Control of chaos in excitable physiological systems: A geometric analysis

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Model-independent chaos control techniques are inherently well-suited for the control of physiological systems for which quantitative system models are unavailable. The proportional perturbation feedback (PPF) control paradigm, which uses electrical stimulation to perturb directly the controlled system variable (e.g., the interbeat or interspike interval), was developed for excitable physiological systems that do not have an easily accessible system parameter. We develop the stable manifold placement (SMP) technique, a PPF-type technique which is simpler and more robust than the original PPF control algorithm. We use the SMP technique to control a simple geometric model of a chaotic system in the neighborhood of an unstable periodic orbit (UPO). We show that while the SMP technique can control a chaotic system that has UPO dynamics which are characterized by one stable manifold and one unstable manifold, the success of the SMP technique is sensitive to UPO parameter estimation errors. © 1997 American Institute of Physics.

I. INTRODUCTION

Deterministic chaos is characterized by dynamical behavior that appears to be random but actually is governed by a nonlinear deterministic system. Although chaos is unpredictable over long time periods, its deterministic nature often can be exploited by control techniques to obtain desired results. Chaos control techniques, which are “model-independent” because they do not require knowledge of a system’s underlying equations, have been applied successfully to a wide range of physical systems. Such success has fostered interest in applying model-independent control techniques to stabilize the fluctuations of excitable physiological systems, which are often well-understood qualitatively, but for which quantitative relationships between system components are usually incomplete. In this study, we use a geometric modeling analysis to demonstrate that model-independent control of excitable physiological systems is, in fact, dynamically feasible.

II. OGY CHAOS CONTROL

In the seminal work in the area of model-independent feedback control, Ott, Grebogi, and Yorke (OGY) developed a control technique for chaotic dynamical systems. The OGY technique is based on the fact that the state point $\xi$ of a chaotic system fluctuates continuously as it moves between infinitely many unstable periodic orbits (UPOs) embedded within the geometrically-finite state space known as the chaotic attractor. The goal of the OGY control technique is to stabilize the state point within one of these UPOs.

The OGY technique (see Fig. 1) exploits the fact that $\xi$ always approaches the UPO $\xi^\# = [x^*, x^*]^T$ (where superscript $T$ denotes transpose and $[x^*, x^*]$ is a $2 \times 1$ column vector) along a characteristic path (the stable manifold) and departs from $\xi^*$ along a different characteristic path (the unstable manifold). Although the system is nonlinear, the stable and unstable manifolds are linear in the neighborhood of $\xi^*$, where they can be approximated by the vectors $e_s$ and $e_u$, respectively. Once $\xi$ wanders into the neighborhood of the desired UPO (which is guaranteed because the chaotic attractor is ergodic), the OGY technique perturbs an accessible system parameter $p$ by an amount $\delta p$ such that $\xi^*$, $e_s$, and $e_u$ are shifted to $\tilde{\xi}^*$, $\tilde{e}_s$, and $\tilde{e}_u$, respectively. The resulting attracting force (due to $\tilde{e}_s$) and repelling force (due to $\tilde{e}_u$) move the state point from its current position $\xi_n$ to its next position $\xi_{n+1}$. At the end of the current control cycle (i.e., when $x$ is re-iterated and $p$ is returned to its initial value), the
III. PPF-TYPE CONTROL OF EXCITABLE PHYSIOLOGICAL SYSTEMS

The OGY technique, and other similar model-independent chaos control techniques, have been applied to a wide range of physical systems, including magneto-elastic ribbons, electronic circuits, lasers, chemical reactions, driven single pendulums, and a driven double pendulum. The success of chaos control in stabilizing physical systems has fostered interest in applying these techniques to excitable physiological systems. In the first such application, Garfinkel et al. stabilized irregular cardiac rhythms in tissue from the interventricular septum of a rabbit heart. Unlike the physical systems controlled by the OGY technique, no system parameter was readily-accessible for perturbation in the rabbit heart system. Garfinkel et al. therefore developed a technique known as proportional perturbation feedback (PPF) control, which is a modification of the original OGY technique. With PPF control, perturbations are made directly to the controlled system variable $x$ [where $x$ is an interbeat (cardiac) or interspike (neuronal) interval], rather than to a system parameter.

PPF control attempts to force the system’s state point towards the UPO by placing it directly onto the stable manifold. This goal is accomplished by inducing premature beats via suprathreshold electrical stimulation, thereby shortening the expected value of $x$ to a value which places it onto the stable manifold. The state point is subsequently attracted towards the UPO via the stable manifold. It is important to note that because suprathreshold electrical stimuli induce premature firings in excitable physiological systems (e.g., cardiac cells and neurons), control of this type cannot lengthen $x$. Thus, control stimuli can only be applied prior to long intervals.

