solve \( \lambda \) with the gradient methods provided in [3], [24], the difficulty in making a good and meaningful initial guess of the costate \( \lambda \) and the time-consuming nature make it not practical for online implementation. Moreover, it does not provide the switching and final time explicitly. Such information is critical to optimize the total visiting time when moving through a collection of states.

\[ \text{III. Switching Time and Final Time} \]

To simplify the presentation, we make the following definitions and assumptions for the calculation of the switching and final time. Define:

\[
\begin{align*}
  t_s &: \text{switching time,} \\
  t_f &: \text{final time,} \\
  \Phi(t, \tau) &: \text{state transition matrix.}
\end{align*}
\]

We assume that there is only one switch in the control. For concreteness, we arbitrarily assume that the control signal is \( u_{\text{max}} \) at the beginning of the bang-bang control, switching to \( u_{\text{min}} \) at \( t_s \) and staying at \( u_{\text{min}} \) until the final time \( t_f \). (The
other cases follow in similar fashion.) Define variables \( v, z, m, n, \hat{x}_{01} \) and \( \hat{x}_{02} \) as follows:

\[
egin{align*}
AX_r + Bu_{\text{max}} &= \begin{bmatrix} v & z \end{bmatrix}, \\
AX_r + Bu_{\text{min}} &= \begin{bmatrix} m & n \end{bmatrix}, \\
\hat{X}_0 &= \begin{bmatrix} \hat{x}_{01} & \hat{x}_{02} \end{bmatrix}^T.
\end{align*}
\]  

(19a)

(19b)

(19c)

From the variation of constants formula, the system state at time \( t_s \) is given by

\[
\hat{X}(t_s) = \Phi(t_s, 0)\hat{X}_0 + \int_0^{t_s} \Phi(t_s, t) \begin{bmatrix} v \\ z \end{bmatrix} dt.
\]

(20)

From time \( t_s \) to \( t_f \), the system is driven by the control signal \( u_{\text{min}} \) with initial state \( \hat{X}(t_s) \). So, the system state at time \( t_f \) is given by

\[
\hat{X}(t_f) = \Phi(t_f - t_s, 0)\hat{X}(t_s) + \int_{t_s}^{t_f} \Phi(t_f - t_s, t) \begin{bmatrix} m \\ n \end{bmatrix} dt.
\]

(21)

Substituting (15) into (21) and multiplying both sides by \( \Phi(t_f, t_s) \) yields

\[
-\int_0^{t_f - t_s} \Phi(0, t) \begin{bmatrix} m \\ n \end{bmatrix} dt = \Phi(t_f, 0)\hat{X}_0 + \int_0^{t_f} \Phi(t_f, t) \begin{bmatrix} v \\ z \end{bmatrix} dt.
\]

(22)

Now, consider the three terms in (22). From (9), it can be shown that

\[
\Phi(t_f, t) = P e^{A(t_f - t)} P^{-1},
\]

(23)

so that

\[
\int_0^{t_f} \Phi(t_f, t) dt = P \begin{bmatrix} \frac{1}{A} & 0 \\ 0 & \frac{1}{A} \end{bmatrix} e^{A(t_f - t_s)} P^{-1},
\]

(24)

and

\[
-\int_0^{t_f - t_s} \Phi(0, t) dt = P \begin{bmatrix} \frac{1}{A} & 0 \\ 0 & \frac{1}{A} \end{bmatrix} e^{A(t_f - t_s)} P^{-1}.
\]

(25)

Define

\[
\tau = t_f - t_s, \quad R(t) = e^{-\frac{A}{2}t}.
\]

(26)

Substituting (6) and (8) into (23), (24) and (25) and carrying out some straightforward but tedious calculations yields

\[
\Phi(t_f, 0)\hat{X}_0 = \begin{bmatrix} \hat{x}_{01} \cos(\omega \tau) + \frac{a_1 \hat{x}_{01} + 2\hat{x}_{02}}{2\omega} \sin(\omega \tau) \\ \hat{x}_{02} \cos(\omega \tau) - \frac{2\hat{x}_{01} + a_1 \hat{x}_{02}}{2\omega} \sin(\omega \tau) \end{bmatrix} R(\tau),
\]

(27)

\[
\int_0^{t_f} \Phi(t_f, t) \begin{bmatrix} v \\ z \end{bmatrix} dt = \begin{bmatrix} \frac{a_{11} + z}{a_2} \\ -v \end{bmatrix} +
\]

\[
\begin{bmatrix} -\frac{a_{11} + z}{a_2} \cos(\omega \tau) + \frac{v(4\omega^2 - a_1^2) - 2a_1 z}{4a_2^{2\omega}} \sin(\omega \tau) \\ v \cos(\omega \tau) + \frac{a_{11} + z}{2a_2} \sin(\omega \tau) \end{bmatrix} R(\tau),
\]