In addition to the fact that the PPF control technique applies perturbations to the variable under control rather than to a system parameter, two other important differences — the frequency and size of perturbations — exist between the OGY and PPF control techniques. The OGY control technique exploits the exponential sensitivity of chaos to initial conditions by using only small perturbations to stabilize the desired UPO. However, because the exponential sensitivity causes rapid departure from the UPO, OGY perturbations must be applied at each Poincaré-map intersection to keep the system’s state point from escaping from the UPO neighborhood. Similarly, the PPF control technique could, in principle, use frequent small perturbations (as often as every other beat) to constrain the chaotic intervals within a small neighborhood around the UPO. However, such an approach is not ideal from a physiological perspective because a large percentage of action potentials would be stimulus-induced premature firings in excitable physiological systems.

Therefore require frequent stimuli to achieve a desired control result. Thus, the preferable PPF-control approach is to allow the system’s state point to wander away from the UPO along the unstable manifold until it reaches a maximum allowable distance (the control threshold) from the UPO. Once it crosses the control threshold, a PPF-interval perturbation (one which is large in comparison to the scale of perturbations required by OGY control) is introduced to force the state point onto the stable manifold. With this approach, PPF control uses relatively large, but infrequent, perturbations.

A. The PPF control technique

To place the system’s state point onto the stable manifold of a UPO, the PPF control technique requires seven steps. Initially, the algorithm must estimate (from pre-
recorded data): 1) The UPO $\xi^*$ (see Refs. 27–30 for UPO-estimation methods and related techniques), 2) the stable manifold $e_s$, 3) the unstable manifold $e_u$, and 4) the sensitivity of the UPO to perturbations:

$$ g = \delta \xi^*/\delta x. $$

(1)

Then, at each control intervention [depicted geometrically in Fig. 2(a)], the algorithm must: 5) Predict the next system value $\hat{x}_{n+1}$ from $x_n$ as:

$$ \hat{x}_{n+1} = \lambda_u (x_n - x^*) + x^*, $$

(2)

where $\lambda_u$ is the eigenvalue of the unstable manifold $e_u$. 6) Compute $\delta x$, the difference between the predicted next system value $\hat{x}_{n+1}$, and the desired next system value $x_{n+1}$ (located on the stable manifold) as:

$$ \delta x = \frac{1}{\lambda_u - 1} \left( \frac{g}{f_u} \right) (\xi_n - \xi^*) \cdot f_u, $$

(3)

where $f_u$ is the unstable contravariant eigenvector given by $f_u \cdot e_u = 1$ and $f_u \cdot e_s = 0$, and 7) Determine the intervention (i.e., stimulation) time as:

$$ x_{n+1} = \hat{x}_{n+1} + \delta x. $$

(4)

Note that because $g$ and $e_u$ are estimated quantities, the computation of $\delta x$ by Eq. (3) does not guarantee that $\xi_{n+1}$ will be placed directly onto $e_s$ via Eq. (4).

B. The SMP control technique

Here, we introduce an alternative to PPF control, called stable manifold placement (SMP). We will refer to the SMP and PPF control techniques as PPF-type control techniques, because both techniques place the system’s state point onto the stable manifold of the desired UPO by delivering a perturbation directly to the variable under control. While PPF control uses an OGY-based method for determining the intervention time, SMP control simply computes the desired intervention time directly from the algebraic equation of the stable manifold. Initially, SMP control must estimate (from pre-recorded data): 1) The UPO $\xi^*$, and 2) the stable manifold $e_s$. Then, at each control intervention [depicted geometrically in Fig. 2(b)], the algorithm only must: 3) Determine the intervention (i.e., stimulation) time:

$$ x_{n+1} = \lambda_s (x_n - x^*) + x^*, $$

(5)

where $\lambda_s$ is the eigenvalue of the stable manifold $e_s$. Note that, as shown in Figs. 2(a) and (b), the intervention times for PPF control [Eq. (4)] and SMP control [Eq. (5)] are, in principle, the same. However, because SMP control does not rely on estimation of $e_u$ or $g$, it guarantees (unlike PPF control) that $\xi_{n+1}$ is always computed such that it will be placed directly onto the estimated stable manifold $e_s$. (With PPF control, estimation errors in $e_u$ or $g$ can lead to intervention-time errors.) Furthermore, the SMP technique requires only two estimations ($\xi^*$ and $e_s$), compared to the four estimations ($\xi^*$, $e_s$, $e_u$, and $g$) required by the PPF technique. Thus, SMP control is more robust (because it guarantees that $\xi_{n+1}$ is always placed directly onto $e_s$) and less computationally intensive (because it requires the estimation of fewer system quantities) than PPF control.

IV. THE DYNAMICS OF PPF-TYPE CONTROL

In addition to the arrhythmic rabbit heart study, the PPF control technique has been used to control irregular in-
terspike intervals (ISIs) from rat hippocampal neuronal networks\textsuperscript{17} and the FitzHugh-Nagumo neuronal model.\textsuperscript{15} A shortcoming of these previous studies\textsuperscript{15–18} is that no distinction was made (in figures depicting control trials) between spontaneous action potentials and those induced via control stimulation. The lack of such a distinction eliminates the possibility of viewing the dynamical results of individual control perturbations, e.g., observing whether a perturbation is followed by the expected stable manifold approach to the UPO. Without such a distinction, it is impossible to determine whether ‘successful’ control is the result of the PPF UPO. Such dynamics are similar to those reported in the experimental applications of PPF control.\textsuperscript{16,17} This geometric model represents the system’s state point once it has entered into the linear neighborhood of a given UPO, and thus is not valid for regions of the attractor outside of that neighborhood. Nevertheless, because chaos control techniques are designed for control only within the linear UPO neighborhood, this model is appropriate for the purpose of observing PPF-type control dynamics. In this model, the system’s state point is defined as:

\[
\xi_n = [x_{n-1}, x_n]^T.
\]

We used the SMP control technique to constrain \(\xi_n\) within the linear control region around the period-1 UPO \(\xi^*\), which is characterized by one stable manifold (\(e_s\), with eigenvalue \(\lambda_s\)) and one unstable manifold (\(e_u\), with eigenvalue \(\lambda_u\)). Such dynamics are similar to those reported in the experimental applications of PPF control.\textsuperscript{16,17} This geometric model provides the freedom to vary the UPO location and the manifold eigenvalues, and therefore enables one to test the effectiveness of the control technique for varying system dynamics. With this model, the next system value \(x_{n+1}\) is mapped from the current system value \(x_n\) according to the equation of an ‘effective’ manifold (\(e_{\text{eff}}\), with eigenvalue \(\lambda_{\text{eff}}\)):

\[
x_{n+1} = \lambda_{\text{eff}}(x_n - x^*) + x^* + \xi_n,
\]

where \(\xi_n\) is an iterate of a Gaussian white-noise time series with zero mean and standard deviation \(\sigma_\xi\), and \(\lambda_{\text{eff}}\) is given by:

\[
\lambda_{\text{eff}} = \frac{b\lambda_s + a\lambda_u}{a+b},
\]

where \(a\) and \(b\) are the Euclidean distances between \(\xi_n\) and \(e_s\), and \(b\) is the Euclidean distance between \(\xi_n\) and \(e_u\). The use of such an ‘effective’ manifold to map the state point makes the assumption that the stable and unstable manifolds exert attracting and repelling forces, respectively, on \(\xi_n\) that are proportional to their distances from \(\xi^*\).\textsuperscript{32} Note that if \(\xi_n\) lies on the stable or unstable manifold, then the mapping of Eq. (7) is simply the algebraic equation of the stable or unstable manifold. If \(\xi_n\) lies between \(e_s\) and \(e_u\), as depicted in Fig. 3, then \(e_{\text{eff}}\) [as computed via Eq. (8)] will not necessarily pass through \(\xi_n\) [as in the case depicted in Fig. 3].

For this study, the state point was chosen to start at a random location on the unstable manifold near the UPO. Thus, the mapping of Eq. (7) initially dictated that the state point march away from the UPO at the exponential rate \(\lambda_u\). The state point was permitted to progress outward along the unstable manifold until \(\xi_n\) moved below the control threshold, whereupon an SMP control perturbation was applied. This control perturbation placed the next state point \(\xi_{n+1}\) onto the stable manifold according to Eq. (5). From that point, the mapping of Eq. (7) pulled \(\xi_n\) toward \(\xi^*\) via the stable manifold until the additive noise \(\xi_n\) caused Eq. (7) to repel the state point from \(\xi^*\) via the unstable manifold. When the state point crossed the control threshold again, it was pulled back onto the stable manifold with an SMP perturbation.

Figure 4(a) shows the first-return map of the first 25 points of a trial with \(\lambda_u = -0.25, \lambda_s = -1.5, \sigma_\xi = 0.0005,\) and \(\xi^* = [0.500, 0.500]^T\). Figure 4(a) shows that the state point marched away (on alternate sides) from \(\xi^*\) along \(e_u\). When the state point moved below the control threshold, an SMP control intervention that placed the next state point...
onto $e$, was applied. The state point then progressed toward $\xi^*$ until it was repelled via the unstable manifold. This pattern was repeated indefinitely, with quantitative differences between interventions resulting from the additive noise $\xi_n$. The time-domain depiction of the first 50 points of this sequence is shown in Fig. 4(b). In this depiction, the alternating, exponential divergence from $\xi^*$ along the unstable manifold, followed by an SMP perturbation-induced approach to $\xi^*$ along the stable manifold, can be clearly seen. This dynamical pattern is the expected result of successful PPF-type control.