(28)

Now, substitute (27), (28) and (29) into (22) and define

\[
X(t) = R(t) \cos(\omega t), \quad Y(t) = R(t) \sin(\omega t),
\]

(30)

\[
A_1 = \begin{bmatrix} \frac{a_1 \hat{x}_{01} + 2\hat{x}_{02}}{2\omega} \\ -\frac{2\hat{x}_{01} + a_1 \hat{x}_{02}}{2\omega} \end{bmatrix},
\]

(31)

\[
A_2 = \begin{bmatrix} \frac{a_1 \hat{x}_{01} + 2\hat{x}_{02}}{2\omega} \\ v \end{bmatrix},
\]

(32)

\[
A_3 = \begin{bmatrix} \frac{a_1 \hat{x}_{01} + 2\hat{x}_{02}}{2\omega} \\ m \end{bmatrix},
\]

(33)

With these definitions, (22) can be written as

\[
A_1 \begin{bmatrix} X(t_f) \\ Y(t_f) \end{bmatrix} + A_2 \begin{bmatrix} X(t_f) \\ Y(t_f) \end{bmatrix} + B_2 = A_3 \begin{bmatrix} X(-t) \\ Y(-t) \end{bmatrix} + B_3.
\]

(34)

Finally let

\[
A_0 = A_3^{-1}(A_1 + A_2),
\]

(35)

\[
B_0 = A_3^{-1}(B_2 - B_3).
\]

(36)

Then (34) can be rewritten as

\[
A_0 \begin{bmatrix} X(t_f) \\ Y(t_f) \end{bmatrix} + B_0 = \begin{bmatrix} X(-t) \\ Y(-t) \end{bmatrix}.
\]

(36)

The switching and final time of the bang-bang control can be calculated by solving (36) for \( t_s \) and \( t_f \). This can be interpreted as finding the crossing point of the curves defined by the two sides of (36). Note that the left hand-side is an affine transformation of the curve on the right hand side. From (36), we see one curve is defined by an increasing parameter and one by a decreasing parameter. We therefore split the curve into two pieces: the switching time curve for \( t > 0 \) and the final-time curve for \( t < 0 \). This is illustrated graphically in Fig. 1, which shows an \((X,Y)\) trajectory of a rest-to-rest switching between two set-points. In this figure, the crossing point is denoted \( M \). The unit circle denotes the splitting of the spiral into the switching-time curve \( ED \), located inside of the unit circle, and the final-time curve \( EM \), located outside of the unit circle.

Note that the spiral curve is given by (30) and is thus determined only by the characteristic polynomial of the system. The affine mapping, however, is defined by \((A_0, B_0)\) in (36) and depends not only on the system dynamics, but also on
the initial state, target state and control bounds of the system. Referring to Fig. 1, $FM$ is the image of the switching-time curve $ED$ under the affine mapping. The parameter value $t = 0$, corresponds to the point $E = (1, 0)$. As the parameter is increased, the vector $OE$ rotates anticlockwise along $ED$ and its image $O\bar{F}$ rotates anticlockwise along $FM$ until it crosses the final-time curve $EM$ at point $M$. Under the mapping, this point corresponds to the point $D$. By finding the coordinates of point $D$ and point $M$, the switching time and final time of the bang-bang control can be calculated from (26) and (30).

Unfortunately, there is no analytic solution for the coordinates of the points $D$ and $M$. However, as illustrated in Fig. 2, they can be solved efficiently using the following numerical algorithm.

Step 0: Defining a stopping criterion $\epsilon > 0$ and give an initial guess of $\angle EOD$ as $\angle EOD_1$.

Step 1: Calculate the coordinates of points $D_1$ and its image point $M_1$ under the affine mapping.

Step 2: Calculate the coordinates of points $M_2$ and $\angle M_1NM_2$.

Step 3: Let $\angle EOD_1 = \angle EOD_2 \angle M_1NM_2$, where $\gamma$ is a gain factor.

Step 4: Repeat step 1 to 3 until $\|M_1 - M_2\| < \epsilon$.

IV. SIMULATION AND EXPERIMENTAL RESULTS

We ran both simulations and physical experiments to demonstrate the feasibility of our scheme. We applied the above method to calculate the bang-bang control to drive a single axis of a 3-D piezoelectric nanopositioning stage (Nano-PDQ, Mad City Labs) from one set point to another. The stage is equipped with a position sensor with accuracy on the order of picometers as reported by manufacturer. It was operated in closed-loop mode with a proportional-integral (PI) feedback controller provided by manufacturer. A data acquisition card (NI-6259, National Instrument) was used to output the command signal to the stage controller and to sample stage position from its position sensor. Both the input and output signals of the data acquisition card range were limited to $[0, 10]$ volts. The stage position was represented by the output voltage of the position sensor. We considered a step from the set point 5 volts to the set point 6 volts.