Figure 5 (where $\lambda_s = -0.25$, $\lambda_u = -1.5$, $\sigma_\xi = 0.0005$, and $\xi^* = [0.500, 0.500]^T$) illustrates an example of the effects of finite estimation errors, which are invariably associated with experimental preparations. In this example, the real stable manifold $e_s$ and the real UPO $\xi^*$ were misestimated as $\overline{e}_s$ and $\overline{\xi}^*=[x^*, x^*]^T=[0.505, 0.505]^T$, respectively. Because of the misestimation, the SMP control perturbations were determined by substituting $\overline{x}$ for $x^*$ in Eq. (5) such that:

$$x_{n+1} = \lambda_s(x_n - \overline{x}) + \overline{x}^*.$$  

(9)

In Fig. 5(a) it can be seen that the SMP perturbations placed the state point onto the misestimated stable manifold $\overline{e}_s$. After each perturbation, the state point obeyed the “real” mapping of Eq. (7) (i.e., obeying $e_s$ and $\xi^*$, rather than $\overline{e}_s$ and $\overline{\xi}^*$) until it crossed the control threshold and the next SMP perturbation was applied. Due to the misestimation, the state point never approached the UPO as in the “successful” trial depicted in Fig. 4. In fact, the system was controlled in a “quasi” period-4 rhythm, in which the state point jumped sequentially from regions $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$ [shown in Fig. 5(a)]. This behavior can be seen in the time-domain depiction of

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**FIG. 4.** Geometric depiction of SMP control of a chaotic system in the linear control region around a UPO $\xi^*$, which is characterized by a stable manifold $e_s$ (with stable eigenvalue $\lambda_s = -0.25$) and an unstable manifold $e_u$ (with unstable eigenvalue $\lambda_u = -1.5$). The system is iterated according to Eq. (7) with $\sigma_\xi = 0.0005$, and $\xi^*=[0.500, 0.500]^T$. An SMP perturbation [according to Eq. (5)] was introduced whenever $x_n$ crossed below the control threshold ($x=0.49$). (a) Shows the first-return map of the first 25 intervals $x_n$ of the iterated sequence. (b) Shows the time-domain progression of the first 50 intervals, with SMP perturbation-induced pulses annotated by solid squares.

**FIG. 5.** Geometric depiction of SMP control of a chaotic system in the linear control region around a UPO $\xi^*$, which is characterized by a stable manifold $e_s$ (with stable eigenvalue $\lambda_s = -0.25$) and an unstable manifold $e_u$ (with unstable eigenvalue $\lambda_u = -1.5$). The system is iterated according to Eq. (7) with $\sigma_\xi = 0.0005$, and $\xi^*=[0.500, 0.500]^T$. In this trial, the real stable manifold $e_s$ and the real UPO $\xi^*$ were misestimated as $\overline{e}_s$ and $\overline{\xi}^*$, respectively. An SMP perturbation [according to Eq. (9)] was introduced whenever $x_n$ crossed below the control threshold ($x=0.49$). (a) Shows the first-return map of the first 25 intervals $x_n$ of the iterated sequence. The system was controlled in a ‘quasi’ period-4 rhythm, in which the state point jumped sequentially from regions $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$. (b) Shows the time-domain progression of the first 50 intervals, with SMP perturbation-induced pulses annotated by solid squares.
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23 Two control techniques (Refs. 19 and 20) that also directly perturb the controlled system's variable have been developed control of cardiac chaos modeled on a onedimensional map system. These techniques are only applicable to one-dimensional systems, in contrast to the PPF control technique which is applicable to higher dimensional systems.
24 Recently, an OGY-type control technique has been used to control a pathological cardiac rhythm in a numerical model (Ref. 21) and in an in vitro rabbit heart preparation (Ref. 22).
25 For PPF control, it is important to draw a distinction between the suprathreshold stimulus and the perturbation. The stimulus is clearly a large perturbation to the action-potential dynamics (as noted in Ref. 19); however, the PPF perturbation (Δt) can be quite small.
26 With demand pacing, stimuli are used to prevent x from exceeding the pre-determined value. See: R. M. Neumann, in Medical Instrumentation Application and Design, 2nd ed., edited by J. G. Webster (Houghton Mifflin, Boston, 1992), pp. 699–700.
32 Note that the results of this study are not dependent on the assumption of a linear relationship between manifold force (on E) and the distance from the manifold. Equation (7) simply provides a method for mapping E when it does not lie exactly on e or e' .
33 UPOs appear as unstable periodic fixed points on a first-return map.