A. Bang-bang Control Design

The transfer function of the stage was estimated by driving the stage in closed-loop mode with a step signal from 5 to 6 volts. The step signal and stage response are shown in Fig. 3. The identified stage transfer function was

$$G(s) = \frac{-261.82s + 1.8143 \times 10^6}{s^2 + 1983.3s + 1.8118 \times 10^6}. \quad (37)$$
Given the stage transfer function (37), the canonical controllable state-space model can be written as

\[
A = \begin{bmatrix} 0 & 1 \\ -1.8118 \times 10^6 & -1983.3 \end{bmatrix},
\]
(38a)

\[
B = \begin{bmatrix} -261.82 \\ 2.3336 \times 10^6 \end{bmatrix}.
\]
(38b)

The initial and target state corresponding to the set points 5 and 6 volts respectively were calculated as

\[
X_0 = \begin{bmatrix} 5.0072 \\ 1309.1 \end{bmatrix}, \quad X_r = \begin{bmatrix} 6.0086 \\ 1570.9 \end{bmatrix}.
\]
(39)

With the control signal bounds \( u_{\text{min}} = 0 \) volts and \( u_{\text{max}} = 10 \) volts, the affine mapping defined by \( A_0 \) and \( B_0 \) was calculated from (31),(32),(33) and (35) as

\[
A_0 = \begin{bmatrix} -0.8333 & 0 \\ 0 & -0.8333 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1.6667 \\ 0 \end{bmatrix}.
\]
(40)

The coordinates of point \( D \) and point \( M \) were calculated numerically as

\[
D = \begin{bmatrix} 0.5623 \\ 0.2631 \end{bmatrix}, \quad M = \begin{bmatrix} 1.1981 \\ -0.2192 \end{bmatrix}.
\]
(41)

From (26) and (30), we have

\[
t_s = 0.48075 \text{ ms}, \quad t_f = 0.67958 \text{ ms}
\]

B. Simulation Results

A simulation program was constructed in Matlab to simulate the response and state-space trajectory of the stage under the above step and bang-bang control signal. Fig. 4 shows the step response of the stage. Note that it matches very well
with the experimental result in Fig. 3. The simulated stage response under the bang-bang control signal is also shown in Fig. 5. The stage rests at the target position at the end of the control, demonstrating the feasibility of our approach. The transition time for the set point change of the stage from 5 volts to 6 volts was reduced from approximately 6 ms using the step input to 0.68 ms using the bang-bang control. The state trajectories under the step and the bang-bang control are shown in Fig. 6. The green line indicates the steady state values corresponding to the set points of the stage. The blue curve is the state trajectory under the step input for $t \in [0, t_f]$. The red curve is the state trajectory under the bang-bang control signal for $t \in [0, t_f]$. The two circle markers on the green line are the initial and target state respectively. The top circle marks the switching state of the bang-bang control.

C. Experimental Results

We also applied the designed bang-bang control signal to drive the real stage. The bang-bang control signal was output to the stage controller and the stage position sampled with a sampling frequency of $f_s = 500$ kHz by the data acquisition card. The result in Fig. 7 shows that the stage was driven to the target position at the end of the control. However, due to the modeling error and parameter uncertainty, the stage passed over the target position. Due to the closed loop controller, the stage did settle gradually to the target position after the residual vibration damped out. To suppress the unwanted overshoot and residual vibration caused by the unmodeled dynamics and parameter uncertainty, one can apply input shaping or other filtering techniques to smooth the input signal to get proximate time-optimal response with minimal excitation of unmodeled high-order modes [25], [10].

V. CONCLUSIONS

We have proposed a new approach to calculate the switching and final time of the bang-bang control with only one switch for a stable second-order system. We have shown that the switching-time curve and the final-time curve are spirals determined by the natural frequency and damping ratio of the second-order system. An affine mapping, determined by the system dynamics, the initial state and the control signal bounds, determines the switching time and final time. This can be readily calculated by finding the crossing point of the final-time curve with the image of the switching-time curve under the mapping. Both the simulation and experimental results demonstrated the feasibility of this approach. This approach tells us not only the switching time for the time-optimal control, but also how long the transition takes. This is particularly important when we want to optimize the visiting sequence to minimize the total visiting time over a collection of states before they are actually visited by the system. The future works include extension to unstable second-order system, multiple switchings and non-resting target state.

VI. ACKNOWLEDGEMENTS

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